ISSN 2451-7100

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AN EXTENSION OF NEWTON-SOBOLEV SPACES FOR CURVES NOT MEASURED BY ARC LENGTH

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IMAL PREPRINT # 2014-0016

Publication date: July 4, 2014

Editorial: Instituto de Matemática Aplicada del Litoral
IMAL (CCT CONICET Santa Fe – UNL)
http://www.imal.santafe-conicet.gov.ar

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ISSN 2451-7100
IMAL PREPRINT # 2014-0016 Publication date: July 04, 2014

An extension of Newton-Sobolev spaces for curves not measured by arc length

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Abstract

Newton-Sobolev spaces, as presentend by N. Shanmugalingam, describe a way to extend Sobolev spaces to the metric setting, for metric spaces with 'sufficient' paths of finite length. We generalize some of this results to spaces where the 'length' of a path is measured differently.

1 Introduction

If Ω is an open set in $\mathbb{R}^n$ and f is a smooth function defined on Ω, the Fundamental Theorem of Calculus for Line Integrals implies that for every piecewise smooth path γ with endpoints x, y we get

$$|f(x) - f(y)| \leq \int_\gamma |\nabla f| d|s|.$$

Nonnegative functions that satisfy this inequality in place of $|\nabla f|$ are referred to as *upper gradients* (see for example [HeK]).

In [Sh], N. Shanmugalingam describes, via upper gradients, a way to characterize Sobolev spaces $W^{1,p}$ in open sets of $\mathbb{R}^n$ that extends to metric measure spaces, defining Newton-Sobolev spaces $N^{1,p}$. If the space has 'sufficient' rectifiable paths (in the sense that the set of rectifiable paths has nonzero p-modulus), an interesting theory of Sobolev functions can be developed, but if the set of rectifiable paths is negligible, this 'Sobolev space' is just L^p.

Easy enough examples of metric measure spaces with no paths of dimension 1 can be constructed. For instance, take $X = \mathbb{R}$ with $d(x, y) = |x - y|^{1/2}$, and we get that paths are either 0-dimensional (trivial paths) or 2-dimensional. While 'classical' Newton-Sobolev theory in such a space would be nonsensical, a good theory could be developed if we measured path 'length' by Hausdorff

*The author was suportted by Consejo Nacional de Investigaciones Científicas y Técnicas and Universidad Nacional del Litoral.
Keywords and phrases: Sobolev Spaces, Spaces of Homogeneous Type
2010 Mathematics Subject Classification: Primary 43A85.

ISSN 2451-7100
IMAL PREPRINT # 2014-0016 Publication date: July 04, 2014

2-dimensional measure $\mathcal{H}^2$.

In this note, following the ideas in [Sh], we develop a more general theory of Newton-Sobolev spaces by replacing Hausdorff 1-dimensional measure by an arbitrary measure μ as a way of measuring path 'lengths'.

In sections 2 and 3 we generalize all the machinery needed to construct Newton-Sobolev spaces. In section 4 we define this spaces and prove they are complete. In section 5 we define some additional properties, such as Poincaré inequality, needed to prove some more interesting results, as Lipschitz density or Sobolev embeddings. We also compare Newton-Sobolev spaces with another kind of Sobolev space in metric spaces: Hajłasz-Sobolev spaces.

2 μ-arc legth and upper gradients

Given a metric space (X, d) and a (compact) path $\gamma : [a, b] \to X$ (i.e. a continuous function from $[a, b]$ into X), its length is defined as

$$l(\gamma) = \sup_{(t_i)_i} \sum_i d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all partitions of $[a, b]$.

We use the notation $|\gamma|$ for $Im(\gamma)$. We say that $\tilde{\gamma}$ is a sub-path of γ if it's the restriction of of γ to a subinterval $[a, b]$.

The concept of arc legth of a path is similar to, but not equal to, Hausdorff one-dimensional measure $\mathcal{H}^1$ of its image, but they do coincide for injective paths (see [Fa]). From this result, for injective paths and for Borel nonnegative measurable functions we get that

$$\int_\gamma g = \int_{|\gamma|} g d\mathcal{H}^1,$$

and from this we can think of changing the measure $\mathcal{H}^1$ for another Borel measure, as $\mathcal{H}^s$.

