IMAL preprints

ADAPTIVE CONTROL OF LOCAL ERRORS FOR ELLIPTIC PROBLEMS USING WEIGHTED SOBOLEV NORMS

By

Eduardo M. Garau and Pedro Morin

I M A L

CONICET
UNL

Adaptive control of local errors for elliptic problems using weighted Sobolev norms

Eduardo M. Garau[*] and Pedro Morin

Instituto de Matemática Aplicada del Litoral, UNL, CONICET, FIQ. Santa Fe, Argentina

April 25, 2016

Abstract

We develop an a posteriori error estimator which focuses on the local H^1 error on a region of interest. The estimator bounds a weighted Sobolev norm of the error and is efficient up to oscillation terms. The new idea is very simple and applies to a large class of problems. An adaptive method guided by this estimator is implemented and compared to other local estimators, showing an excellent performance.

Keywords: elliptic problems, point sources, a posteriori error estimates, finite elements, local estimates, weighted Sobolev spaces

1 Introduction

In many practical problems, the *region of interest* is much smaller than the whole domain where a differential equation must be solved to obtain meaningful results. The goal of this article is to design a new adaptive method for the approximation of general second order elliptic problems, with the adaptive strategy focused on controlling the error in a localized region of interest. Throughout this article, *local error* stands for the $H^1(\Omega_0)$ error and Ω_0 is an open subset of the whole domain Ω which we refer to as *region of interest*. Adaptive methods and a posteriori error estimators for controlling local errors in elliptic problems have already been developed and studied [LN03, Dem07, Dem10]. In these papers the local error is bounded by the sum of two terms. The H^1-type estimators on a region slightly larger than Ω_0 and a weaker norm of the error on the whole domain; due to the so-called *pollution effect*. Demlow [Dem10] considers Poisson's equation $-\Delta u = f$ and bounds the latter by the $L^2(\Omega)$ error, which in turn is bounded by the $L^p(\Omega)$ estimators for some $p > 2$ —when the $L^2(\Omega)$ estimators are not available due to geometry constraints (e.g. reentrant corners). Recall that all known L^2-type a posteriori error estimators are not a guaranteed upper bound for the $L^2(\Omega)$ error unless the problem is H^2-regular, because the only available proofs are based on duality techniques. Liao and Nochetto [LN03], instead, resort to estimators for the error in a weighted $L^2(\Omega)$ norm, with weights that are singular at the reentrant corners; their result is valid for the equation $-\nabla \cdot (A\nabla u) = f$ with A smooth.

In this work we bound the local H^1 error by a weighted H^1 error on the whole domain, and then by a posteriori error estimators of residual type. The weight is chosen such that the global weighted error is an upper bound of the local H^1 error, but does not overestimate it by too much. This is a simple idea, motivated by the need to use weighted norms when working

---

[*]Corresponding author: egarau@santafe-conicet.gov.ar

with point sources, and has the advantage of allowing us to work with the general second order linear elliptic PDE on Lipschitz polygonal/polyhedral domains in 2D/3D without convexity constraints, allowing also discontinuous coefficients. We consider the following problem on a Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with a polygonal/polyhedral boundary $\partial\Omega$:

$$\begin{cases} -\nabla \cdot \left(\mathcal{A}\nabla u \right) + \boldsymbol{b} \cdot \nabla u + cu = g, & \text{in } \Omega \\ \qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathcal{A} \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ is symmetric, piecewise $W^{1,\infty}$ and uniformly positive definite over Ω, i.e., there exist constants $0 < \gamma_1 \leq \gamma_2$ such that

$$\gamma_1|\xi|^2 \leq \xi^T \mathcal{A}(x)\xi \leq \gamma_2|\xi|^2, \quad \forall x \in \Omega, \ \xi \in \mathbb{R}^n;$$

$\boldsymbol{b} \in W^{1,\infty}(\Omega; \mathbb{R}^n)$, and $c \in L^\infty(\Omega)$ with $c - \frac{1}{2}\operatorname{div}(\boldsymbol{b}) \geq 0$.

