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By

Hugo Aimar, Eleonor Harboure and Miguel Marcos

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Publications Director: Dr. Rubén Spies
E-mail: rspies@santafe-conicet.gov.ar

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An Extension Theorem for Besov functions on metric spaces

Hugo Aimar, Eleonor Harboure and Miguel Marcos *

Instituto de Matemática Aplicada del Litoral (CONICET-UNL)
Departamento de Matemática (FIQ-UNL)

Abstract

As the first part of a wider program related to traces of Besov functions to sets of lower dimentsion in metric measure spaces, in this note we prove an extension theorem for Besov functions on general spaces of homogeneous type.

1 Introduction and Main Result

Given a set X and a subset F, the problem of extending functions from a certain Banach space $B_1(F)$ of functions defined in F to another one $B_2(X)$ of functions defined in X naturally arises. This means finding a bounded linear operator

$$\mathcal{E} : B_1(F) \to B_2(X),$$

satisfying that an 'extended' function $\mathcal{E}f$ recovers f when restricted to F.

When F is a closed subset of $\mathbb{R}^n$, in 1934 Whitney developed a method to define differentiable functions in F. This method can adjust to different notions of smoothness. The strategy consists in partitioning the complement of F in *Whitney cubes* with diameter comparable to its distance to F, then with those cubes building a partition of unity and use it to build the extension operator.

In [S], Stein defines Besov spaces in $\mathbb{R}^n$ for $0 < \alpha < 1$ and $1 \leq p, q \leq \infty$, $\Lambda_\alpha^{p,q}$, as those $f \in L^p$ with

$$\|f\|_p + \left(\int_0^\infty (t^{-\alpha}\omega_p f(t))^q \frac{dt}{t} \right)^{1/q} < \infty$$

where $\omega_p f(t) = \sup_{|h|<t} \|\Delta_h f\|_p$ is the modulus of continuity, and with the usual modification for $q = \infty$.

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For *d-sets* F of $\mathbb{R}^n$, Jonsson and Wallin (see [JW]) define the Besov space $B_\alpha^{p,q}(F)$ as those functions $f \in L^p(F, \mathcal{H}^d)$ with

$$\left(\sum_k 2^{k\alpha} \int_F \fint_{F \cap B(s,2^{-k})} |f(s) - f(t)|^p d\mathcal{H}^d(t) d\mathcal{H}^d(t) \right)^{1/q} < \infty$$

and the usual modification for $q = \infty$. Here $\mathcal{H}^d$ denotes Hausdorff d-dimensional measure.

They then prove that there exists an extension operator

$$\mathcal{E} : B_\beta^{p,q}(F) \to \Lambda_\alpha^{p,q},$$

where $0 < \alpha = \beta + (n - d)/p < 1$.

We prove a similar extension theorem for a certain kind of Besov Spaces on metric measure spaces. Instead of asking for dimensions n and d for the space and the subspace, we ask for a doubling condition in both spaces, and the existence of a *'local difference of dimensions'* $n - d$. The precise statement of the theorem is as follows:

Theorem 1.1. *Let (X, d, m) be a doubling metric measure space, and let $F \subset X$ be closed with $m(F) = 0$. If μ is a nontrivial Borel measure with support F which is doubling for balls centered in F, and if there exists $\gamma > 0$ and $R_0 > 0$ such that*

$$\frac{m(B)}{\mu(B)} \sim r_B^\gamma \tag{1}$$

for balls B centered in F with radius $r_B < R_0$, then there is an extention operator $\mathcal{E}$ for functions $f \in L^1_{loc}(F, \mu)$ that satisfies, for $\beta > 0$, $1 \le p < \infty$ and $1 \le q \le \infty$

$$\mathcal{E} : B^\beta_{p,q}(F, \mu) \to B^\alpha_{p,q}(X, m)$$

for $\alpha = \beta + \gamma/p$ if $\beta < 1 - \gamma/p$ and $1 \le q \le \infty$ or $\alpha = 1$ if $\beta = 1$ and $q = \infty$.