Let μ be a non-atomic Borel measure in X. Define Γ^μ as the set of all non trivial injective paths γ in X such that $0 < \mu(|\tilde{\gamma}|) < \infty$ for all non trivial subpaths of γ. For nonnegative Borel functions $g : X \to [0, \infty]$ we define

$$\int_\gamma g = \int_{|\gamma|} g d\mu.$$

Now, for a path $\gamma : [a, b] \to X$ in Γ^μ, we define $h(\gamma) = \mu(|\gamma|)$ and its μ-arc length $\nu_\gamma : [a, b] \to \mathbb{R}$ as

$$\nu_\gamma(x) = h(\gamma|_{[a,x]}).$$

Lemma 2.1. *For paths $\gamma : [a, b] \to X$ in Γ^μ, we have that ν_γ is strictly increasing, continuous, onto $[0, h(\gamma)]$, and besides*

$$h(\gamma) = h(\gamma|_{[a,x]}) + h(\gamma|_{[x,b]}).$$

ISSN 2451-7100
IMAL PREPRINT # 2014-0016 Publication date: July 04, 2014

Proof. ν_γ is clearly increasing. Continuity follows from μ being non-atomic, and surjectivity follows from it being continuous and increasing.
The fact that ν_γ is strictly increasing follows from the fact that every non trivial subcurve of γ has positive measure, as $\gamma \in \Gamma^\mu$. $\square$

Theorem 2.2. *For $\gamma : [a, b] \to X$ in Γ^μ, there exists a unique $\gamma_h : [0, h(\gamma)] \to X$ such that*

$$\gamma = \gamma_h \circ \nu_\gamma,$$

$|\gamma| = |\gamma_h|$ and $\nu_{(\gamma_h)}(t) = t$ in $[0, h(\gamma)]$ (therefore $\gamma_h = \gamma_h \circ \nu_{\gamma_h}$). We call this the μ-arc length parametrization of γ.

Proof. As $\nu_\gamma : [a, b] \to [0, h(\gamma)]$ is strictly increasing and onto, it's a bijection between $[a, b]$ and $[0, h(\gamma)]$ and we can define

$$\gamma_h = \gamma \circ \nu_\gamma^{-1}.$$

We inmediately see that $|\gamma| = |\gamma_h|$, and

$$\nu_{(\gamma_h)}(t) = \mu(\gamma_h([0, t])) = \mu(\gamma(\nu_\gamma^{-1}([0, t]))) = \mu(\gamma([a, \nu_\gamma^{-1}(t)])) = \nu_\gamma(\nu_\gamma^{-1}(t)) = t.$$

$\square$

Theorem 2.3. *If $\gamma : [0, h] \to X$ is a path in Γ^μ parametrized by μ-arc length, then for every Borel set B of $[0, h]$, we have*

$$\mu(\gamma(B)) = l(B).$$

Furthermore, if $g : X \to \mathbb{R}$ is nonnegative and Borel measurable, then for each subpath $\tilde{\gamma} = \gamma|_{[a,b]}$ we have

$$\int_{\tilde{\gamma}} g = \int_a^b g \circ \tilde{\gamma}.$$

Finally, we get the same result as with rectifiable curves:

Theorem 2.4. *Given a function $f : X \to \mathbb{R}$ and a path $\gamma : [0, h] \to X$ in Γ^μ parametrized by μ-arc length, if there exists a Borel measurable nonnegative $\rho : X \to \mathbb{R}$ satisfying*

$$|f(\gamma(s)) - f(\gamma(t))| \le \int_{\gamma|_{[s,t]}} \rho < \infty$$

for every $0 \le s < t \le h$, then $f \circ \gamma : [0, h] \to \mathbb{R}$ is absolutely continuous.

Proof. Let $\epsilon > 0$. As $\rho \in L^1(|\gamma|, \mu)$, by absolute continuity of the integral there exists $\delta > 0$ such that for every $E \subset |\gamma|$ with $\mu(E) < \delta$ we have $\int_E \rho d\mu < \epsilon$. Then if $0 \le a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n \le h$ satisfy $\sum_i |b_i - a_i| < \delta$,

$$\mu(\cup_i \gamma([a_i, b_i])) = \sum_i \nu_\gamma(b_i) - \nu_\gamma(a_i) = \sum_i b_i - a_i < \delta$$

and therefore

$$\sum_i |f \circ \gamma(b_i) - f \circ \gamma(a_i)| \le \sum_i \int_{\gamma|_{[a_i,b_i]}} \rho = \int_{\cup_i \gamma([a_i,b_i])} \rho d\mu < \epsilon.$$

$\square$

ISSN 2451-7100
IMAL PREPRINT # 2014-0016 Publication date: July 04, 2014

A nonnegative Borel measurable function ρ satisfying

$$|f(x) - f(y)| \leq \int_\gamma \rho$$

for every $\gamma \in \Gamma^\mu$ with endpoints x, y, for every pair of points x, y with $f(x), f(y)$ finite is called a μ-upper gradient for f.
As 2.4 shows, a function f is absolutely continuous over every path on which it has an upper gradient with finite integral over that path.

3 Modulus of a path family and p-weak upper gradients

Let now m be a Borel measure on X. As in [Sh], we adjust the definition of modulus of a set of measures in [Fu] to path families.

