ISSN 2451-7100

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IMAL PREPRINT # 2016-0032

Publication date: March 15, 2016

Editorial: Instituto de Matemática Aplicada del Litoral
IMAL (CCT CONICET Santa Fe – UNL)
http://www.imal.santafe-conicet.gov.ar

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ISSN 2451-7100
IMAL PREPRINT # 2016-0032 Publication date: March 15, 2016

DEMOCRACY OF HAAR TYPE SYSTEMS IN SPACES OF HOMOGENEOUS TYPE

HUGO AIMAR, ANA BERNARDIS, AND LUIS NOWAK

ABSTRACT. We explore the relation of the geometric structure of the underlying space and the democratic character of Haar systems in Lorentz spaces.

1. Introduction

Let $(\mathbb{B}, \|.\|_{\mathbb{B}})$ be given a Banach space. A countable set $\mathcal{B}$ of the unit sphere of $\mathbb{B}$ is said to be democratic in $\mathbb{B}$ if for some positive constant D the inequality

$$\left\| \sum_{h\in F_1} h \right\|_{\mathbb{B}} \leq D \left\| \sum_{h\in F_2} h \right\|_{\mathbb{B}}, \tag{1.1}$$

holds for every choice of finite subsets F_1 and F_2 of $\mathbb{B}$ with $|F_1| = |F_2|$. Here, as usual, we write $|F|$ to denote the number of elements in F.

The most classical example of democratic system is provided by any orthonormal sequence in a Hilbert space. In fact, in such case (1.1) is an identity with $D = 1$.

As Temlyakov showed in [15], democracy in Lebesgue spaces on Euclidean spaces, L^p with $1 < p < \infty$, is a common property of all wavelet bases that are equivalent to the Haar basis.

In [6] Garrigós, Hernández and Martell showed that the only Orlicz spaces on $\mathbb{R}^n$ for which wavelet bases are democratic, are precisely Lebesgue spaces. In [16] Wojtaszczyk showed that this is true also among all rearrangement invariant spaces for the Haar system in [0,1] equipped with Lebesgue measure.

In [1] (see also [2]) was defined Haar type systems $\mathcal{H}$ (see Section 2 for the definitions) in the context of the space of homogeneous type (X, d, μ) and was proved that the systems $\mathcal{H}$ are unconditional bases of the Lebesgue spaces $L^p(X, \mu)$, $1 < p < \infty$. Later on, in [14] was proved that the result can be extended to the Lorentz spaces $L^{p,q}(X, \mu)$ with $1 < p, q < \infty$. The aim of this paper is to show that the results of democracy given in [16] and [9] for the classical Haar system and the Lorentz spaces in the euclidean context can be generalized to the more general context of the spaces of homogeneous type.

For Lebesgue spaces the result is the following

Theorem 1.1. *Let (X, d, μ) be a space of homogeneous type and let $\mathcal{H}$ be a Haar type system in X. Then $\mathcal{H}$ is a democratic basis for $L^p(X, \mu)$ with $1 < p < \infty$.*

2000 *Mathematics Subject Classification.* 42C15, 42B20, 28C15 .

Key words and phrases. Haar basis, Democratic bases, Lorentz spaces, spaces of homogeneous type.

IMAL PREPRINT # 2016-0032
ISSN 2451-7100
Publication date: March 15, 2016

Moreover

$$\left\| \sum_{h\in F} \frac{h}{\|h\|_p} \right\|_p \approx |F|^{1/p}, \tag{1.2}$$

for each finite subset F of $\mathcal{H}$.

In order to obtain a result for the Lorentz spaces analogous to the euclidean, some particular distribution of mass is required. We call it the growth property $\mathcal{G}$ for $\mathcal{D}$. The precise definition is given in Section 2 and the growth property is denoted by $\mathcal{G}$. Following the lines of the proof given in [9] we can prove the next statement.

Theorem 1.2. *Let (X, d, μ) be a space of homogeneous type that admits a dyadic family $\mathcal{D}$ satisfying property $\mathcal{G}$ and let $\mathcal{H}$ be a Haar system associate to $\mathcal{D}$. If $\mathcal{H}$ is democratic in $L^{p,q}(X,\mu)$ with $1 < p, q < \infty$, then necessarily $p = q$.*

When the dyadic family $\mathcal{D}$, and therefore the Haar system associated to $\mathcal{D}$, does not verify the condition $\mathcal{G}$, it is possible to define a Haar type system which is democratic on $L^{p,q}(X,\mu)$ with $p \neq q$, as shown in the following proposition.

Proposition 1.3. *Let X be the set of real numbers of the form $x_n = \frac{1}{2^n}$, $n = 1, 2, \ldots$. On the subsets E of X we define the measure $\mu(E) = \sum_{x_n\in E} \frac{1}{2^n}$. In the measure space (X,μ) the class of Lorentz spaces $L^{p,q}$ is well defined, see Section 2. For each positive integer i we define the function $\mathbf{h}^i : X \longrightarrow \mathbb{R}$ by*

$$\mathbf{h}^i_n = \mathbf{h}^i(x_n) = \begin{cases} 2^{\frac{i-1}{2}} & \textit{if } i < n \\ -2^{\frac{i-1}{2}} & \textit{if } i = n \\ 0 & \textit{if } i > n \end{cases}$$

We shall use $\mathbf{H}$ to denote the family of all functions $\mathbf{h}_i$ with $i \in \mathbb{Z}^+$. Then

(1.1.1) *(X, d, μ) is a space of homogeneous type when d is the restriction to X of the standard distance in $\mathbb{R}$.*
(1.1.2) *$\mathbf{H}$ is an orthonormal basis for $L^2_0(X,\mu) = \{f \in L^2(X,\mu) : \int_X f d\mu = 0\}$.*
(1.1.3) *For any $1 < p, q < \infty$ we have that $\left\| \sum_{\mathbf{h}\in F} \frac{\mathbf{h}}{\|\mathbf{h}\|_{p,q}} \right\|_{p,q} \approx |F|^{1/q}$, for each finite subset F of $\mathbf{H}$ where the equivalence constants are independent of F.*
(1.1.4) *The system $\mathbf{H}$ is democratic in each $L^{p,q}(X,\mu)$ when $1 < p, q < \infty$.*
(1.1.5) *No Lorentz space $L^{p,q}(X,\mu)$ with $p \neq q$ is a Lebesgue space on (X,μ).*

Notice that in the above proposition, if we interpret the cubes in the dyadic family in X as the intersection of standard dyadic intervals with the set X, we see that at each level $j \in \mathbb{Z}^+$ we have only one of such cubes with nontrivial offspring: $(0, 2^{-j}] \cap X$. The example is related to some results about the subsequences of the Haar system given in [16].

The paper is organized in the following way: in Section 2 we give the definitions and known results. Sections 3 and 4 are devoted to prove Theorems 1.1 and 1.2, respectively. In Section 5 we prove the statements in Proposition 1.3. Finally, in Section 6 we show that Theorem 1.1 can be generalized to the weighted spaces as in [11].

IMAL PREPRINT # 2016-0032
ISSN 2451-7100
Publication date: March 15, 2016

2. Haar systems $\mathcal{H}$ and characterization of Lorentz spaces

Let us recall the basic properties of the general theory of spaces of homogeneous type. Assume that X is a set, a nonnegative symmetric function d on $X \times X$ is called a quasi-distance if there exists a constant K such that

$$d(x, y) \leq K[d(x, z) + d(z, y)], \tag{2.1}$$

for every $x, y, z \in X$, and $d(x, y) = 0$ if and only if $x = y$.