Certainly, Theorem 1.1 contains the classical result in Theorem 1, Chapter VI from [JW] for the case $0 < \alpha < 1$.

To ilustrate our result, we observe that the cases in which X and F are Ahlfors n-regular and d-regular, respectively, satisfy the quotient realtion 1 and therefore the theorem applies. In another context, if F is a doubling measure space and Y si Ahlfors γ-regular, the spaces $X = F \times Y$ and F also satisfy the hypotheses, if we take the product metric and the product measure for X.

In section 2 we introduce the basic terminology and some auxiliary results needed to prove the theorem, and in section 3 we present our proof.

2 Preliminaries and auxiliary results

We say that (X, d, m) is a metric measure space if (X, d) is a metric space and m is a Borel measure on X that is positive and finite for all balls on X. Furthermore, we say that m is doubling if there exists a constant A such that

$$m(2B) \le Am(B),$$

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where B is any ball in X and κB is the ball with the same center and κ times the radius.

Throughout this paper, constants are labeled C or c, and their specific value can change from line to line. We say that two objects (functions or numbers) r, s are equivalent, $r \sim s$, if there exist constants $c, C > 0$ such that $cr \leq s \leq Cr$.

In $\mathbb{R}^n$, one can check that the modulus of continuity satisfies, for $1 \leq p < \infty$,

$$\omega_p f(t) \sim \left(\fint_{B(0,t)} \|\Delta_h f\|_p^p dh \right)^{1/p},$$

and, as [GKS] show, changing the order of integration,

$$\fint_{B(0,t)} \|\Delta_h f\|_p^p dh = \int_{\mathbb{R}^n} \fint_{B(x,t)} |f(y) - f(x)|^p dy dx.$$

Using this, one can define (as [GKS] do) non-homogeneous Besov spaces $B^\alpha_{p,q}$, $\alpha > 0$, in an arbitrary metric measure space (X, d, m) as those $f \in L^p(X)$ with

$$\|f\|_p + \left(\int_0^\infty (t^{-\alpha} E_p f(t))^q \frac{dt}{t} \right)^{1/q} < \infty,$$

where the modulus of continuity is now defined as

$$E_p f(t) = \left(\int_X \fint_{B(x,t)} |f(y) - f(x)|^p dm(y) dm(x) \right)^{1/p}$$

(with the usual modifications if $p = \infty$ or $q = \infty$).

For other definitions of Besov Spaces in metric spaces and their relationships, see [HS], [MY] and [GKS]. One equivalent form we need is the following:

Lemma 2.1. *If m is doubling,*

$$\|t^{-\alpha} E_p f(t)\|_{L^q((0,\infty), \frac{dt}{t})} \sim \left\| \left(2^{l\alpha} E_p f(2^{-l})\right)_{l \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})}.$$

Proof. Assume first $1 \leq p < \infty$. For $2^{-l-1} \leq t < 2^{-l}$, as m is doubling we have that

$$E_p f(t)^p = \int_X \fint_{B(x,t)} |f(y) - f(x)|^p dm(y) dm(x) \leq$$

$$\leq \int_X \frac{1}{m(B(x, 2^{-l-1}))} \int_{B(x,2^{-l})} |f(y) - f(x)|^p dm(y) dm(x) \leq$$

$$\leq A \int_X \fint_{B(x,2^{-l})} |f(y) - f(x)|^p dm(y) dm(x) = A E_p f(2^{-l})^p,$$

and similarly

$$E_p f(t)^p \geq \frac{1}{A} E_p f(2^{-l-1})^p.$$

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We also have that

$$E_\infty f(2^{-l-1}) \le E_\infty f(t) \le E_\infty f(2^{-l}).$$

Finally, as we have

$$c2^{\alpha(l+1)} E_p f(2^{-(l+1)}) \le t^{-\alpha} E_p f(t) \le C 2^{\alpha l} E_p f(2^{-l}),$$

we get the conclusion. □

If m is doubling, then (X, d) satisfies the weak homogeneity property and the following two results hold (see [A]):