For every family $\Gamma \subset \Gamma^\mu$ and $0 < p < \infty$, we define its p-modulus as

$$Mod_p(\Gamma) = \inf \int_X g^p dm$$

where the infimum is taken over all nonnegative Borel measurable functions $g : X \to \mathbb{R}$ satisfying

$$\int_\gamma g \geq 1$$

for every $\gamma \in \Gamma$.

Theorem 3.1. *Mod_p is an outer measure on Γ^μ.*

Proof. The fact that $Mod_p(\emptyset) = 0$ and its monotonicity are immediate. For σ-subaditivity, if $\Gamma = \cup_i \Gamma_i$, given $\epsilon > 0$ we take g_i with $\int_\gamma g_i \geq 1$ for every $\gamma \in \Gamma_i$ and such that

$$\int_X g_i^p dm \leq Mod_p(\Gamma_i) + 2^{-i}\epsilon$$

Now, if $g = \sup_i g_i$, g satisfies $\int_\gamma g \geq 1$ for every $\gamma \in \Gamma$, and

$$Mod_p(\Gamma) \leq \int_X g^p dm \leq \sum_i \int_X g_i^p dm \leq \sum_i Mod_p(\Gamma_i) + \epsilon.$$

$\square$

As expected, we say that a property holds for *p-almost every path* $\gamma \in \Gamma^\mu$ if the set Γ where it doesn't hold has $Mod_p(\Gamma) = 0$. A useful property of sets of p-modulus zero is the following.

Lemma 3.2. *$Mod_p(\Gamma) = 0$ if and only if there exists a nonnegative Borel measurable function g satisfying $\int_X g^p dm < \infty$ and*

$$\int_\gamma g = \infty$$

for every $\gamma \in \Gamma$.

ISSN 2451-7100
IMAL PREPRINT # 2014-0016 Publication date: July 04, 2014

Proof. For the 'if' part, for every n, $g_n = \frac{1}{n}g$ satisfies $\int_\gamma g_n = \infty \geq 1$ for every $\gamma \in \Gamma$. Then

$$Mod_p(\Gamma) \leq \int_X g_n^p dm = \frac{1}{n^p}\int_X g^p dm \to 0.$$

Now, if $Mod_p(\Gamma) = 0$, then for each n we can find g_n satisfying $\int_\gamma g_n \geq 1$ for every $\gamma \in \Gamma$ and $\int_X g_n^p dm < 4^{-n}$. Then if we define $g = (\sum_n 2^n g_n^p)^{1/p}$, g is Borel measurable, nonnegative and $\int_X g^p dm = \sum_n 2^n \int_X g_n^p dm \leq 1 < \infty$, and besides

$$\int_\gamma g \geq \int_\gamma 2^{n/p} g_n \geq 2^{n/p}$$

for every n, therefore $\int_\gamma g = \infty$ for every $\gamma \in \Gamma$. □

We also need the following result:

Lemma 3.3. *If $\int |g_n - g|^p dm \to 0$, there exists a subsequence $(g_{n_k})_k$ such that $\int_\gamma |g_{n_k} - g| \to 0$ for p-almost every $\gamma \in \Gamma^\mu$.*

Proof. Without loss of generality we assume $g_n \geq 0$ and $\int g_n^p dm \to 0$, and we need to prove $\int_\gamma g_{n_k} \to 0$ for some subsequence (g_{n_k}). We take a subsequence satisfying

$$\int_X g_{n_k}^p dm < 2^{-k(p+1)}.$$

Let now $\Gamma_k = \{\gamma : \int_\gamma g_{n_k} \geq 2^{-k}\}$ and $\Gamma = \limsup_k \Gamma_k$. Clearly $\int_\gamma 2^k g_{n_k} \geq 1$ for each $\gamma \in \Gamma_k$, and therefore

$$Mod_p(\Gamma_k) \leq \int_X 2^{kp} g_{n_k}^p dm < 2^{-k},$$

and for every j,

$$Mod_p(\Gamma) \leq Mod_p(\cup_{k>j}\Gamma_k) \leq \sum_{k>j} Mod_p(\Gamma_k) < 2^{-j}$$

and $Mod_p(\Gamma) = 0$. Finally, if $\gamma \notin \Gamma$, there exists j such that for $k > j$, $\int_\gamma g_{n_k} < 2^{-k}$ and we have what we wanted. □

Given a set $E \subset X$ we define

$$\Gamma_E = \{\gamma \in \Gamma^\mu : |\gamma| \cap E \neq \emptyset\}$$

$$\Gamma_E^+ = \{\gamma \in \Gamma^\mu : \mu(|\gamma| \cap E) > 0\}$$

and we have the following lemma

Lemma 3.4. *If $m(E) = 0$, then $Mod_p(\Gamma_E^+) = 0$.*

Proof. Trivial, as $g = \infty\chi_E$ satisfies $g = 0$ m-almost everywhere, but $\int_\gamma g = \infty$ for every $\gamma \in \Gamma_E^+$. □

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IMAL PREPRINT # 2014-0016 Publication date: July 04, 2014

A nonnegative Borel measurable function ρ satisfying

$$|f(x) - f(y)| \leq \int_\gamma \rho$$

for p-almost every $\gamma \in \Gamma^\mu$ is called a p-weak upper gradient for f.