We shall say that (X, d, μ) is a space of homogeneous type if d is a quasi-distance on X, μ is a positive Borel measure defined on a σ-algebra of subsets of X which contains the balls, and there exists a constant C such that the inequalities

$$0 \ < \ \mu(B(x, 2r)) \ \leq \ C \ \mu(B(x, r)) \ < \ \infty$$

hold for every $x \in X$ and every $r > 0$.

It is well known that the d-balls are generally not open sets. Moreover, sometimes some balls are not even Borel measurable subsets of X. Nevertheless in [13], R. Macias and C. Segovia prove that if d is a quasi-distance on X, then there exist a distance ρ and a number $\alpha \geq 1$ such that d is equivalent to ρ^α. Hence we shall assume along this paper that (X, d, μ) is a space of homogeneous type with d a distance on X, in other words that $K = 1$ in (2.1). In order to be able to apply Lebesgue Differentiation Theorem we shall also assume that continuous functions are dense in $L^1(X, \mu)$.

The construction of dyadic type families of subsets in metric or quasi-metric spaces with some inner and outer metric control of the sizes of the dyadic sets is given in [5]. These families satisfy all the relevant properties of the usual dyadic cubes in $\mathbb{R}^n$. Actually the only properties of Christ's cubes needed in our further analysis are contained in the next definition which we borrow from [3].

Definition 2.1. The class $\mathfrak{D}(\delta)$ of all dyadic families. We say that $\mathcal{D} = \bigcup_{j \in \mathbb{Z}} \mathcal{D}^j$ is a dyadic family on X with parameter $\delta \in (0, 1)$, briefly that $\mathcal{D}$ belong $\mathfrak{D}(\delta)$, if each $\mathcal{D}^j$ is a family of open subsets Q of X, such that

- *(d.1) For every $j \in \mathbb{Z}$ the cubes in $\mathcal{D}^j$ are pairwise disjoints.*
- *(d.2) For every $j \in \mathbb{Z}$ the family $\mathcal{D}^j$ covers almost all X in the sense that $\mu(X - \bigcup_{Q \in \mathcal{D}^j} Q) = 0$.*
- *(d.3) If $Q \in \mathcal{D}^j$ and $i < j$, then there exists a unique $\tilde{Q} \in \mathcal{D}^i$ such that $Q \subseteq \tilde{Q}$.*
- *(d.4) If $Q \in \mathcal{D}^j$ and $\tilde{Q} \in \mathcal{D}^i$ with $i \leq j$, then either $Q \subseteq \tilde{Q}$ or $Q \cap \tilde{Q} = \emptyset$.*
- *(d.5) There exist two constants a_1 and a_2 such that for each $Q \in \mathcal{D}^j$ there exists a point $x \in Q$ for which $B(x, a_1\delta^j) \subseteq Q \subseteq B(x, a_2\delta^j)$.*

The main properties of a dyadic family $\mathcal{D}$ in the class $\mathfrak{D}(\delta)$ are contained in the following result.

Proposition 2.2. *Let $\mathcal{D}$ be a dyadic family in the class $\mathfrak{D}(\delta)$. Then*

- *(d.6) There exists a positive integer N depending only on the doubling constant such that for every $j \in \mathbb{Z}$ and all $Q \in \mathcal{D}^j$ the inequalities $1 \leq \#(\mathcal{O}(Q)) \leq N$ hold, where $\mathcal{O}(Q) = \{Q' \in \mathcal{D}^{j+1} : Q' \subseteq Q\}$.*
- *(d.7) X is bounded if and only if there exists a dyadic cube Q in $\mathcal{D}$ such that $X = Q$.*
- *(d.8) The families $\tilde{\mathcal{D}}^j = \{Q \in \mathcal{D}^j : \#(\{Q' \in \mathcal{D}^{j+1} : Q' \subseteq Q\}) > 1\}$, $j \in \mathbb{Z}$ are pairwise disjoints.*

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IMAL PREPRINT # 2016-0032 Publication date: March 15, 2016

(d.9) *There exists a finite and positive constant C such that $\mu(Q^{'}) \leq \mu(Q) \leq C\mu(Q^{'})$, for all $j \in \mathbb{Z}$, every dyadic cube $Q \in \mathcal{D}_j$ and each $Q^{'} \in \mathcal{O}(Q)$.*

(d.10) *There exists a constant $\alpha > 1$ such that for every $Q \in \tilde{\mathcal{D}}$ and all $Q^{'} \in \mathcal{O}(Q)$ we have that $\mu(Q) \geq \alpha\mu(Q^{'})$.*

For proofs of *(d.6)* to *(d.8)* see [3]. From *(d.5)*, *(d.6)* and the doubling property for μ we get *(d.9)*. Finally from *(d.5)*, *(d.8)* and the doubling property for μ we get *(d.10)*.

We shall now introduce the growth property $\mathcal{G}$ of a dyadic system $\mathcal{D}$.

Definition 2.3. We shall say that a dyadic family $\mathcal{D}$ in the class $\mathfrak{D}(\delta)$ satisfies the growth property $\mathcal{G}$ if

$$\limsup_{|i|\to\infty} |\hat{\mathcal{D}}_i| = \infty,$$

where $\hat{\mathcal{D}}_i = \{Q \in \tilde{\mathcal{D}} : \delta^{i+1} < \mu(Q) \leq \delta^i\}$.

This concept allows us to obtain from $\tilde{\mathcal{D}}$ two types of finite sequences of disjoint cubes,

(a) for each positive integer M a sequence of M disjoint dyadic cubes can be selected belonging to the same level $\hat{\mathcal{D}}_j$ for $|j|$ as large as desired,

(b) for each positive integer M a sequence of M disjoint dyadic cubes can be selected belonging to different levels $\hat{\mathcal{D}}_{j_i}$, $i = 1, ..., M$ with all the $|j_i|$ as large as desired.

Even when we introduced property $\mathcal{G}$ as a property for a particular dyadic family, it is not difficult to see that it actually is a property of the space. In fact, if $\mathcal{D}_1$ satisfies $\mathcal{G}$ then also any other $\mathcal{D}_2$ in $\mathfrak{D}(\delta)$ satisfies $\mathcal{G}$.

In [1] (see also [2]) the authors show that for a given dyadic family $\mathcal{D}$ in the class $\mathfrak{D}(\delta)$ there exist Haar type bases $\mathcal{H}$, of Borel measurable simple real functions h, satisfying the following properties.

(h.1) *For each $h \in \mathcal{H}$ there exists a unique $j \in \mathbb{Z}$ and a cube $Q = Q(h) \in \tilde{\mathcal{D}}^j$ such that $\{x \in X : h(x) \neq 0\} \subseteq Q$, and this property does not hold for any cube in $\mathcal{D}^{j+1}$. Moreover, each function h is constant in each cube $Q^{'} \in \mathcal{O}(Q)$.*

(h.2) *For every $Q \in \tilde{\mathcal{D}} = \bigcup_{j\in\mathbb{Z}} \tilde{\mathcal{D}}^j$ there exist exactly $M_Q = \#(\mathcal{O}(Q)) - 1 \geq 1$ functions $h \in \mathcal{H}$ such that (h.1) holds. We shall write $\mathcal{H}_Q$ to denote the set of all these functions h.*

(h.3) *For each $h \in \mathcal{H}$ we have that $\int_X h d\mu = 0$.*

(h.4) *For each $Q \in \tilde{\mathcal{D}}$ let V_Q denote the vector space of all functions on Q which are constant on each $Q^{'} \in \mathcal{L}(Q)$. Then the system $\{\frac{\chi_Q}{(\mu(Q))^{1/2}}\} \bigcup \mathcal{H}_Q$ is an orthonormal basis for V_Q.*

From *(h.1)* to *(h.4)* and *(d.9)*, we get the following two additional properties. As usual, for a measurable function f, we write $\|f\|_\infty = sup\ ess|f|$.