Lemma 2.2. ***Whitney type covering + partition of unity.*** *Let (X, d) be a metric space with the weak homogeneity property. Let F be a closed subset of X and $\Omega = \{x \in X : 0 < d(x, F) < 1\}$. Then there exists a (countable) collection $\{B_i = B(x_i, r_i)\}_i$ of balls satisfying*

1. *$\{B_i\}$ are pairwise disjoint;*
2. *$\cup_i 3B_i = \Omega$;*
3. *$6B_i \subset \Omega$ for each i;*
4. *$6r_i \le d(x, F) \le 18r_i$ for each $x \in 6B_i$, for each i;*
5. *for each i there exists $y_i \in F$ satisfying $d(x_i, y_i) < 18r_i$;*

Furthermore, there exists a collection $(\varphi_i)_i$ of real functions satisfying

1. *$3B_i \subset supp\varphi_i \subset 6B_i$;*
2. *$0 \le \varphi_i \le 1$;*
3. *$\sum_i \varphi_i = \chi_\Omega$;*
4. *$\varphi_i \equiv 1$ in B_i;*
5. *for each i, $|\varphi_i(x) - \varphi_i(y)| \le \frac{C}{r_i} d(x, y)$ with C independent of i.*

Lemma 2.3. ***Bounded overlap.*** *Let (X, d) be a metric space with the weak homogeneity property and let $1 \le a < b$, $\kappa > 1$. There exists a constant C such that, if $\{B_i = B(x_i, r_i)\}_i$ is a family of disjoint balls, and $r > 0$,*

$$\sum_{i: ar \le r_i \le br} \chi_{\kappa B_i} \le C.$$

We also need the following discrete version of Hardy's Inequality:

Lemma 2.4. ***Hardy's Inequality.*** *Let (b_n) be a sequence of nonnegative terms, $\gamma > 0$ and $a > 0$, then there exists $C > 0$ such that*

$$\sum_{n=0}^{\infty} 2^{-na} \left(\sum_{k=0}^{n} b_k \right)^{\gamma} \le C \sum_{n=0}^{\infty} 2^{-na} b_n^{\gamma}.$$

Proof. If $\gamma \le 1$, the result is trivial:

$$\sum_{n=0}^{\infty} 2^{-na} \left(\sum_{k=0}^{n} b_k \right)^{\gamma} \le \sum_{n=0}^{\infty} \sum_{k=0}^{n} 2^{-na} b_k^{\gamma} = \sum_{k=0}^{\infty} b_k^{\gamma} 2^{-ka} \sum_{n=k}^{\infty} 2^{-(n+k)a} = C \sum_{k=0}^{\infty} b_k^{\gamma} 2^{-ka}.$$

See [L] for the case $\gamma > 1$. □

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3 Proof of Theorem 1.1

Without loss of generality, we assume all balls used in the proof satisfy 1, this can always be done by modifying the set Ω used in Whitney's partition.

Let us first define the extension operator: let $\{B_i, \varphi_i\}_i$ be as in 2.2. If $f \in L^1_{loc}(F, \mu)$ and $x \in X \backslash F$, we define

$$\mathcal{E}f(x) = \sum_i \varphi_i(x) \fint_{18B_i} f d\mu.$$

Extension part.

First we need to check that $\mathcal{E}f$ is an extension of f, i.e. that $\mathcal{E}f|_F = f$. This is, μ-almost every point in F is an m-Lebesgue point of $\mathcal{E}f$, and for those points $\mathcal{E}f(x) = f(x)$.