As in Shanmugalingam's case, we don't lose much by restricting ourselves to weak upper gradients:

Proposition 3.5. *If ρ is a p-weak upper gradient for f and $\epsilon > 0$, there exists an upper gradient for f ρ_ϵ such that $\rho_\epsilon \geq \rho$ and $\|\rho - \rho_\epsilon\|_p < \epsilon$.*

Proof. Let Γ be the set of paths where the inequality for ρ doesn't hold ($Mod_p(\Gamma) = 0$). Then there exists $g \geq 0$ Borel measurable with $\int_X g^p dm < \infty$ but $\int_\gamma g = \infty$ for every $\gamma \in \Gamma$. We define

$$\rho_\epsilon = \rho + \frac{\epsilon}{1 + \|g\|_p} g$$

and it's clear that $\rho_\epsilon \geq \rho$, $\int_\gamma \rho_\epsilon \geq 1$ for every γ, so ρ_ϵ is an upper gradient for f, and finally

$$\|\rho_\epsilon - \rho\|_p = \epsilon \frac{\|g\|_p}{1 + \|g\|_p} < \epsilon.$$

□

As seen in 2.4, functions with 'small' upper gradients are absolutely continuous on curves. We say that a function f is ACC_p or *absolutely continuous over p-almost every path* if $f \circ \gamma_h : [0, h(\gamma)] \to \mathbb{R}$ is absolutely continuous por p-almost every γ.

Lemma 3.6. *If a function f has a p-weak upper gradient $\rho \in L^p$, it is ACC_p.*

Proof. Let Γ_0 be the set of all paths γ such that $|f(x) - f(y)| > \int_\gamma \rho$ and let Γ_1 be the set of all paths with a subpath in Γ_0. As ρ is a weak upper gradient, $Mod_p(\Gamma_0) = 0$, but if g satisfies $\int_\gamma g \geq 1$, it also satisfies $\int_{\tilde{\gamma}} g \geq 1$ for every subpath $\tilde{\gamma}$ of γ, and therefore

$$Mod_p(\Gamma_1) \leq Mod_p(\Gamma_0) = 0.$$

Let Γ_2 be the set of all paths γ with $\int_\gamma \rho = \infty$. Then as $\rho \in L^p$, $Mod_p(\Gamma_2) = 0$. For paths not in $\Gamma_1 \cup \Gamma_2$, we can apply 2.4 and we conclude the lemma. □

We will also need the following lemma later on:

Lemma 3.7. *If f is ACC_p and $f = 0$ m-almost everywhere, then the family*

$$\Gamma = \{\gamma \in \Gamma^* : f \circ \gamma \not\equiv 0\}$$

has p-modulus zero.

Proof. Let $E = \{x : f(x) \neq 0\}$, then $m(E) = 0$ and $\Gamma = \Gamma_E$. As Γ_E^+ has modulus zero (because $m(E) = 0$), we only need to see that $\Gamma_E \backslash \Gamma_E^+$ also has modulus zero. But if $\gamma \in \Gamma_E \backslash \Gamma_E^+$, $|\gamma| \cap E \neq \emptyset$ but $\mu(|\gamma| \cap E) = 0$, therefore $\gamma_h^{-1}(E)$ has length 0 in $\mathbb{R}$ and $f \circ \gamma_h$ is nonzero in a set of length 0, and if $E \neq \emptyset$ this set is not empty and $f \circ \gamma_h$ cannot be absolutely continuous. Therefore $Mod_p(\Gamma_E \backslash \Gamma_E^+) = 0$. □

ISSN 2451-7100
IMAL PREPRINT # 2014-0016 Publication date: July 04, 2014

4 Extended Newton-Sobolev spaces $N^{1,p}$

We define the space $\tilde{N}^{1,p}_{\mu}$ as the space of all functions f with finite p-norm with a p-weak upper gradient with finite p-norm. We equip it with the norm

$$\|f\|_{N^{1,p}} = \|f\|_p + \inf_{\rho} \|\rho\|_p,$$

where the infimum is taken over all p-weal upper gradients of f.

It immediately follows from the definition that $(\tilde{N}^{1,p}_{\mu}, \|\cdot\|_{N^{1,p}})$ is a semi-normed vector space. Morover, if $f, g \in \tilde{N}^{1,p}_{\mu}$, then $|f|, \min\{f,g\}, \max\{f,g\} \in \tilde{N}^{1,p}_{\mu}$. As seen before, every function in $\tilde{N}^{1,p}_{\mu}$ is ACC_p.