We consider both the case of regular sources, i.e., $g \in L^2(\Omega)$, and the case of a singular point source assuming that $g = f + \nu\delta_{x_0}$, where $f \in L^2(\Omega)$, $\nu \in \mathbb{R}$ and δ_{x_0} is the Dirac delta distribution supported at an inner point x_0 of Ω. Applications arise in different areas, such as in the study of pollutant diffusion in aquatic media [ABR07], in the mathematical modeling of electromagnetic fields [J75], or in optimal control of elliptic problems with state constraints [C86]. Other applications involve the coupling between reaction-diffusion problems taking place in domains of different dimension, which arise in tissue perfusion models [DQ08].

When $\nu = 0$ ($g = f \in L^2(\Omega)$) we say that $u \in H_0^1(\Omega) := W_0^{1,2}(\Omega)$ is a weak solution of (1) if

$$\mathcal{B}[u, v] = F(v), \quad \forall v \in H_0^1(\Omega),$$

where $\mathcal{B}$ is the bilinear form given by

$$\mathcal{B}[u, v] = \int_\Omega \mathcal{A}\nabla u \cdot \nabla v + \boldsymbol{b} \cdot \nabla u \, v + c \, u \, v, \tag{2}$$

and $F(v) := \int_\Omega fv$. If $\nu \neq 0$, the solution u of (1) does not belong to $H_0^1(\Omega)$, but defining the weight $\omega(x) \cong |x - x_0|^{2\alpha}$ for certain values of $\alpha > 0$, the following weak formulation is well-posed [AGM14, Theorem 2.3]

$$u \in H_0^1(\Omega, \omega) : \qquad \mathcal{B}[u, v] = F(v), \quad \forall v \in H_0^1(\Omega, \omega^{-1}),$$

where $F(v) = \int_\Omega fv + \nu\delta_{x_0}(v)$ and $\mathcal{B}[\cdot, \cdot]$ is given by (2). Here, $H_0^1(\Omega, \omega)$ and $H_0^1(\Omega, \omega^{-1})$ denote weighted Sobolev spaces which will be explicitly defined in the next section.

The main results of this article are presented in Section 3.2 where a posteriori error estimators are presented for the weighted error, and the reliability and efficiency are discussed. The numerical results of Section 4 show an excellent performance of an adaptive method guided by these a posteriori estimators.

The rest of the article is organized as follows. In Section 2 we discuss the formulation of elliptic problems in weighted Sobolev spaces, including the design of weights which *localize* the error and cope with the singularities due to the point source. In Section 3 we state the discrete formulation of the problem and present a posteriori error estimators for the weighted norm of the error with their reliability and efficiency. In Section 4 we present numerical experiments.

2 Linear elliptic problems in weighted spaces

We start this section briefly introducing some notions about weighted Sobolev spaces, which are useful for the purpose of this article.

2.1   Weighted Sobolev spaces

We consider weights belonging to the Muckenhoupt class A_2, which are defined as the set of positive functions $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that their A_2-constant

$$\sup_B \left(\frac{1}{|B|} \int_B \omega(x)\, dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{-1}\, dx \right)$$

is finite, where the supremum is taken over all balls B in $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. If $\omega \in A_2$, we denote by $L^2(\Omega, \omega)$ the space of measurable functions u such that

$$\|u\|_{L^2(\Omega,\omega)} := \left(\int_\Omega |u(x)|^2 \omega(x) dx \right)^{\frac{1}{2}} < \infty.$$

Notice that $L^2(\Omega, \omega)$ is a Hilbert space equipped with the scalar product

$$\langle u, v \rangle_{\Omega, \omega} := \int_\Omega u(x) v(x) \omega(x) dx.$$

We also define the weighted Sobolev space $H^1(\Omega, \omega)$ of weakly differentiable functions u such that $\|u\|_{H^1(\Omega,\omega)} < \infty$, where

$$\|u\|_{H^1(\Omega,\omega)} := \|u\|_{L^2(\Omega,\omega)} + \|\nabla u\|_{L^2(\Omega,\omega)}.$$

Finally, $H^1_0(\Omega, \omega)$ is the closure of $C^\infty_0(\Omega)$ in $H^1(\Omega, \omega)$. By [FKS82, Theorem 1.3] we have that a Poincaré inequality holds in $H^1_0(\Omega, \omega)$, and thus,

$$\|u\|_{H^1_0(\Omega,\omega)} := \|\nabla u\|_{L^2(\Omega,\omega)}$$

is a norm in $H^1_0(\Omega, \omega)$ equivalent to the inherited norm $\|u\|_{H^1(\Omega,\omega)}$. More precisely, there exists a constant $C_P > 0$, depending on n, the diameter of Ω, and the A_2-constant of ω such that

$$\|u\|_{H^1_0(\Omega,\omega)} \le \|u\|_{H^1(\Omega,\omega)} \le C_P \|u\|_{H^1_0(\Omega,\omega)}, \qquad u \in H^1_0(\Omega, \omega). \tag{3}$$