(h.5) *There exist two positive constants c_1 and c_2 such that the inequalities*

$$c_1\mu(Q^{'})^{-1/2} \leq \|h\|_\infty \leq c_2\mu(Q(h))^{-1/2},$$

hold for each $h \in \mathcal{H}$ and each $Q^{'} \in \mathcal{O}(Q(h))$.

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(h.6) For each $h \in \mathcal{H}$ we have that

$$\|h\|_\infty \chi_{Q'_h}(x) \leq |h(x)| \leq \|h\|_\infty \chi_{Q(h)}(x),$$

for all point $x \in X$ and some dyadic cube $Q'_h \in \mathcal{O}(Q(h))$.

Thus, from *(h.5)* and *(h.6)*, there exist two positive constants C_1 and C_2 such that

$$C_1 \mu(Q'_h)^{\frac{1}{p}-\frac{1}{2}} \leq \|h\|_{p,q} \leq C_2 \mu(Q(h))^{\frac{1}{p}-\frac{1}{2}}, \tag{2.2}$$

for each function $h \in \mathcal{H}$ where Q'_h and $Q(h)$ are the dyadic cubes given in *(h.6)* and *(h.1)* respectively. Also, from *(h.1)* to *(h.4)* we obtain the following result (see [2], [3]).

Theorem 2.4. *Let $\mathcal{D}$ be a dyadic family on X such that $\mathcal{D}$ belong to class $\mathfrak{D}(\delta)$. Then every Haar type system $\mathcal{H}$ associated to $\mathcal{D}$ is an orthonormal basis in $\mathcal{L}^2(X, \mu)$.*

In the above Theorem and in the sequel we write $\mathcal{L}^p(X, \mu), (p \geq 1)$ to denote the space $L^p(X, \mu)$ of all measurable functions f such that $\|f\|_p^p = \int_X |f|^p d\mu < \infty$ when $\mu(X) = \infty$ and the space $L_0^p = \{f \in L^p(X, \mu) : \int_X f d\mu = 0\}$ if $\mu(X) < \infty$.

Now we introduce the Lorentz spaces in the general setting of measure spaces (X, μ) such that μ is a σ-finite measure. We shall restrict our attention only to the scale $L^{p,q}$ with $1 < p, q < \infty$. The basic source for the general theory of Lorentz spaces is the paper of R. Hunt [10]. See also [12], [8] or [4].

Given a measurable real valued function f defined on X, we denote with λ_f the distribution function of f, that is, $\lambda_f(s) = \mu(\{x \in X : |f(x)| > s\})$. The non increasing rearrangement of f is the function given by $f^*(t) = inf\{s > 0 : \lambda_f(s) \leq t\}$, for $t \geq 0$.

For $1 < p, q < \infty$, the $L^{p,q}(X, \mu) = L^{p,q}$ space is defined as the linear space of all measurable functions f on X such that $\|f\|_{p,q}^* < \infty$, where

$$\|f\|_{p,q}^* = \left(\frac{q}{p} \int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q}.$$

Notice that the Lorentz space $L^{p,p}$ is the classical Lebesgue space L^p. The quantity $\|.\|_{p,q}^*$ is not a norm. However, R. Hunt introduce in [10] a norm $\|.\|_{p,q}$ on $L^{p,q}(X, \mu)$ such that the topology given by $\|.\|_{p,q}^*$ is equivalent to topology induced by the norm.

The following statements collect the main properties of Lorentz spaces that we shall use later. For the proofs see [10] and [7].

- $(L1)$ $(L^{p,q}, \|.\|_{p,q})$ is a Banach space and $\|f\|_{p,q} \approx \|f\|_{p,q}^*$ for each function f.
- $(L2)$ For every measurable set E we have that $\|\chi_E\|_{p,q}^* = \mu(E)^{1/p}$, where χ_E denote the indicator function of E.
- $(L3)$ If f and g are two measurable functions defined on X such that $|f| \leq |g|$ μ-a.e., then $\|f\|_{p,q} \leq \|g\|_{p,q}$.
- $(L4)$ For any $\alpha > max\{1, 2^{\frac{1}{p}-\frac{1}{q}}\}$ there exist two positive and finite constants C_1 and C_2 such that

$$C_1 \|f\|_{p,q}^* = \left(\sum_{k \in \mathbb{Z}} \alpha^{kq} \mu(\{x : \alpha^k \leq |f(x)| < \alpha^{k+1}\})^{q/p} \right)^{1/q} = C_2 \|f\|_{p,q}^*.$$

We shall denote by $\mathcal{L}^{p,q}(X, \mu)$ the Lorentz space $L^{p,q}(X, \mu)$ if $\mu(X) = \infty$ and those functions in the space $L^{p,q}(X, \mu)$ with vanishing integral if $\mu(X) < \infty$

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ISSN 2451-7100
Publication date: March 15, 2016

In [14] one of the authors obtain the following characterization of Lorentz spaces on a space of homogeneous type in terms of these Haar systems.

Theorem 2.5. *Let (X, d, μ) be a space of homogeneous type and let $\mathcal{H}$ be a Haar system associated to a dyadic family $\mathcal{D}$ in $\mathfrak{D}(\delta)$. If $1 < p, q < \infty$, then $\mathcal{H}$ is an unconditional basis for $\mathcal{L}^{p,q}(X, \mu)$. Moreover, there exist two positive constants C_1 and C_2 such that for all $f \in \mathcal{L}^{p,q}(X, \mu)$ we have that*

$$C_1\|f\|_{p,q} \;\leq \left\| \left(\sum_{h\in\mathcal{H}} | < f, h > |^2 |h|^2 \right)^{1/2} \right\|_{p,q} \;\leq\; C_2\|f\|_{p,q},$$

where $< f, h >= \int_X fh \; d\mu$.

It is important to observe that, with $p = q$, Theorem 2.5 also give a characterization of Lebesgue spaces L^p with $1 < p < \infty$.

3. Proof of Theorem 1.1

First, we introduce some notation. Given a dyadic cube $Q \in \mathcal{D}$, we shall consider the non-trivial ancestors of Q, $(Q_n : n \in I) \subseteq \mathcal{D}$ satisfying the following properties

(1) I is an initial interval of non negative integer $\mathbb{Z}_0^+$, which is finite if and only if $\mu(X) < \infty$;
(2) $Q_0 = Q$;
(3) $Q_n \in \tilde{\mathcal{D}}$ for each $n \geq 1$;
(4) $Q_n \subseteq Q_{n+1}$ for each $n \in I$;
(5) if $Q^{'} \in \mathcal{D}$ is such that $Q_n \subseteq Q^{'} \subseteq Q_{n+1}$ for some n, then $Q^{'} = Q_n$ or $Q^{'} = Q_{n+1}$.

From *(d.2)*, *(d.7)* and *(d.8)*, each dyadic cube $Q \in \mathcal{D}$ has an unique sequence of non-trivial ancestors of Q.