$\tilde{N}^{1,p}_{\mu}$ is not a normed space, as two distinct functions can be equal almost everywhere, but also because a function may be in $\tilde{N}^{1,p}_{\mu}$ while a function equal almost everywhere to it may not. We do have the following as a corolary of 3.7:

Corollary 4.1. *If $f, g \in \tilde{N}^{1,p}$ and $f = g$ m-a.e., then $\|f - g\|_{N^{1,p}} = 0$.*

Finally, we define the equivalence relation $f \sim g$ iff $\|f - g\|_{N^{1,p}} = 0$, and the quotient space $N^{1,p} = \tilde{N}^{1,p}/\sim$. We will show, as [Sh], that this is a Banach space, but first a lemma:

Lemma 4.2. *Let $F \subset X$ be such that*

$$\inf\left\{ \|f\|_{N^{1,p}} : f \in \tilde{N}^{1,p}(X) \wedge f|_F \geq 1 \right\} = 0.$$

Then $Mod_p(\Gamma_F) = 0$.

Proof. For every n we take $v_n \in \tilde{N}^{1,p}(X)$ with $v_n|_F \geq 1$ and $\|v_n\|_{N^{1,p}} < 2^{-n}$, and take weak upper gradients ρ_n of v_n with $\|\rho_n\|_p < 2^{-n}$. Take $u_n = \sum_1^n |v_k|, g_n = \sum_1^n \rho_k$ (each g_n will be a weak upper gradient of u_n) and $u = \sum |v_n|$ (observe that $u|_F = \infty$), $g = \sum \rho_n$.
Every u_n turns to be in $\tilde{N}^{1,p}$, and $(u_n), (g_n)$ are Cauchy in L^p, therefore convergent in L^p to functions $\tilde{u}, \tilde{g}$ respectively. Then $u = \tilde{u}, g = \tilde{g}$ a.e. and we have $\int |u|^p < \infty$.
Let $E = \{x \in X : u(x) = \infty\}$, then $m(E) = 0$ (as $\int_X |u|^p < \infty$) and $F \subset E$.
If we take

$$\Gamma = \left\{ \gamma : \int_\gamma g = \infty \vee \int_\gamma g_n \not\to \int_\gamma g \right\}$$

then $Mod_p(\Gamma) = 0$ from 3.2 and 3.3.
If $\gamma \notin \Gamma \cup \Gamma_E^+$ $(Mod_p(\Gamma_E^+) = 0)$, then there exists $y \in |\gamma| \backslash E$, and if $x \in |\gamma|$,

$$|u_n(x)| \leq |u_n(y)| + \int_\gamma g_n \leq |u(y)| + \int_\gamma g,$$

therefore $|u(x)| < \infty$ and $\gamma \notin \Gamma_E$, and we have

$$Mod_p(\Gamma_F) \leq Mod_p(\Gamma_E) \leq Mod_p(\Gamma \cup \Gamma_E^+) = 0.$$

□

ISSN 2451-7100
IMAL PREPRINT # 2014-0016 Publication date: July 04, 2014

Theorem 4.3. *$N^{1,p}$ is Banach.*

Proof. Let (u_n) be Cauchy in $N^{1,p}$. By taking subsequences we can assume

$$\|u_n - u_{n+1}\|_{N^{1,p}} < 2^{-n\frac{p+1}{p}}$$

and take weak upper gradients g_n of $u_n - u_{n+1}$ with

$$\|g_n\|_p < 2^{-n}.$$

Define

$$E_n = \{x \in X : |u_n(x) - u_{n+1}(x)| \geq 2^{-n}\}, E = \limsup E_n.$$

If $x \notin E$, then there exists n_x such that $|u_n(x) - u_{n+1}(x)| < 2^{-n}$ for $n \geq n_x$ and therefore outside of E

$$u(x) = \lim u_n(x)$$

it's well defined.

By Tchebyschev's inequality, $\mu(E_n) \leq 2^{np}\|u_n - u_{n+1}\|_p^p \leq 2^{-n}$, and

$$\mu(E) \leq \sum_n^{\infty} \mu(E_k) \leq 2^{-n} \cdot 2,$$

for every n, and on the other hand

$$\inf\left\{\|f\|_{N^{1,p}} : f \in \tilde{N}^{1,p}(X) \wedge f|_E \geq 1\right\} \leq \sum_n^{\infty} \inf\left\{\|f\|_{N^{1,p}} : f \in \tilde{N}^{1,p}(X) \wedge f|_{E_n} \geq 1\right\}$$

$$\leq \sum_n^{\infty} 2^{np}\|u_n - u_{n+1}\|_{N^{1,p}}^p \leq 2^{-n} \cdot 2$$

for every n.