2.2   Well-posedness

The next result generalizes those given by [AGM14, Theorem 2.3] and [NOS16, Lemma 7.7]. We remark that the only required properties on the weight function ω are:

Assumption 2.1. *The positive weight $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies the following:*

- *$\omega \in A_2$, $\omega \in L^\infty(\Omega)$;*

- *$H^1_0(\Omega) \hookrightarrow L^2(\Omega, \omega^{-1})$, i.e., there exists a constant $C_E > 0$ such that*

$$\|v\|_{L^2(\Omega,\omega^{-1})} \le C_E \|\nabla v\|_{L^2(\Omega)}, \qquad \text{for } v \in H^1_0(\Omega).$$

Remark 2.2. By [NOS16, Theorem 6.1] (see also [CW85]) we have that a weight $\omega \in A_2$ fulfills $H^1_0(\Omega) \hookrightarrow L^2(\Omega, \omega^{-1})$ if there exists a constant $C_\omega > 0$ such that the following *compatibility condition* holds:

$$\left(\fint_{B(x,r)} \omega^{-1} \right) \left(\fint_{B(x,R)} \omega^{-1} \right)^{-1} \le C_\omega \left(\frac{r}{R} \right)^{n-2}, \qquad \text{for all } x \in \Omega \text{ and } r \le R. \tag{4}$$

3

Theorem 2.3. *Let $\mathcal{B}[\cdot,\cdot] : H_0^1(\Omega,\omega) \times H_0^1(\Omega,\omega^{-1}) \to \mathbb{R}$ be as in* (2), *and let ω be a weight satisfying Assumption* 2.1. *Then, for each $F \in (H_0^1(\Omega,\omega^{-1}))'$, there exists a unique solution $u \in H_0^1(\Omega,\omega)$ of*

$$\mathcal{B}[u,v] = F(v), \quad \forall\, v \in H_0^1(\Omega,\omega^{-1}),$$

which satisfies

$$\|u\|_{H_0^1(\Omega,\omega)} \le C_*\|F\|_{(H_0^1(\Omega,\omega^{-1}))'},$$

where the constant $C_* = \frac{2}{\gamma_1}\left(1 + \|\omega\|_{L^\infty(\Omega)}^{\frac{1}{2}} \frac{C_P C_E}{\gamma_1} \max\{\|\boldsymbol{b}\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}\}\right)$.

Moreover, the following inf-sup condition holds:

$$\inf_{w \in H_0^1(\Omega,\omega)} \sup_{v \in H_0^1(\Omega,\omega^{-1})} \frac{\mathcal{B}[w,v]}{\|w\|_{H_0^1(\Omega,\omega)}\|v\|_{H_0^1(\Omega,\omega^{-1})}} \ge \frac{1}{C_*}. \tag{5}$$

The proof of this theorem is identical to that of [AGM14, Theorem 2.3] and will thus be omitted in this article.

2.3   Global weighted norms to localize the energy norm

Let the *region of interest* Ω_0 be a fixed open subset of Ω. We are interested in estimating $\|e\|_{H^1(\Omega_0)}$, where e is the error between the weak solution of problem (1) and its finite element approximation. We will develop a posteriori error estimators and propose adaptive methods oriented towards reducing the local error $\|e\|_{H^1(\Omega_0)}$ with the least amount of degrees of freedom.

The weight in charge of *localizing* the H^1-norm is φ_0 which we only assume to be in $L_{\text{loc}}^1(\mathbb{R}^n)$ and satisfy the following properties:

(i) $\varphi_0(x) > 0$ for a.e. $x \in \mathbb{R}^n$ and $\varphi_0 \in A_2$.

(ii) $\varphi_0(x) \le 1$, for a.e. $x \in \Omega$ and $\varphi_0(x) = 1$, for a.e. $x \in \Omega_0$.