Let $t_0 \in F$ and $r > 0$. As

$$\mathcal{E}f(x) - f(t_0) = \sum_i \varphi_i(x) \fint_{18B_i} (f(t) - f(t_0)) d\mu(t),$$

we have

$$|\mathcal{E}f(x) - f(t_0)| \leq \sum_i \chi_{6B_i}(x) \fint_{18B_i} |f(t) - f(t_0)| d\mu(t)$$

and for $d(x, F) \sim 2^{-k}$, $d(x, t_0) < r$,

$$|\mathcal{E}f(x) - f(t_0)| \leq \int_F |f(t) - f(t_0)| \left(\sum_{r_i \sim 2^{-k}} \chi_{18B_i}(t) \frac{\chi_{6B_i}(x)}{\mu(18B_i)} \right) d\mu(t) \leq$$

$$\leq \int_{B(t_0, r + c2^{-k})} |f(t) - f(t_0)| \left(\sum_{r_i \sim 2^{-k}} \chi_{18B_i}(t) \frac{\chi_{18B_i}(x)}{\mu(18B_i)} \right) d\mu(t) \leq$$

$$\leq \int_{B(t_0, cr)} |f(t) - f(t_0)| \left(\sum_{r_i \sim 2^{-k}} \chi_{18B_i}(t) \frac{\chi_{18B_i}(x)}{\mu(18B_i)} \right) d\mu(t);$$

then by bounded overlap,

$$\int_{B(t_0,r)} |\mathcal{E}f(x) - f(t_0)| dm(x) \leq \sum_{2^{-k} \leq cr} \int_{x \in B(t_0,r), d(x,F) \sim 2^{-k}} |\mathcal{E}f(x) - f(t_0)| dm(x) \leq$$

$$\leq \sum_{2^{-k} \leq cr} \int_{B(t_0, cr)} |f(t) - f(t_0)| \left(\sum_{r_i \sim 2^{-k}} \chi_{18B_i}(t) \frac{m(18B_i)}{\mu(18B_i)} \right) d\mu(t) \leq$$

$$\leq C \left(\sum_{2^{-k} \leq cr} 2^{-k\gamma} \right) \int_{B(t_0, cr)} |f(t) - f(t_0)| d\mu(t) \leq Cr^\gamma \int_{B(t_0, cr)} |f(t) - f(t_0)| d\mu(t) \leq$$

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$$\leq C\frac{m(B(t_0,cr))}{\mu(B(t_0,cr))}\int_{B(t_0,cr)}|f(t)-f(t_0)|d\mu(t).$$

In other words,

$$⨍_{B(t_0,r)}|\mathcal{E}f(x)-f(t_0)|dm(x)\leq C⨍_{B(t_0,cr)}|f(t)-f(t_0)|d\mu(t),$$

and as the right side of the inequality tends to zero as $r\to 0$ for $\mu-$almost every $t_0\in F$ (because μ is doubling and $f\in L^1_{loc}(F)$), so does the left side and we have $\mathcal{E}f(t_0)=f(t_0)$ for $\mu-$almost every $t_0\in F$.

L^p part.

We need to check now that there exists $C>0$ such that $\|\mathcal{E}f\|_{p,m}\leq C\|f\|_{p,\mu}$ for every $f\in L^p(F,\mu)$.

We have

$$|\mathcal{E}f(x)|=\left|\sum_i\varphi_i(x)⨍_{18B_i}fd\mu\right|\leq\sum_i\chi_{6B_i}(x)\left(⨍_{18B_i}|f|^pd\mu\right)^{1/p},$$

and so by Hölder's inequality,

$$|\mathcal{E}f(x)|^p\leq C\sum_i\chi_{6B_i}(x)⨍_{18B_i}|f|^pd\mu\leq C\int_F|f(t)|^p\left(\sum_i\chi_{18B_i}(t)\frac{\chi_{18B_i}(x)}{\mu(18B_i)}\right)d\mu(t).$$

Once again by bounded overlap

$$\int_X|\mathcal{E}f(x)|^pdm(x)\leq C\int_F|f(t)|^p\left(\sum_i\chi_{18B_i}(t)\frac{m(18B_i)}{\mu(18B_i)}\right)d\mu(t)\leq$$

$$\leq C\int_F|f(t)|^p\left(\sum_k\sum_{r_i\sim 2^{-k}}\chi_{18B_i}(t)\frac{m(18B_i)}{\mu(18B_i)}\right)d\mu(t)\leq$$

$$\leq C\int_F|f(t)|^p\left(\sum_k 2^{-k\gamma}\right)d\mu(t)\leq C\|f\|^p_{p,\mu}.$$

Besov part.