By the previous lemma, $Mod_p(\Gamma_E) = 0$, and if we define $u|_E \equiv 0$, as (u_n) is Cauchy in L^p and $u_n \to u$ a.e., we have $\int |u|^p < \infty$. Finally for $\gamma \notin \Gamma_E$ with endpoints x, y we have

$$|(u - u_n)(x) - (u - u_n)(y)| \leq \sum_n^{\infty} |(u_{k+1} - u_k)(x) - (u_{k+1} - u_k)(y)| \leq \sum_n^{\infty} \int_\gamma g_k,$$

and we get that $\sum_n^\infty g_k$ is a p-weak upper gradient of $u - u_n$ (which tends to 0 in L^p), and we have $u \in N^{1,p}$ and

$$\|u - u_n\|_{N^{1,p}} \leq \|u - u_n\|_p + \|\sum_n^{\infty} g_k\|_p \to 0.$$

□

ISSN 2451-7100
IMAL PREPRINT # 2014-0016 Publication date: July 04, 2014

5 Poincaré Inequality

If there's no relationship between the 'space measure' m and the 'path measure' μ, most results about $N^{1,p}$ couldn't be proven. The standard way of relating them is by Poincaré inequality.

We say that X supports a $(1,p)$-*Poincaré inequality* if there exists $C > 0, \lambda \geq 1$ such that for every ball B and every pair f, ρ defined in B such that $f \in L^1(B)$ and ρ is an upper gradient of f in B, we have

$$\fint_B |f - f_B| dm \leq C diam(B) \left(\fint_{\lambda B} \rho^p \right)^{1/p}.$$

In Shanmugalingam's case, this property suffices for proving that Lipschitz functions are dense in $N^{1,p}$. One crucial fact for proving this is that the length of a path is always greater than or equal to the distance between any pair of points over the curve, but in our context this may not be the case. We ask the family Γ^μ to have the following property:

$$\exists C_\mu > 0 : \forall \gamma \in \Gamma^\mu, \forall \tilde{\gamma} \text{ non trivial subpath of } \gamma, diam(|\tilde{\gamma}|) \leq C_\mu \mu(|\tilde{\gamma}|) \qquad (1)$$

Lemma 5.1. *Let f be ACC_p such that $f|_F = 0$ m-a.e., for F a closed subset of X. If ρ is an upper gradient of f, then $\rho\chi_{X \setminus F}$ is a p-weak upper gradient of f.*

Proof. Let Γ_0 be the set of paths for which $f \circ \gamma_h$ is not absolutely continuous, and let $E = \{x \in F : f(x) \neq 0\}$, so $Mod_p(\Gamma_0 \cup \Gamma_E^+) = 0$. Now, if $\gamma \notin \Gamma_0 \cup \Gamma_E^+$ has endpoints x, y:

- If $|\gamma| \subset (X \setminus F) \cup E$, then $|f(x) - f(y)| \leq \int_\gamma \rho = \int_\gamma \rho\chi_{X \setminus F}$ as $\mu(|\gamma| \cap E) = 0$.
- If $x, y \in F \setminus E$, then $f(x) = f(y) = 0$ and $|f(x) - f(y)| \leq \int_\gamma \rho\chi_{X \setminus F}$ holds trivially.
- If $x \in (X \setminus F) \cup E$ (or the same for y) but $|\gamma|$ is not completely in $(X \setminus F) \cup E$, as $(f \circ \gamma_h)^{-1}(\{0\})$ is a closed set of $[0, h(\gamma)]$ ($f \circ \gamma_h$ is continuous), it has a minimum a and maximum b (with $f \circ \gamma_h(a) = f \circ \gamma_h(b) = 0$). Then:

  $$|f(x) - f(y)| \leq |f(x) - f(\gamma_h(a))| + |f(\gamma_h(a)) - f(\gamma_h(b))| + |f(\gamma_h(b)) - f(y)| \leq$$

  $$\leq \int_{\gamma_h|_{[0,a]}} \rho + \int_{\gamma_h|_{[b,h(\gamma)]}} \rho \leq \int_\gamma \rho\chi_{X \setminus F}$$

  as $\gamma_h([0,a])$ and $\gamma_h([b, h(\gamma)])$ intersect F in a set of μ-measure zero.