Notice that given a finite subset F of $\mathcal{H}$, the function $g_F = \sum_{h\in F} \frac{h}{\|h\|_p}$ belongs to $\mathcal{L}^p(X, \mu)$. Thus, from the characterization via Haar wavelets of Lebesgue spaces on spaces of homogeneous type and from the orthogonality of the Haar system $\mathcal{H}$ we get that

$$\|g_F\|_p \approx \left\| \left(\sum_{h\in F} \frac{|h|^2}{\|h\|_p^2} \right)^{1/2} \right\|_p . \tag{3.1}$$

On the other hand, using *(h.5)*, *(h.6)*, (2.2) and *(d.9)* we get that there exist two positive constants c_1 and c_2 such that

$$c_1 \sum_{h\in F} \mu(Q_h^{'})^{-2/p} \chi_{Q_h^{'}}(x) \leq \sum_{h\in F} \frac{|h(x)|^2}{\|h\|_p^2} \leq c_2 \sum_{h\in F} \mu(Q(h))^{-2/p} \chi_{Q(h)}(x). \tag{3.2}$$

We first prove that there exists a positive constant C such that

$$\left\| \left(\sum_{h\in F} \mu(Q_h^{'})^{-2/p} \chi_{Q_h^{'}} \right)^{1/2} \right\|_p^p \geq C|F|. \tag{3.3}$$

ISSN 2451-7100
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For each $y \in E_F^{'} = \bigcup_{h\in F} Q_h^{'}$ we write $Q^{'}(y)$ to denote the smallest dyadic cube $Q_h^{'}$ with $h \in F$ such that $y \in Q_h^{'}$. Then

$$\left(\sum_{h\in F} \mu(Q_h^{'})^{-2/p} \chi_{Q_h^{'}}(y)\right)^{p/2} \geq \mu(Q^{'}(y))^{-1} \tag{3.4}$$

for every $y \in E_F^{'}$. Now, for each $Q^{'}(y)$ with $y \in E_F^{'}$, let $(Q_n^{'}(y) : n \in I)$ the sequence of non-trivial ancestor of $Q^{'}(y)$. Then, from *(d.10)* we get that

$$\mu(Q_n^{'}(y)) \ \geq \alpha^n \ \mu(Q_0^{'}(y)) \ = \ \alpha^n \ \mu(Q^{'}(y)), \tag{3.5}$$

for each $n \in I$. Notice that for each $y \in E_F^{'}$ we have that $\{Q_h^{'} \ni y : h \in F\} \subseteq \{Q_n^{'}(y) : n \in I\}$. So that, from *(h.2)*, *(d.6)*, (3.5) and the fact that $\alpha > 1$ we get that

$$\begin{aligned} \sum_{h\in F} \mu(Q_h^{'})^{-1} \chi_{Q_h^{'}}(y) &\leq N \sum_{n\in I} \mu(Q_n^{'}(y))^{-1} \\ &\leq N \sum_{n\in I} \alpha^{-n} \mu(Q^{'}(y))^{-1} \\ &= C\mu(Q^{'}(y))^{-1}. \end{aligned} \tag{3.6}$$

Hence, from (3.4) and (3.6) we obtain that

$$\begin{aligned} \left\| \left(\sum_{h\in F} \mu(Q_h^{'})^{-2/p} \chi_{Q_h^{'}}\right)^{1/2} \right\|_p &\geq C \left(\int_{E_F^{'}} \mu(Q^{'}(y))^{-1} d\mu(y)\right)^{1/p} \\ &\geq C \left(\int_{E_F^{'}} \sum_{h\in F} \mu(Q_h^{'})^{-1} \chi_{Q_h^{'}}(y) d\mu(y)\right)^{1/p} \\ &= C \left(\sum_{h\in F} \int_{E_F^{'}} \mu(Q_h^{'})^{-1} \chi_{Q_h^{'}}(y) d\mu(y)\right)^{1/p} \\ &= C|F|^{1/p}, \end{aligned}$$

which proves (3.3).

Now we shall prove that there exists a positive constant C such that

$$\left\| \left(\sum_{h\in F} \mu(Q(h))^{-2/p} \chi_{Q(h)}\right)^{1/2} \right\|_p^p \leq C|F|. \tag{3.7}$$

Let us start by estimating $\sum_{h\in F} \mu(Q(h))^{-2/p} \chi_{Q(h)}(x)$. For each point $x \in E_F = \bigcup_{h\in F} Q(h)$, we write $Q(x)$ to denote the smallest dyadic cube $Q(h)$ with $h \in F$ such that $x \in Q(h)$. Set $(Q_n(x) : n \in I)$ the sequence of non-trivial ancestor of $Q(x)$. Notice that from *(d.10)* we have that $\mu(Q_n(x)) \geq \alpha^n \mu(Q(x))$ for each $n \in I$. Then, from *(h.2)* and *(d.6)*, with the same argument as in (3.6), we get that

$$\sum_{h\in F} \mu(Q(h))^{-2/p} \chi_{Q(h)}(x) \ \leq \ N \sum_{n\in I} \mu(Q_n(x))^{-2/p} \leq C\mu(Q(x))^{-2/p}. \tag{3.8}$$

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On the other hand, for each $h \in F$ we define the set $S(h) = \{x \in E_F : Q(x) = Q(h)\}$. Then $E_F = \bigcup_{h\in F} S(h) = \bigcup_{h\in F} Q(h)$, $Q(x) = Q(h)$ if $x \in S(h)$, and $S(h) \subseteq Q(h)$ for each $h \in F$. Hence, from (3.8) we get

$$\begin{aligned}\left\|\left(\sum_{h\in F}\mu(Q(h))^{-2/p}\chi_{Q(h)}\right)^{1/2}\right\|_p &\leq C\left(\int_{E_F}\mu(Q(x))^{-1}d\mu(x)\right)^{1/p}\\ &\leq C\left(\sum_{h\in F}\int_{S(h)}\mu(Q(x))^{-1}d\mu(x)\right)^{1/p}\\ &= C\left(\sum_{h\in F}\int_{S(h)}\mu(Q(h))^{-1}d\mu(x)\right)^{1/p}\\ &\leq C\left(\sum_{h\in F}\int_{Q(h)}\mu(Q(h))^{-1}d\mu(x)\right)^{1/p}\\ &= C\,|F|^{1/p}.\end{aligned}$$

Finally, from (3.1), (3.2), (3.3) and (3.7) we conclude the proof of Theorem 1.1.