For this last part, we need to check that there exists $C>0$ such that $[\mathcal{E}f]_{B^\alpha_{p,q}(X,m)}\leq C[f]_{B^\beta_{p,q}(F,\mu)}$ for every $f\in B^\beta_{p,q}(F,\mu)$.

We will first prove that

$$E_p\mathcal{E}f(2^{-l})^p\leq C2^{-lp}\sum_{k\leq l}2^{k(p-\gamma)}E_pf(c2^{-k})^p+C2^{-l\gamma}E_pf(c2^{-l})^p.$$

For this, we split $E_p\mathcal{E}f(2^{-l})^p$ in two parts:

$$E_p\mathcal{E}f(2^{-l})^p\leq\int_{d(x,F)\lesssim 2^{-l}}⨍_{B(x,2^{-l})}|\mathcal{E}f(x)-\mathcal{E}f(y)|^pdm(y)dm(x)+$$

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$$+\int_{d(x,F)\gtrsim 2^{-l}} \fint_{B(x,2^{-l})} |\mathcal{E}f(x)-\mathcal{E}f(y)|^p dm(y)dm(x) = I + II.$$

For II, $d(x,F)\sim 2^{-k}$ for some $k\leq l$, and we can write

$$\mathcal{E}f(x)-\mathcal{E}f(y) = \sum_i (\varphi_i(x)-\varphi_i(y)) \fint_{18B_i}\fint_{B(x,2^{-k})} f(s)-f(t)d\mu(t)d\mu(s),$$

and for $x\in 3B_j$, $r_j\sim 2^{-k}$,

$$\int_{3B_j}\fint_{B(x,2^{-l})} |\mathcal{E}f(x)-\mathcal{E}f(y)|^p dm(y)dm(x) \leq$$

$$\leq C\int_{3B_j}\fint_{B(x,2^{-l})}\sum_i |\varphi_i(x)-\varphi_i(y)|^p \fint_{18B_i}\fint_{B(x,2^{-k})} |f(s)-f(t)|^p d\mu(t)d\mu(s)dm(y)dm(x) \leq$$

$$\leq C\int_{3B_j}\fint_{B(x,2^{-l})}\sum_{i:x\vee y\in supp\varphi_i} d(x,y)^p r_i^{-p} \fint_{cB_j}\fint_{B(s,c2^{-k})} |f(s)-f(t)|^p d\mu(t)d\mu(s)dm(y)dm(x) \leq$$

$$\leq C2^{kp}2^{-lp}\frac{m(3B_j)}{\mu(cB_j)}\int_{cB_j}\fint_{B(s,c2^{-k})} |f(s)-f(t)|^p d\mu(t)d\mu(s) \leq$$

$$\leq C2^{kp}2^{-lp}2^{-k\gamma}\int_{cB_j}\fint_{B(s,c2^{-k})} |f(s)-f(t)|^p d\mu(t)d\mu(s),$$

so now we add up in $j : r_j\sim 2^{-k}$ and $k\leq l$, to get

$$II \leq \sum_{k\leq l}\sum_{r_j\sim 2^{-k}}\int_{3B_j}\fint_{B(x,2^{-l})} |\mathcal{E}f(x)-\mathcal{E}f(y)|^p dm(y)dm(x) \leq$$

$$\leq C2^{-lp}\sum_{k\leq l} 2^{kp}2^{-k\gamma}\int_F \left(\sum_{r_j\sim 2^{-k}} \chi_{cB_j}(s)\right)\fint_{B(s,c2^{-k})} |f(s)-f(t)|^p d\mu(t)d\mu(s) \leq$$