$\square$

Lemma 5.2. *If Γ^μ has property 1, then Lipschitz functions are absolutely continuous over every curve of Γ^μ.*

Proof. Let $\gamma : [0, h] \to X$ be a path in Γ^μ parametrized by μ-arc length, and let $f : X \to \mathbb{R}$ be Lipschitz with constant L. If $\epsilon > 0$ and $0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \leq h$ satisfies $\sum_i |b_i - a_i| < \frac{\epsilon}{LC_\mu}$, then

$$\sum_i |f(\gamma(b_i)) - f(\gamma(a_i))| \leq L \sum_i d(\gamma(b_i), \gamma(a_i)) \leq L \sum_i diam(\gamma([a_i, b_i])) \leq$$

ISSN 2451-7100
IMAL PREPRINT # 2014-0016 Publication date: July 04, 2014

$$\leq LC\sum_i \mu(\gamma([a_i,b_i])) = LC\sum_i |b_i - a_i| < \epsilon.$$

□

Lemma 5.3. *If Γ^μ has property 1 and $f: X \to \mathbb{R}$ is a Lipschitz function with constant L, then $CL\chi_{supp(f)}$ is an upper gradient of f. In particular if $supp(f)$ is compact we have $f \in \tilde{N}^{1,p}$.*

Proof. Let $\gamma : [a,b] \to X$ have endpoints x, y. Consider the following cases:

- $|\gamma| \subset supp(f)$. Then $|f(x) - f(y)| \leq Ld(x,y) \leq C_\mu L\mu(|\gamma|) = \int_\gamma LC = \int_\gamma CL\chi_{supp(f)}$.
- $|\gamma| \cap supp(f) = \emptyset$. Then $|f(x) - f(y)| = 0 = \int_\gamma CL\chi_{supp(f)}$.
- $x \in supp(f)$ but $|\gamma| \not\subset supp(f)$. Then as $(f\circ\gamma)^{-1}(\{0\})$ is closed in $[a,b]$, it has minimum $a_0 > a$ and maximum $b_0 \leq b$. We have that $\gamma([a,a_0])$ and $\gamma([b_0,b])$ are subsets of $supp(f)$ and $f(\gamma(a_0)) = f(\gamma(b_0)) = 0$ so,

$$|f(x)-f(y)| \leq |f(x)-f(\gamma(a_0))|+|f(\gamma(a_0))-f(\gamma(b_0))|+|f(\gamma(b_0))-f(y)| \leq$$

$$\leq Ld(x,\gamma(a_0)) + Ld(\gamma(b_0),y) \leq$$

$$\leq LC_\mu\mu(\gamma([a,a_0])) + LC_\mu\mu(\gamma([b_0,b]) \leq \int_\gamma LC_\mu\chi_{supp(f)}.$$

Finally if $supp(f)$ is compact, $f, CL\chi_{supp(f)} \in L^p(m)$ for every p. □

Theorem 5.4. *If m is doubling, X supports a $(1,p)$-Poincaré inequality and Γ^μ satisfies property 1, then Lipschitz functions are dense in $N^{1,p}$.*

Proof. Let $f \in \tilde{N}^{1,p}$ and let $g \in L^p$ be an upper gradient of f. Assume f is bounded (bounded functions are clearly dense in $N^{1,p}$). We define

$$E_k = \{x \in X : Mg^p(x) > k^p\},$$

where M is the uncentered Hardy-Littlewood maximal function. As m is doubling, M is weak type $1,1$, and

$$m(E_k) \leq \frac{C}{k^p}\int_X g^p \to 0 \quad \text{as} \quad k \to \infty.$$

Let $F_k = X\backslash E_k$ (which is closed as E_k is open). If $x \in F_k$, $r > 0$ and $B = B(x,r)$,

$$\fint_B |f - f_B| \leq Cr(\fint_B g^p)^{1/p} \leq Cr(Mg^p(x))^{1/p} \leq Crk.$$

Then if we define $f_n(x) = f_{B(x,2^{-n}r)}$, we have

$$|f_{n+j}(x)-f_n(x)| \leq \sum_{i=1}^{j} |f_{n+i+1}(x)-f_{n+i}(x)| \leq \sum_{i=1}^{j} \fint_{B(x,2^{-(n+i+1)}r)} |f-f_{B(x,2^{-(n+i)}r)}| \leq$$

$$\leq C\sum_{i=1}^{j} \fint_{B(x,2^{-(n+i)}r)} |f - f_{B(x,2^{-(n+i)}r)}| \leq Ckr2^{-n}\sum_{i=1}^{j} 2^{-i} \leq Ckr2^{-n},$$

ISSN 2451-7100
IMAL PREPRINT # 2014-0016 Publication date: July 04, 2014

and therefore $f_n(x)$ is Cauchy for each $x \in F_k$. Now, we define for $x \in F_k$,

$$f^k(x) = \lim f_n(x).$$

Observe that for Lebesgue points of f in F_k we have $f^k(x) = f(x)$. Let's verify that f^k is Lipschitz:
Given $x, y \in F_k$, take $r = d(x, y)$, $B_n = B(x, 2^{-n}r)$, $B'_n = B(y, 2^{-n}r)$, and

$$|f^k(x) - f^k(y)| \le \sum_{n=0}^{\infty} |f_n(x) - f_{n+1}(x)| + |f_0(x) - f_0(y)| + \sum_{n=0}^{\infty} |f_n(y) - f_{n+1}(y)| \le$$

$$\le \sum_{n=0}^{\infty} C \fint_{B_n} |f - f_{B_n}| + C \fint_{2B_0} |f - f_{2B_0}| + \sum_{n=0}^{\infty} C \fint_{B'_n} |f - f_{B'_n}| \le$$

$$\le Ckr \sum_{n=0}^{\infty} 2^{-n} + Crk \le Ckr = Ckd(x, y).$$

Now, f^k can be extended to all of X as a Lipschitz function, and we can assume it's bounded by Ck.