(iii) There exists a constant $C_{\varphi_0} > 0$ such that

$$\left(\int_{B(x,r)} \varphi_0^{-1}\right)\left(\int_{B(x,R)} \varphi_0^{-1}\right)^{-1} \le C_{\varphi_0}\left(\frac{r}{R}\right)^{n-2}, \qquad \text{for all } x \in \Omega \text{ and } r \le R. \tag{6}$$

A simple way to construct such a weight is to let $\varphi_0(x) := \varphi(\text{dist}(x,\Omega_0))$, with $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a decreasing positive function such that $\varphi(0) = 1$. In Section 4 we make some particular choices of φ and φ_0.

Recall that the source term of problem (1) is $g = f + \nu\delta_{x_0}$, with $f \in L^2(\Omega)$, $\nu \in \mathbb{R}$ and δ_{x_0} is the Dirac delta distribution supported at an inner point x_0 of Ω.

The use of a weight will also allow us to overcome the difficulty produced by a Dirac delta source term. We thus consider two cases depending on the actual presence of a point source:

• Case 1: *There is no point source.* If $\nu = 0$, let

$$\omega(x) = \varphi_0(x), \quad x \in \mathbb{R}^n. \tag{7a}$$

• Case 2: *There is a point source.* If $\nu \ne 0$, let

$$\omega(x) = \min\left(\left(\frac{\mathrm{d}_{x_0}(x)}{D_0}\right)^{2\alpha}, \varphi_0(x)\right), \quad x \in \mathbb{R}^n. \tag{7b}$$

Here, $D_0 := \text{dist}(x_0,\Omega_0) = \inf_{x \in \Omega_0}|x - x_0| > 0$, $\mathrm{d}_{x_0}(x) := |x - x_0|$, for $x \in \mathbb{R}^n$, and α is fixed, with $\alpha \in \mathbb{I}$, where $\mathbb{I} = (n/2 - 1, n/2)$ if $\boldsymbol{b} = 0$ and $c = 0$ and $\mathbb{I} = (n/2 - 1, 1)$ otherwise.

Since $\mathrm{d}_{x_0}^{\beta} \in A_2$ if and only if $-n < \beta < n$ [AGM14], and since $\varphi_0 \in A_2$ we conclude that in both cases $\omega \in A_2$, because—as can be easily checked—the minimum of two A_2 weights also belongs to A_2. Furthermore, since $\omega = 1$ in Ω_0,

$$\|u\|_{H^1(\Omega_0)} \leq \|u\|_{H^1(\Omega,\omega)},$$

i.e., the usual local energy $\|u\|_{H^1(\Omega_0)}$ is bounded above by the global weighted norm $\|u\|_{H^1(\Omega,\omega)}$. This bound becomes sharper when φ_0 is chosen smaller on $\Omega \setminus \Omega_0$.

In Section 3.2 we develop a posteriori estimators for the error $\|e\|_{H^1(\Omega,\omega)}$, they will be used to guide an adaptive method, its goal is to reduce the desired quantity $\|e\|_{H^1(\Omega_0)}$.

2.4   Weak formulation

In order to state a variational formulation for the linear elliptic problem (1), we first note that, due to the definition of the weight ω given by (7), if $\nu \neq 0$, $H_0^1(\Omega, \omega^{-1}) \subset H_0^1(\Omega, \mathrm{d}_{x_0}^{-2\alpha})$. Thus, in view of [KMR97, Lema 7.1.3] we have that $\langle g, v \rangle := \int_\Omega fv + \nu \langle \delta_{x_0}, v \rangle$ is a bounded a linear functional on $H_0^1(\Omega, \omega^{-1})$, for $f \in L^2(\Omega)$, whenever $\frac{n}{2} - 1 < \alpha < \frac{n}{2}$. We consider the following weak formulation of problem (1):

$$u \in H_0^1(\Omega, \omega): \qquad \mathcal{B}[u, v] = \int_\Omega fv + \nu \langle \delta_{x_0}, v \rangle, \quad \forall\, v \in H_0^1(\Omega, \omega^{-1}). \tag{8}$$

Recall that $\mathcal{B}$ is the bilinear form given by (2), which is clearly well-defined and bounded in $H_0^1(\Omega, \omega) \times H_0^1(\Omega, \omega^{-1})$ due to Hölder inequality. Moreover, the compatibility condition (4) holds for ω defined in (7) due to assumption (6) on φ_0 and the fact that it also holds for the weight $\mathrm{d}^{2\alpha}$, for $\alpha \in \mathbb{I}$, as can be easily checked. Therefore, as a consequence of Theorem 2.3, the bilinear form $\mathcal{B}[\cdot, \cdot]$ satisfies the inf-sup condition (5), which yields the existence and uniqueness of solution to the variational problem (8).