4. Proof of Theorem 1.2

We shall show that, for each positive integer M, there exist two subsets F_1 and F_2 of $\mathcal{H}$ with $|F_i| = M, i = 1, 2$ such that

$$C_1 M^{1/p} \leq \left\|\sum_{h\in F_1}\frac{h}{\|h\|_{p,q}}\right\|_{p,q} \leq C_2 M^{1/p} \tag{4.1}$$

and

$$C_1 M^{1/q} \leq \left\|\sum_{h\in F_2}\frac{h}{\|h\|_{p,q}}\right\|_{p,q} \leq C_2 M^{1/q}, \tag{4.2}$$

for some constants C_1 and C_2. Let M be a positive integer. Let us start by obtaining a set F_1 such that (4.1) holds. Since the dyadic family $\mathcal{D}$ in the class $\mathfrak{D}(\delta)$ satisfies the growth property $\mathcal{G}$, then there exist an integer $i = i(M)$ and a set $F_1 = \{h_1, ..., h_M\} \subseteq \mathcal{H}$ such that $Q(h_j) \in \hat{\mathcal{D}}_i$ for each $j = 1, ..., M$ and the dyadic cubes $Q(h_j)$ are disjoint. From the doubling property of μ and from the definition of $\hat{\mathcal{D}}_i$, we clearly have that

$$\mu(Q') \approx \mu(Q(h)) \approx \delta^i, \tag{4.3}$$

for every $Q' \in \mathcal{O}(Q(h))$ and all $h \in F_1$. On the other hand, as in (3.2), from *(h.5)*, *(h.6)*, (2.2) and *(d.9)*, we get that

$$c_1\sum_{h\in F}\mu(Q'_h)^{-2/p}\chi_{Q'_h}(x) \leq \sum_{h\in F}\frac{|h(x)|^2}{\|h\|^2_{p,q}} \leq \sum_{h\in F}\mu(Q(h))^{-2/p}\chi_{Q(h)}(x), \tag{4.4}$$

for some positive constants c_1 and c_2 and every finite subset F of $\mathcal{H}$. Then, from Theorem 2.5, the left hand side of inequality (4.4), the fact that the cubes $Q(h_j)$

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are disjoints, $(L3)$, (4.3) and $(L2)$ we get that

$$\begin{aligned}
\left\|\sum_{h\in F_1}\frac{h}{\|h\|_{p,q}}\right\|_{p,q} &\approx \left\|\left(\sum_{h\in F_1}\frac{|h(x)|^2}{\|h\|_{p,q}^2}\right)^{1/2}\right\|_{p,q}\\
&\geq C\left\|\sum_{h\in F_1}\mu(Q_h^{'})^{-1/p}\chi_{Q_h^{'}}\right\|_{p,q}\\
&\approx C(\delta^i)^{-1/p}\left\|\sum_{h\in F_1}\chi_{Q_h^{'}}\right\|_{p,q}\\
&= C(\delta^i)^{-1/p}\left\|\chi_{\bigcup_{h\in F_1}Q_h^{'}}\right\|_{p,q}\\
&= C(\delta^i)^{-1/p}\left(\sum_{h\in F_1}\mu(Q_h^{'})\right)^{1/p}\\
&= C\left(\sum_{h\in F_1}\delta^{-i}\mu(Q_h^{'})\right)^{1/p}\\
&\approx C|F_1|^{1/p}.
\end{aligned}$$

Similarly, using the right hand side of inequality (4.4) we get that

$$\begin{aligned}
\left\|\sum_{h\in F_1}\frac{h}{\|h\|_{p,q}}\right\|_{p,q} &\approx \left\|\left(\sum_{h\in F}\frac{|h(x)|^2}{\|h\|_{p,q}^2}\right)^{1/2}\right\|_{p,q}\\
&\leq C\left\|\sum_{h\in F_1}\mu(Q(h))^{-1/p}\chi_{Q(h)}\right\|_{p,q}\\
&\leq C|F_1|^{1/p}.
\end{aligned}$$

Now, let us exhibit a subset F_2 of $\mathcal{H}$ such that (4.2) holds. Since $\mathcal{D}$ in the class $\mathfrak{D}(\delta)$ satisfies the growth property $\mathcal{G}$, we can take a finite sequence of disjoint dyadic cubes $(Q_{i_j} : j = 1, ..., M)$ such that $Q_{i_j} \in \hat{\mathcal{D}}_{i_j}$ and $i_j > i_{j+1}$. Set $F_2 = \{h_j : j = 1, ..., M\}$ such that the dyadic cubes $Q(h_j) = Q_{i_j}$. Notice that, as in (4.3),

$$\mu(Q_{h_j}^{'}) \approx \mu(Q(h_j)) \approx \delta^{i_j}, \tag{4.5}$$

for each $j = 1, ..., M$ where $Q_{h_j}^{'}$ is the cube in *(h.6)*. Then, from Theorem 2.5, the left hand side of inequality (4.4), the disjointness of $Q(h_j)$, $(L3)$, (4.5) and $(L4)$ with $\alpha = \delta^{-1/p}$ we get that

$$\begin{aligned}
\left\|\sum_{h\in F_2}\frac{h}{\|h\|_{p,q}}\right\|_{p,q}^q &\approx \left\|\left(\sum_{h\in F_2}\frac{|h(x)|^2}{\|h\|_{p,q}^2}\right)^{1/2}\right\|_{p,q}^q\\
&\geq C\left\|\sum_{h\in F_2}\mu(Q_h^{'})^{-1/p}\chi_{Q_h^{'}}\right\|_{p,q}^q
\end{aligned}$$

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$$\begin{aligned}
&= C\left\|\sum_{j=1}^{M}\mu(Q'_{h_j})^{-1/p}\chi_{Q'_{h_j}}\right\|_{p,q}^{q}\\
&\approx C\left\|\sum_{j=1}^{M}(\delta^{i_j})^{-1/p}\chi_{Q'_{h_j}}\right\|_{p,q}^{q}\\
&\approx \sum_{k\in\mathbb{Z}}\delta^{\frac{-kq}{p}}\mu(\{x:\delta^{-k/p}\leq\sum_{j=1}^{M}(\delta^{i_j})^{-1/p}\chi_{Q'_{h_j}}(x)<\delta^{\frac{-(k+1)}{p}}\})^{q/p}\\
&= \sum_{j=1}^{M}\delta^{\frac{-i_jq}{p}}\mu(Q'_{h_j})^{q/p}\\
&\approx \sum_{j=1}^{N}\delta^{\frac{-i_jq}{p}}(\delta^{i_j})^{q/p}\\
&= M.
\end{aligned}$$

Analogously, using the right hand side of inequality (4.4) we get that

$$\begin{aligned}
\left\|\sum_{h\in F_1}\frac{h}{\|h\|_{p,q}}\right\|_{p,q}^{q} &\approx \left\|\left(\sum_{h\in F}\frac{|h(x)|^2}{\|h\|_{p,q}^2}\right)^{1/2}\right\|_{p,q}^{q}\\
&\leq C\left\|\sum_{h\in F_2}\mu(Q(h))^{-1/p}\chi_{Q(h)}\right\|_{p,q}^{q}\\
&= M.
\end{aligned}$$

5. Proof of Proposition 1.3

Let (X,μ) be the measure space given in the example.

Proof of (1.1.1)**.** Let d be the restriction to X of the standard distance in $\mathbb{R}$. We have to show that for some constant C, every $x\in X$ and all $r>0$ we have that

$$0<\mu(B(x,2r))\leq C\mu(B(x,r))<\infty, \tag{5.1}$$

where $B(x,r)=\{y\in X: d(x,y)<r\}$. Notice that every ball in X with center x and radius $r>0$ has the form $B(x,r)=\{\frac{1}{2^j}: j\in J\}$, where

(*a*) $J=\mathbb{Z}^+$; or
(*b*) $J=\{s,s+1,...,m\}$ with s and m two positive integers such that $s<m$; or
(*c*) $J=\{s,s+1,...\}$ for some positive integer s.