$$\leq C2^{-lp}\sum_{k\leq l} 2^{kp}2^{-k\gamma}E_pf(c2^{-k})^p.$$

Now for I, as $d(x,F)\lesssim 2^{-l}$ and $d(x,y)<2^{-l}$, we have $d(y,F)\lesssim 2^{-l}$ and there exist $k,m\geq l$ such that $d(x,F)\sim 2^{-k}, d(y,F)\sim 2^{-m}$ and, as we can write

$$\mathcal{E}f(x)-\mathcal{E}f(y) = \sum_i\sum_j \varphi_i(x)\varphi_j(y)\fint_{18B_i}\fint_{18B_j} f(s)-f(t)d\mu(t)d\mu(s),$$

we have that

$$|\mathcal{E}f(x)-\mathcal{E}f(y)|^p \leq$$

$$\leq C\sum_{r_i\sim 2^{-k}}\sum_{r_j\sim 2^{-m}} \chi_{6B_i}(x)\chi_{6B_j}(y)\fint_{B(x,c2^{-k})}\fint_{B(y,c2^{-m})} |f(s)-f(t)|^p d\mu(t)d\mu(s) \leq$$

$$\leq C\fint_{B(x,c2^{-k})}\fint_{B(y,c2^{-m})} |f(s)-f(t)|^p d\mu(t)d\mu(s).$$

Integrating first with respect to y,

$$\int_{y\in B(x,2^{-l}),d(y,F)\sim 2^{-m}} |\mathcal{E}f(x)-\mathcal{E}f(y)|^p dm(y) \leq$$

$$\leq C \fint_{B(x,c2^{-k})} \int_{y\in B(x,2^{-l}),d(y,F)\sim 2^{-m}} \fint_{B(y,c2^{-m})} |f(s)-f(t)|^p d\mu(t)dm(y)d\mu(s),$$

but

$$\int\limits_{y\in B(x,2^{-l}),d(y,F)\sim 2^{-m}} \fint\limits_{B(y,c2^{-m})} |f(s)-f(t)|^p d\mu(t)dm(y) \leq$$

$$\leq \sum_{r_h\sim 2^{-m}} \int\limits_{B(x,2^{-l})\cap cB_h} \fint\limits_{B(y,c2^{-m})} |f(s)-f(t)|^p d\mu(t)dm(y) \leq$$

$$\leq C \int\limits_{B(x,2^{-l}+c2^{-m})} |f(s)-f(t)|^p \int\limits_{B(t,c2^{-m})} \frac{1}{\mu(B(y,c2^{-m}))} \left(\sum_{r_j\sim 2^{-m}} \chi_{aB_j}(y)\right) dm(y)d\mu(t) \leq$$

$$\leq C2^{-m\gamma} \int\limits_{B(x,c2^{-l})} |f(s)-f(t)|^p d\mu(t),$$

so we get

$$\int_{y\in B(x,2^{-l}),d(y,F)\sim 2^{-m}} |\mathcal{E}f(x)-\mathcal{E}f(y)|^p dm(y) \leq$$

$$\leq C2^{-m\gamma} \fint\limits_{B(x,c2^{-k})} \int\limits_{B(x,c2^{-l})} |f(s)-f(t)|^p d\mu(t)d\mu(s);$$

and now integrating in x,

$$\sum_{r_i\sim 2^{-k}} \int_{3B_i} \frac{1}{m(B(x,2^{-l}))} \int\limits_{y\in B(x,2^{-l}),d(y,F)\sim 2^{-m}} |\mathcal{E}f(x)-\mathcal{E}f(y)|^p dm(y)dm(x) \leq$$

$$\leq C2^{-m\gamma} \sum_{r_i\sim 2^{-k}} \int\limits_{3B_i} \frac{2^{l\gamma}}{\mu(B(x,c2^{-l}))} \fint\limits_{B(x,c2^{-k})} \int\limits_{B(x,c2^{-l})} |f(s)-f(t)|^p d\mu(t)d\mu(s)dm(x) \leq$$