$$\int_X |f - f^k|^p = \int_{E_k} |f - f^k|^p \le C \int_{E_k} |f|^p + Ck^p m(E_k) \to 0$$

as $k \to \infty$, so f^k tends to f in L^p. As f y f^k are ACC_p, $(g + \tilde{C}k)\chi_{E_k}$ is a p-weak upper gradient of $f - f^k$, and as it's in L^p and tends to 0 when $k \to \infty$, $f - f^k \in N^{1,p}$ for every k and $\|f - f^k\|_{N^{1,p}} \to 0$. $\square$

If X is doubling and supports a $(1, q)$ Poincaré inequality for some $1 \le q < p$, then we have that every function in $N^{1,p}$ has a Hajłasz gradient in L^p, i.e. $N^{1,p} \hookrightarrow M^{1,p}$ with $\| \cdot \|_{M^{1,p}} \le C \| \cdot \|_{N^{1,p}}$ (see [Ha], [KM], [Sh]). The converse embedding holds true in general for Shanmugalingam's case. In our case we need property 1.

Lemma 5.5. *If Γ^μ satisfies property 1, then every continuous function f satisfying*

$$|f(x) - f(y)| \le d(x, y)(g(x) + g(y))$$

for every x, y, for some nonnegative measurable function g, then there exists $C > 0$ such that Cg is an upper gradient for f.

Proof. Let $\gamma : [0, h] \to X$ be a path in Γ^μ parametrized by μ-arc length with endpoints x, y. If $\int_\gamma g = \infty$ we are done. Otherwise, for each n we take $\gamma_i = \gamma|_{[\frac{i}{n}, \frac{i+1}{n}]}$, $0 \le i \le n - 1$, as γ is a μ-arc length parametrization we have that $\mu(|\gamma_i|) = \mu(|\gamma|)/n = h/n$. For each i, there exists $x_i \in |\gamma_i|$ with $g(x_i) \le \fint_{\gamma_i} g$, and property 1 implies that $d(x_i, x_{i+1}) \le C\mu(|\gamma_i|)$, then

$$|f(x_1) - f(x_{n-1})| \le \sum_i |f(x_i) - f(x_{i+1})| \le \sum_i d(x_i, x_{i+1})(g(x_i) + g(x_{i+1})) \le$$

$$\le C \sum_i \left(\int_{\gamma_i} g + \int_{\gamma_{i+1}} g \right) \le$$

ISSN 2451-7100
IMAL PREPRINT # 2014-0016 Publication date: July 04, 2014

$$\leq C \int_\gamma g.$$

Taking $n \to \infty$, $x_0 \to x, x_{n-1} \to y$ and

$$|f(x) - f(y)| \leq C \int_\gamma g$$

and we have what we needed. □

As continuous functions are dense, we have that

Corollary 5.6. *If Γ^μ satisfies property 1, then $M^{1,p} \hookrightarrow N^{1,p}$, with $\|\cdot\|_{N_{1,p}} \leq C\|\cdot\|_{M^{1,p}}$.*

Theorem 5.7. *If X is doubling and supports a $(1,q)$ Poincaré inequality for some $1 \leq q < p$, and Γ^μ satisfies property 1, then $M^{1,p} = N^{1,p}$, with equivalent norms.*

Finally, as in [Sh], we have the following versions of the classical Sobolev embedding theorems:

Theorem 5.8. *If m is doubling and satisfies*

$$m(B(x,r)) \geq Cr^N$$

for C, N independent of $x \in X, 0 < r < 2diam(X)$, and if X supports a $(1,p)$ Poncaré inequality for $p > N$, then functions in $N^{1,p}$ are Lipschitz of exponent $\alpha = 1 - N/p$.

Theorem 5.9. *If X is bounded and satisfies*

$$cr^N \leq m(B(x,r)) \leq Cr^N$$

with c, C, N independent of $x \in X, 0 < r < 2diam(X)$ (i.e. X is Ahlfors N-regular), and if X supports a $(1,q)$ Poincaré inequality for $q > 1$, then for p satisfying $q < p < Nq$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{Nq}$ we have that every $f \in N^{1,p}$ with upper gradient g,

$$\|u - u_X\|_{p^*} \leq Cdiam(X)^{\beta - 1/q}\|g\|_p.$$

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ISSN 2451-7100
IMAL PREPRINT # 2014-0016 Publication date: July 04, 2014

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