In the case (a) we have that (5.1) holds since $\mu(B(x,2r))=\mu(B(x,r))=\mu(X)$. For the case (b) we first notice that if $B(x,r)=\{\frac{1}{2^m},\frac{1}{2^{m-1}},...,\frac{1}{2^{s+1}},\frac{1}{2^s}\}$ with $s<m$, then $x=\frac{1}{2^s}$ and $r<\frac{1}{2^s}-\frac{1}{2^{m+1}}$. So that $B(x,2r)=\{\frac{1}{2^s},\frac{1}{2^{s+1}},...\}$. Hence

$$\mu(B(x,2r))=\mu(\{\frac{1}{2^{m+1}},\frac{1}{2^{m+2}},...\})+\mu(B(x,r))\leq 2\mu(B(x,r)),$$

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since $\mu(\{\frac{1}{2^{m+1}}, \frac{1}{2^{m+2}}, ...\}) = \frac{1}{2^m} \leq \frac{1}{2^s} \leq \mu(B(x,r))$. Finally, in the case (c) we have that $r < \frac{1}{2^{s-1}}$ and therefore $2r < \frac{1}{2^{s-2}}$. Then

$$B(x,2r) \subseteq \{\frac{1}{2^{s-1}}, \frac{1}{2^s}, \frac{1}{2^{s+1}}, ...\} = \{\frac{1}{2^{s-1}}\} \cup B(x,r).$$

So that

$$\begin{aligned} \mu(B(x,2r)) &\leq \mu(\{\frac{1}{2^{s-1}}\}) + \mu(B(x,r)) \\ &= 3\mu(B(x,r)). \end{aligned}$$

Hence, taking $C = 3$ in (5.1)we get that (X, d, μ) is a metric space of homogeneous type.

Proof of (1.1.2)**.** For each positive integer i the function $\mathbf{h}^i$ is a Haar function in the sense that it is a simple function of vanishing integral with $L^2(X,\mu)$ norm equal to one. Moreover, the system $\mathbf{H}$ is an orthonormal basis for $L^2_0(X,\mu) = \{f \in L^2(X,\mu) : \int_X f d\mu = 0\}$. It is easy to show that $\mathbf{H}$ is an orthonormal system in $L^2_0(X,\mu)$. We only have to prove that, for $f \in L^2_0(X,\mu)$, we have the identity

$$f = \sum_{j=1}^{\infty} < f, \mathbf{h}^j > \mathbf{h}^j$$

which reads as

$$f_m = \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} f_k \mathbf{h}^j_k \frac{1}{2^k} \right) h^j_m, \tag{5.2}$$

for each positive integer m, where $f_i = f(x_i)$. Notice first that, since $\mathbf{h}^i_n = 0$ for all $n < i$, then

$$\begin{aligned} < f, \mathbf{h}^i > &= \int_X f\mathbf{h}^i \, d\mu \\ &= \sum_{n=1}^{\infty} f_n \mathbf{h}^i_n \frac{1}{2^n} \\ &= \sum_{n=i+1}^{\infty} f_n \frac{2^{\frac{i-1}{2}}}{2^n} - f_i \frac{2^{\frac{i-1}{2}}}{2^i}, \end{aligned} \tag{5.3}$$

for each positive integer i. On the other hand, for each positive integer m we have that $\mathbf{h}^i_m \neq 0$ only when $i \leq m$. Therefore, writing $S_{\mathbf{H}} f = \sum_{i=1}^{\infty} < f, \mathbf{h}^i > \mathbf{h}^i$, we have that

$$(S_{\mathbf{H}} f)_m = S_{\mathbf{H}} f(x_m) = \sum_{i=1}^{m} < f, \mathbf{h}^i > \mathbf{h}^i_m, \tag{5.4}$$

for each $m \in \mathbb{Z}^+$. Thus, from (5.4), (5.3), the definitions of $\mathbf{h}^i$ and μ and the fact that $\sum_{i=1}^{\infty} f_i \frac{1}{2^i} = \int_X f d\mu = 0$, we get that

$$\begin{aligned} (S_{\mathbf{H}} f)_m &= \sum_{i=1}^{m-1} < f, \mathbf{h}^i > \mathbf{h}^i_m + < f, \mathbf{h}^m > \mathbf{h}^m_m \\ &= \sum_{i=1}^{m-1} \left(\sum_{n=i+1}^{\infty} \frac{f_n}{2^{n-(i-1)}} - \frac{f_i}{2} \right) - \left(\sum_{n=m+1}^{\infty} \frac{f_n}{2^{n-(m-1)}} - \frac{f_m}{2} \right) \end{aligned}$$

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$$= \sum_{i=1}^{m-1}\left(-\sum_{n=1}^{i}\frac{f_n}{2^{n-(i-1)}}-\frac{f_i}{2}\right)+\sum_{n=1}^{m}\frac{f_n}{2^{n-(m-1)}}+\frac{f_m}{2}$$

for each $m\in\mathbb{Z}^+$. Therefore, to obtain (5.2) we shall prove that

$$\sum_{i=1}^{m-1}\left(-\sum_{n=1}^{i}\frac{f_n}{2^{n-(i-1)}}-\frac{f_i}{2}\right)+\sum_{n=1}^{m}\frac{f_n}{2^{n-(m-1)}}=\frac{f_m}{2}.$$

But

$$\sum_{n=1}^{m}\frac{f_n}{2^{n-(m-1)}}=\sum_{n=1}^{m-1}\frac{f_n}{2^{n-(m-1)}}+\frac{f_m}{2}.$$

Hence we only need shows that

$$\sum_{i=1}^{m-1}\left(\sum_{n=1}^{i}\frac{f_n}{2^{n-(i-1)}}+\frac{f_i}{2}\right)=\sum_{n=1}^{m}\frac{f_n}{2^{n-(m-1)}},$$

which we obtain using a discrete version of Fubini Theorem as follows.

$$\begin{aligned}\sum_{i=1}^{m-1}\left(\sum_{n=1}^{i}\frac{f_n}{2^{n-(i-1)}}+\frac{f_i}{2}\right) &= \sum_{n=1}^{m-1}\frac{f_n}{2^n}\sum_{i=n}^{m-1}2^{i-1}+\sum_{i=1}^{m-1}\frac{f_i}{2}\\ &= \sum_{n=1}^{m-1}\frac{f_n}{2^n}(2^{m-1}-2^{n-1})+\sum_{i=1}^{m-1}\frac{f_i}{2}\\ &= \sum_{n=1}^{m}\frac{f_n}{2^{n-(m-1)}}.\end{aligned}$$

So that (5.2) holds. Actually the system $\mathbf{H}$ is a Haar system in (X,d,μ) as described in Section 3 (see also [ABN]).

Proof of (1.1.3) **and** (1.1.4)**.** In the sequel we shall write Q_i to denote the support of $\mathbf{h}^i$ on X. That is, $Q_i=\{x_i,x_{i+1},...\}$. Then $\mu(Q_i)=2^{-(i-1)}$. Let N be a positive integer and let F be a given a subset of $\mathbf{H}$ such that $|F|=N$. We can assume, without loss of generality, that $F=\{\mathbf{h}^{i_j}: j=1,...,N\}$ with $i_1<i_2<...<i_N$. With the above notation, we have that $Q_{i_j}=\{2^{-(i_j)},2^{-(i_j+1)},...\}$. Set $Q_{i_{N+1}}$ to denote the empty set. Thus, from the characterization of Lorentz spaces in terms of Haar coefficients that we shall state in Theorem 2.5 in next section and is proved in [N], we have that

$$\left\|\sum_{\mathbf{h}\in F}\frac{\mathbf{h}}{\|\mathbf{h}\|^*_{p,q}}\right\|^*_{p,q} \approx \left\|\left(\sum_{\tilde{\mathbf{h}}\in\mathcal{H}}|<\sum_{\mathbf{h}\in F}\frac{\mathbf{h}}{\|\mathbf{h}\|^*_{p,q}},\tilde{\mathbf{h}}>|^2|\tilde{h}|^2\right)^{1/2}\right\|^*_{p,q}$$