$$\leq C2^{-m\gamma}2^{l\gamma} \int\limits_F \fint\limits_{B(s,c2^{-l})} \frac{|f(s)-f(t)|^p}{\mu(B(s,c2^{-k}))} \int\limits_{B(s,c2^{-k})} \left(\sum_{r_i\sim 2^{-k}} \chi_{3B_i}(x)\right) dm(x)d\mu(t)\mu(s) \leq$$

$$\leq C2^{-m\gamma}2^{l\gamma}2^{-k\gamma} \int_F \fint_{B(s,c2^{-l})} |f(s)-f(t)|^p d\mu(t)\mu(s) = C2^{-m\gamma}2^{l\gamma}2^{-k\gamma} E_pf(c2^{-l})^p.$$

Finally, adding in k,

$$I \leq \sum_{k\geq l} \int_{d(x,F)\sim 2^{-k}} \fint_{B(x,2^{-l})} |\mathcal{E}f(x)-\mathcal{E}f(y)|^p dm(y)dm(x) \leq$$

$$\leq C2^{l\gamma}E_pf(2^{-l})^p \sum_{k\geq l}\sum_{m\geq l} 2^{-m\gamma}2^{-k\gamma} \leq C2^{-l\gamma}E_pf(c2^{-l})^p.$$

Now, we have to consider each case separately. For $q<\infty$ and $\alpha=\beta+\gamma/p$, by Hardy's Inequality ($\nu=q/p$) we have that

$$\sum_l 2^{l\alpha q}(E_p\mathcal{E}f(2^{-l})^p)^{q/p} \leq$$

$$\leq C\sum_l 2^{l\beta q}2^{l\gamma q/p}2^{-lq}\left(\sum_{k\leq l} 2^{k(p-\gamma)}E_pf(c2^{-k})^p\right)^{q/p} + C\sum_l 2^{l\beta q}2^{l\gamma q/p}2^{-l\gamma q/p}E_pf(c2^{-l})^q \leq$$

$$\leq C\sum_l 2^{l\beta q}2^{l\gamma q/p}2^{-lq}2^{l(q-\gamma q/p)}E_pf(c2^{-l})^q + C\sum_l 2^{l\beta q}2^{l\gamma q/p}2^{-l\gamma q/p}E_pf(c2^{-l})^q \leq$$

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$$\leq C \sum_{l} 2^{l\beta q} E_p f(c2^{-l})^q.$$

For $q = \infty$ and $\alpha = \beta + p/q < 1$,

$$2^{-lp} \sum_{k\leq l} 2^{kp} 2^{-k\gamma} E_p f(c2^{-k})^p \leq 2^{-lp} \left(\sup_k 2^{k\beta p} E_p f(c2^{-k})^p \right) \left(\sum_{k\leq l} 2^{kp} 2^{-k\gamma} 2^{-k\beta p} \right) \leq$$

$$\leq C 2^{-lp} 2^{l(1-\alpha)p} \left(\sup_k 2^{k\beta p} E_p f(c2^{-k})^p \right) = C 2^{-l\alpha p} \left(\sup_{k\leq l} 2^{k\beta p} E_p f(c2^{-k})^p \right),$$

so

$$\sup_l 2^{l\alpha p} E_p \mathcal{E} f(2^{-l})^p \leq C \sup_l 2^{l\beta p} E_p f(c2^{-l})^p.$$

Finally, for $q = \infty$ and $\alpha = \beta = 1$,

$$\sup_l 2^{lp} E_p \mathcal{E} f(2^{-l})^p \leq C \sup_k 2^{kp} E_p f(c2^{-k})^p + C \sup_l 2^{lp} E_p f(c2^{-l})^p.$$

$\square$

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E-mail address: `haimar@santafe-conicet.gov.ar`

E-mail address: `harbour@santafe-conicet.gov.ar`

E-mail address: `mmarcos@santafe-conicet.gov.ar`