3   Discrete problem and a posteriori error analysis

3.1   Finite element discretization

Let $\mathcal{T}$ be a conforming triangulation of the domain $\Omega \subset \mathbb{R}^n$. That is, a partition of Ω into n-simplexes such that if two elements intersect, they do so at a full vertex/edge/face of both elements. We define the mesh regularity constant

$$\kappa := \sup_{T \in \mathcal{T}} \frac{\mathrm{diam}(T)}{\rho_T},$$

where $\mathrm{diam}(T)$ is the diameter of T, and ρ_T is the radius of the largest ball contained in it. Also, the diameter of any element $T \in \mathcal{T}$ is equivalent to the local mesh-size $h_T := |T|^{1/n}$, with equivalence constants depending on κ.

On the other hand, we denote the subset of $\mathcal{T}$ consisting of an element T and its neighbors by $\mathcal{N}_T$, and the union of the elements in $\mathcal{N}_T$ by S_T. More precisely, for $T \in \mathcal{T}$,

$$\mathcal{N}_T := \{T' \in \mathcal{T} \mid T \cap T' \neq \emptyset\}, \qquad S_T := \bigcup_{T' \in \mathcal{N}_T} T'.$$

We denote by $\mathcal{E}_\Omega$ the set of sides (edges for $n = 2$ and faces for $n = 3$) of the elements in $\mathcal{T}$ which are inside Ω, and by $\mathcal{E}_{\partial\Omega}$ the set of sides which lie on the boundary of Ω. We define S_E

as the union of the two elements sharing E, if $E \in \mathcal{E}_\Omega$, and as the unique element T_E satisfying $E \subset \partial T_E$ if $E \in \mathcal{E}_{\partial\Omega}$.

For the discretization we consider piecewise polynomial Lagrange finite elements, more precisely, we let

$$\mathbb{V}_\mathcal{T}^\ell := \{V \in H_0^1(\Omega) \mid V_{|_T} \in \mathcal{P}_\ell(T), \; \forall \, T \in \mathcal{T}\}.$$

where $\ell \in \mathbb{N}$ is a fixed polynomial degree. The discrete counterpart of (8) reads:

$$U \in \mathbb{V}_\mathcal{T}^\ell : \qquad \mathcal{B}[U, V] = \int_\Omega fV + \nu\langle\delta_{x_0}, V\rangle, \quad \forall V \in \mathbb{V}_\mathcal{T}^\ell. \tag{9}$$

Notice that it is the standard finite element discretization. The weighted norms have no influence in the formulation of the discrete problem.

3.2   A posteriori estimation of the local error

In this section we derive computable bounds for the error measured in the weighted norm $\|\cdot\|_{H^1(\Omega,\omega)}$, where the weight function ω is given by (7).

Let u be the solution of (8) and let $U \in \mathbb{V}_\mathcal{T}$ be the solution of the discrete problem (9). Integrating by parts on each $T \in \mathcal{T}$ we have that

$$\mathcal{B}[U - u, v] = \sum_{T \in \mathcal{T}} \left(\int_T Rv + \int_{\partial T} Jv \right) - \nu\langle\delta_{x_0}, v\rangle, \qquad \forall v \in H_0^1(\Omega, \omega^{-1}),$$

where R denotes the *element residual* given by

$$R_{|_T} := -\nabla \cdot (\mathcal{A}\nabla U) + \boldsymbol{b} \cdot \nabla U + cU - f, \qquad \forall T \in \mathcal{T},$$

and J is the *jump residual* given by

$$J_{|_E} := \frac{1}{2}\left[\mathcal{A}\nabla U_{|_{T_1}} \cdot \vec{n}_1 + \mathcal{A}\nabla U_{|_{T_2}} \cdot \vec{n}_2\right], \text{ if } E \in \mathcal{E}_\Omega, \quad J_{|_E} = 0, \text{ if } E \in \mathcal{E}_{\partial\Omega}.$$

Here, T_1 and T_2 denote the elements of $\mathcal{T}$ sharing E, and $\vec{n}_1$ and $\vec{n}_2$ are the outward unit normals of T_1 and T_2 on E, respectively.