Now, from the orthonormality in $L^2(X,\mu)$ of $\mathbf{H}$ and $(L2)$, we have

$$\left\|\sum_{\mathbf{h}\in F}\frac{\mathbf{h}}{\|\mathbf{h}\|^*_{p,q}}\right\|^*_{p,q} \approx \left\|\left(\sum_{j=1}^{N}\mu(Q_{i_j})^{-2/p}\chi_{Q_{i_j}}\right)^{1/2}\right\|^*_{p,q}$$

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$$
\begin{aligned}
&= \left\| \left(\sum_{j=1}^{N} \left(\sum_{l=1}^{j} \mu(Q_{i_l})^{-2/p} \right) \chi_{Q_{i_j} \setminus Q_{i_{j+1}}} \right)^{1/2} \right\|_{p,q}^{*} \\
&= \left\| \sum_{j=1}^{N} \left(\sum_{l=1}^{j} \mu(Q_{i_l})^{-2/p} \right)^{1/2} \chi_{Q_{i_j} \setminus Q_{i_{j+1}}} \right\|_{p,q}^{*}.
\end{aligned}
$$

Set

$$
f(x) = \sum_{j=1}^{N} A_j \; \chi_{Q_{i_j} \setminus Q_{i_{j+1}}}(x), \text{ with } \; A_j = \left(\sum_{l=1}^{j} \mu(Q_{i_l})^{-2/p} \right)^{1/2}.
$$

Then,

$$
f^*(t) = \sum_{j=0}^{N-1} A_{N-j} \; \chi_{I_j}(t),
$$

where I_j denotes the real interval $[\mu(Q_{i_{N-j+1}}), \mu(Q_{i_{N-j}}))$. Thus,

$$
\begin{aligned}
\|f\|_{p,q}^q &\approx \frac{q}{p} \int_0^\infty \left(t^{1/p} f^*(t) \right)^q \frac{dt}{t} \\
&= \frac{q}{p} \sum_{j=0}^{N-1} A_{N-j}^q \int_{I_j} t^{\frac{q}{p}-1} dt \\
&= \sum_{j=0}^{N-1} A_{N-j}^q \left(\mu(Q_{i_{N-j}})^{q/p} - \mu(Q_{i_{N-j+1}})^{q/p} \right).
\end{aligned}
$$

Notice that $\mu(Q_{i_{N-j}})^{-1/p} \leq A_{N-j}$ and

$$
\begin{aligned}
A_{N-j} &= \mu(Q_{i_{N-j}})^{-1/p} \left(\sum_{l=1}^{N-j} \left(\frac{\mu(Q_{i_{N-j}})}{\mu(Q_{i_l})} \right)^{2/p} \right)^{1/2} \\
&= \mu(Q_{i_{N-j}})^{-1/p} \left(1 + \sum_{l=1}^{N-j-1} \left(\frac{\mu(Q_{i_{N-j}})}{\mu(Q_{i_l})} \right)^{2/p} \right)^{1/2} \\
&\leq \mu(Q_{i_{N-j}})^{-1/p} \left(1 + \sum_{s=1}^{\infty} \left(\frac{1}{2^s} \right)^{2/p} \right)^{1/2}.
\end{aligned}
$$

Therefore

$$
(5.5) \quad C_1 \|f\|_{p,q}^q \leq \sum_{j=0}^{N-1} \mu(Q_{i_{N-j}})^{-q/p} \left(\mu(Q_{i_{N-j}})^{q/p} - \mu(Q_{i_{N-j+1}})^{q/p} \right) \leq C_2 \|f\|_{p,q}^q,
$$

for some positive constants C_1 and C_2. So that, from (5.5) we get that

$$
\|f\|_{p,q}^q \leq C \sum_{j=0}^{N-1} \mu(Q_{i_{N-j}})^{-q/p} \mu(Q_{i_{N-j}})^{q/p} = N,
$$

or equivalently

$$
\|f\|_{p,q}^q \leq C N^{1/q}.
$$

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On the other hand, since $i_{N-j} < i_{N-j+1}$, there exists a positive integer s such that $i_{N-j} + s = i_{N-j+1}$. Then

$$\begin{aligned}
\mu(Q_{i_{N-j}})^{q/p} - \mu(Q_{i_{N-j+1}})^{q/p} &= \left(\frac{1}{2^{i_{N-j}}}\right)^{q/p} - \left(\frac{1}{2^{i_{N-j+1}}}\right)^{q/p} \\
&= \left(\frac{1}{2^{i_{N-j}}}\right)^{q/p} - \left(\frac{1}{2^{i_{N-j}}}\right)^{q/p}\left(\frac{1}{2^{s}}\right)^{q/p} \\
&= \left(\frac{1}{2^{i_{N-j}}}\right)^{q/p}\left(1 - \left(\frac{1}{2^{s}}\right)^{q/p}\right) \\
&\geq \left(\frac{1}{2^{i_{N-j}}}\right)^{q/p}\left(1 - \left(\frac{1}{2}\right)^{q/p}\right).
\end{aligned}$$

Therefore, from this last inequality and (5.5) we have that

$$\|f\|_{p,q} \geq CN^{1/q}.$$

Thus, for any $1 < p, q < \infty$ we have that $\left\|\sum_{\mathbf{h}\in F} \frac{\mathbf{h}}{\|\mathbf{h}\|_{p,q}}\right\|_{p,q} \approx |F|^{1/q}$, for each finite subset F of $\mathbf{H}$ where the equivalence constants are independent of F. This implies that $\mathbf{H}$ is democratic in $L^{p,q}(X,\mu)$, for any choice of $1 < p, q < \infty$.

Proof of (1.1.5)**.** Even when it is possible to build examples showing that Lorentz are not Lebesgue spaces when $p \neq q$, in our particular setting, a direct argument comes from Theorem 1.1, $(L2)$ and (1.1.3). In fact if for some $1 < p, q, r < \infty$ we have $L^{p,q}(X,\mu) = L^r(X,\mu)$, then from Theorem 1.1 with r instead of p we should have that

$$\left\|\sum_{\mathbf{h}\in F} \frac{\mathbf{h}}{\|\mathbf{h}\|_r}\right\|_r \approx |F|^{1/r},$$

for each finite subset F of $\mathbf{H}$. But from (1.1.3) we have that

$$\left\|\sum_{\mathbf{h}\in F} \frac{\mathbf{h}}{\|\mathbf{h}\|_r}\right\|_r \approx \left\|\sum_{\mathbf{h}\in F} \frac{\mathbf{h}}{\|\mathbf{h}\|_{p,q}}\right\|_{p,q} \approx |F|^{1/q}$$

for each finite subset F of $\mathbf{H}$ and, since $|F|$ can be as large as desired, necessarily $r = q$. On the other hand, since from $(L2)$ the $L^{p,q}(X,\mu)$ norm of χ_E is equivalent to $\mu(E)^{1/p}$, $\|\chi_E\|_r = \mu(E)^{1/r}$, and there exist in X sets of measure as small as desired, if $L^{p,q}(X,\mu) = L^r(X,\mu)$, we should have $p = r$. In other words p has to be q in order to get a Lebesgue space.

6. Further results

The result contained in Theorem 1.1 can be extended to the case of weighted Lebesgue spaces $L^p(X, wd\mu)$ when w belong to a particular class of weights that we proceed to define.

When a dyadic family $\mathcal{D}$ is given we define, as usual, the class of Muckenhoupt type dyadic weight function associated to $\mathcal{D}$. A non-negative, measurable and locally integrable function w defined on the space of homogeneous type (X, d, μ), is

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said to be a Muckenhoupt dyadic weight of class $A_p^{\mathcal{D}}$, $1 < p < \infty$ if the inequality

$$\left(\frac{1}{\mu(Q)}\int_Q w(x)d\mu(x)\right)\left(\frac{1}{\mu(Q)}\int_Q w(x)^{\frac{-1}{p-1}}d\mu(x)\right)^{p-1} \leq C, \tag{6.1}$$

holds for some constant C and every dyadic set $Q \in \mathcal{D}$.

As in the classical case of dyadic weights associated to the usual dyadic cubes in $\mathbb{R}^n$ we obtain the following basic property in the setting of space of homogeneous type: *if $w \in A_p^{\mathcal{D}}$ then $wd\mu$ has the doubling property on dyadic cubes in $\mathcal{D}$, this is there exists a positive constant C such that $w(\tilde{Q}) \leq Cw(Q)$ for all $Q \in \mathcal{D}$ and $\tilde{Q}$ the first-ancestor of Q.* Moreover, holds the following reverse doubling property.

Lemma 6.1. *Let (X, d, μ) be space of homogeneous type and $\mathcal{D}$ a dyadic family. Let w a finite and doubling measure on $\mathcal{D}$. Then, there exists $\alpha > 1$ such that $w(Q) \geq \alpha w(Q^{'})$, for all Q and $Q^{'}$ dyadic cubes in $\mathcal{D}$ with Q the first cube different of $Q^{'}$ such that $Q^{'} \subseteq Q$.*

Proof. Let Q and $Q^{'}$ dyadic cubes with Q the first cube different of $Q^{'}$ such that $Q^{'} \subseteq Q$. As $Q \neq Q^{'}$, there exists $Q^{''}$ dyadic cube such that $Q^{''}, Q^{'} \in \mathcal{D}^{j+1}$ for some $j \in \mathbb{Z}$ and $Q^{''} \subseteq Q \in \mathcal{D}^j$.

First note that there exist two positive constants C_1 and C_2 such that $C_1\, w(Q^{''}) \leq w(Q^{'}) \leq C_2\, w(Q^{''})$. In fact, since $Q^{'}$ and $Q^{''}$ are subsets of Q and w is a doubling measure on $\mathcal{D}$ we obtain the following chain of inequalities

$$w(Q^{''}) \leq w(Q) \leq C\, w(Q^{'}) \leq C\, w(Q) \leq C\, w(Q^{''}).$$

Thus, since $w(Q^{'}) < \infty$ then $w(Q \setminus Q^{'}) = w(Q) - w(Q^{'})$ and therefore we obtain that

$$\begin{aligned} w(Q^{'}) &\leq Cw(Q^{''}) \\ &\leq Cw(Q \setminus Q^{'}) \\ &\leq C\left(w(Q) - w(Q^{'})\right). \end{aligned}$$

Then $\frac{1+C}{C} w(Q^{'}) \leq w(Q)$ and the lemma is proved with $\alpha = \frac{1+C}{C}$. □

Also, in [2] the authors prove the following result.

Theorem 6.2. *Let (X, d, μ) be a space of homogeneous type and let $\mathcal{H}$ be a Haar system associated to a dyadic family $\mathcal{D}$ in $\mathfrak{D}(\delta)$. If $1 < p < \infty$ and $w \in A_p^{\mathcal{D}}$, then $\mathcal{H}$ is an unconditional basis for $\mathcal{L}^p(X, w\mu)$. Moreover, there exist two positive constants C_1 and C_2 such that for all $f \in \mathcal{L}^p(X, w\mu)$ we have that*

$$C_1\|f\|_{\mathcal{L}^p(X,w\mu)} \leq \left\|\left(\sum_{h\in\mathcal{H}} |<f,h>|^2 |h|^2\right)^{1/2}\right\|_{\mathcal{L}^p(X,w\mu)} \leq C_2\|f\|_{\mathcal{L}^p(X,w\mu)},$$

where $<f,h> = \int_X fh\, d\mu$.

Thus, from the above Lemma and the Theorem 6.2 we obtain the following result as in the proof of Theorem 1.1.

Theorem 6.3. *Let (X, d, μ) be a space of homogeneous type and let $\mathcal{H}$ be a Haar type system associated to $\mathcal{D}$. If $w \in A_p^{\mathcal{D}}$, then $\mathcal{H}$ is a democratic basis for $L^p(X, w\mu)$*

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ISSN 2451-7100
Publication date: March 15, 2016

with $1 < p < \infty$. *Moreover*

$$\left\| \sum_{h \in F} \frac{h}{\|h\|_{L^p(X,w\mu)}} \right\|_{L^p(X,w\mu)} \approx |F|^{1/p}, \tag{6.2}$$

for each finite subset F *of* $\mathcal{H}$.

References

[1] H. Aimar, *Construction of Haar type bases on quasi-metric spaces with finite Assouad dimension*, Anal. Acad. Nac. Cs. Ex., F. y Nat., Buenos Aires 54 (2004).

[2] H. Aimar, A. Bernardis and B. Iaffei, *Multiresolution approximation and unconditional bases on weighted Lebesgue spaces on spaces of homogeneous type*, J. Approx. Theory, **148** (2007) 12–34.

[3] H. Aimar, A. Bernardis and L. Nowak *Equivalence of Haar bases associated to different dyadic systems*, J. Geom. Anal. 21 (2011), no. 2, 288–304.

[4] C. Bennett and R. Sharpley, *Interpolation of operators*. Pure and Applied Mathematics, 129. Academic Press, Inc., Boston, MA, 1988.

[5] M. Christ, *A T(b) theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60/61** (2) (1990), 601–628.

[6] G. Garrigós, E. Hernández and J. Martell, *Wavelets Orlicz spaces and greedy bases*. Appl. Comput. Harmon. Anal. **24** (2008), 70–93

[7] G. Garrigós, E. Hernández and M. De Natividade, *Democracy functions and optimal embeddings for approximation spaces*. Preprint

[8] L. Grafakos, *Classical and modern Fourier analysis*. Pearson Education, Inc., Upper Saddle River, NJ, 2004.

[9] E. Hernández, J. Martell and M. De Natividade, *Quantifying democracy for wavelet bases in Lorentz spaces*. Const. Approx. 33 (2011), no.1, 1–14.

[10] R. Hunt, *On $L(p,q)$ spaces*. Enseignement Math. (2) 12 1966 249–276.

[11] M. Izuki, *The Haar wavelets and the Haar scaling function in weighted L^p spaces with $A_p^{dy,m}$ weights*. Hokkaido Math. J. **36** (2007), 417–444.

[12] G. Lorentz, *On the Theory of spaces* Λ. Pacific J. Math. 1 (1951), 411–429.

[13] R. Macias and C. Segovia, *Lipschitz functions on spaces of homogeneous type*. Adv. in Math. **33** (1979), 271-309.

[14] L. Nowak, *Haar type bases in Lorentz spaces via extrapolation*. Manuscript

[15] V. Temlyakov, *The best m-term approximation and greedy algorithms*. Adv. Comput. Math. **8** (1998), 249–265.

[16] P. Wojtaszczyk, *Greediness of the Haar system in rearrangement invariant spaces*. Approximation and probability,Banach Center Publ.,**72** (2006), 385-395.

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