Let $\omega_T := \sup_{x \in S_T} \omega(x)$. We define the *a posteriori local error estimator* η_T by

$$\eta_T^2 := \begin{cases} h_T^2 \omega_T \|R\|_{L^2(T)}^2 + h_T \omega_T \|J\|_{L^2(\partial T)}^2 + \nu^2 D_0^{-2\alpha} h_T^{2\alpha+2-n}, & \text{if } x_0 \in T \\ h_T^2 \omega_T \|R\|_{L^2(T)}^2 + h_T \omega_T \|J\|_{L^2(\partial T)}^2, & \text{if } x_0 \notin T \end{cases} \tag{10}$$

and the *global error estimator* η by $\eta := \left(\sum_{T \in \mathcal{T}} \eta_T^2\right)^{\frac{1}{2}}$.

Notice that for elements T such that $S_T \subset \overline{\Omega_0}$, $\omega_T = 1$ so that the local estimator η_T coincides with the usual local H^1-estimator, i.e.,

$$\eta_T = \left(h_T^2 \|R\|_{L^2(T)}^2 + h_T \|J\|_{L^2(\partial T)}^2\right)^{\frac{1}{2}},$$

whereas for the others elements the usual local H^1-estimator is weakened by the constant ω_T.

The exact same proof of [AGM14, Theorem 5.1] allows us to conclude the assertion of the following theorem.

Theorem 3.1 (Reliability of the global error estimator)**.** *Let ω be defined by* (7)*. Let $u \in H_0^1(\Omega, \omega)$ be the solution of problem* (8) *and let $U \in \mathbb{V}_{\mathcal{T}}^{\ell}$ be the solution of the discrete problem* (9)*. Then, there exists a constant $c_1 = c_1(\kappa, \alpha)$ such that*

$$\|U - u\|_{H^1(\Omega, \omega)} \le c_1 C_* C_P \, \eta,$$

where C_* is the continuous inf-sup constant from (5) *and C_P is the Poincaré constant from* (3)*.*

We now discuss the *efficiency* of the local error estimators. In this context, the boundedness of the bilinear form yields the following: if $C_{\mathcal{B}} := \max\{\gamma_2, \|\boldsymbol{b}\|_{L^\infty}, \|c\|_{L^\infty}\}$, then

$$\left| \mathcal{B}[U, v] - \left(\int_\Omega fv + \nu\langle\delta_{x_0}, v\rangle \right) \right| = |\mathcal{B}[U, v] - \mathcal{B}[u, v]| \le C_{\mathcal{B}}\|U - u\|_{H^1(\Theta, \omega)}\|v\|_{H^1(\Theta, \omega^{-1})},$$

for all $\Theta \subset \overline{\Omega}$ and all $v \in H_0^1(\Omega, \omega^{-1})$ with $\mathrm{supp}(v) \subset \Theta$. This bound and the usual steps with scaled bubble functions allow us to conclude the following result, related to the elements which are *far away* from x_0; the detailed proof is omitted.

Theorem 3.2 (Efficiency of local estimators 1)**.** *Let $u \in H_0^1(\Omega, \omega)$ be the solution of problem* (8) *and let $U \in \mathbb{V}_{\mathcal{T}}^{\ell}$ be the solution of the discrete problem* (9)*. Then, there exists a constant $c_2 = c_2(\kappa)$ such that*

$$c_2 \eta_T \le C_\omega^T C_{\mathcal{B}}\|U - u\|_{H^1(S_T, \omega)} + \mathrm{osc}_T, \tag{11}$$

for all $T \in \mathcal{T}$ such that $x_0 \notin S_T$, where $C_\omega^T := \left(\frac{\max_{S_T} \omega}{\min_{S_T} \omega} \right)^{\frac{1}{2}}$. Here, the local oscillation osc_T *is given by*

$$\mathrm{osc}_T := \left(h_T^2 \omega_T \left\| R - \overline{R} \right\|_{L^2(S_T)}^2 + h_T \omega_T \left\| J - \overline{J} \right\|_{L^2(\partial T)}^2 \right)^{\frac{1}{2}}, \tag{12}$$

where $\overline{R}_{|_{T'}}$ denotes the L^2 projection of R on $\mathcal{P}_{\ell-1}(T')$, for all $T' \in \mathcal{T}$, and for each side E, $\overline{J}_{|_E}$ denotes the L^2 projection of J on $\mathcal{P}_{\ell-1}(E)$.

As it usually happens for residual based error estimators, the lower bound is local, and holds up to some oscillation terms. Notice that for elements T such that $S_T \subset \overline{\Omega_0}$, this result coincides with the usual H^1-local error estimation, because $\omega_{|_{S_T}} \equiv 1$.

Remark 3.3. We notice that the constants in the estimation (11), depend on the weight ω only through the quotient $C_\omega^T = \left(\frac{\max_{S_T} \omega}{\min_{S_T} \omega} \right)^{\frac{1}{2}}$, which tends to one when the meshsize tends to zero, provided ω is continuous.

Now we consider the efficiency of the local estimators asociated to the elements T which are near x_0, more precisely, elements T such that $x_0 \in S_T$. As a consequence of [AGM14, Theorem 5.3] we obtain the following result, which holds if we assume that for such T's, $\omega_{|_{S_T}} = \left(\frac{\mathrm{d}_{x_0}}{D_0} \right)^{2\alpha}$. Taking into account the definition of ω given in (7), we notice that this will be the case as soon as the mesh around x_0 is fairly refined, because $\mathrm{dist}(x_0, \Omega_0) > 0$.

Theorem 3.4 (Efficiency of local estimators 2)**.** *Let $u \in H_0^1(\Omega, \omega)$ be the solution of problem* (8) *and let $U \in \mathbb{V}_{\mathcal{T}}^{\ell}$ be the solution of the discrete problem* (9)*. Then, there exists a constant $c_3 = c_3(\kappa, \alpha) > 0$ such that*

$$c_3 \eta_T \le C_{\mathcal{B}}\|U - u\|_{H^1(S_T, \omega)} + \mathrm{osc}_T,$$

for all $T \in \mathcal{T}$ such that $x_0 \in S_T$.[1]

---

[1] The oscillation osc_T when $x_0 \notin T$ is given by (12). In the case that $x_0 \in T$, the jump oscillations in (12) are considered over all $E \in \mathcal{E}_\Omega$ that touch T, not only those contained in ∂T (Cf. [AGM14]).

4   Numerical experiments and applications

In this section we perform some numerical experiments in two dimensional domains with linear elements ($\ell = 1$), in order to illustrate the performance of our estimators and compare with other already known estimators [AGM14, LN03, Dem10]. We consider a standard adaptive loop of the form

$$\text{SOLVE} \quad \longrightarrow \quad \text{ESTIMATE/MARK} \quad \longrightarrow \quad \text{REFINE}.$$

The step SOLVE consists in solving the discrete system (9) for the current mesh $\mathcal{T}$. For the step ESTIMATE/MARK we consider the computation of different alternative a posteriori error estimators and select in $\mathcal{M}$, for refinement, some elements of $\mathcal{T}$ according to different marking strategies. This gives rise to particular adaptive algorithms that we describe in detail below. The last step REFINE consists in performing two bisections to each marked element, and refining some extra elements in order to keep conformity of the meshes, using the *newest-vertex bisection*. We used a custom implementation in MATLAB.

We now describe in detail the three ESTIMATE/MARK alternatives to be considered in this article. We start with our proposal.

Localized Weighted Estimators (LWE).   We consider two possible choices for the function φ in Section 2.3, that we call φ_1 and φ_2, respectively.

- $\varphi_1(x) := \left(1 + a_1 \frac{x}{L}\right)^{-1}$, where $L := \max_{x \in \Omega} \text{dist}(x, \Omega_0)$ and a_1 is a parameter to be fixed.

- $\varphi_2(x) := \begin{cases} a_2, & x > 0, \\ 1, & x = 0, \end{cases}$ where a_2 is a parameter to be fixed.

Then, for $j = 1, 2$, we denote ω_j the corresponding weight given by (7), where $\varphi_0(x) = \varphi_j(\text{dist}(x, \Omega_0))$ and $\alpha = \frac{1}{2}$. We compute the local estimators η_T given by (10) and apply the Dörfler strategy with parameter $\theta = 0.5$ for marking, i.e., we collect in $\mathcal{M}$ those elements $T \in \mathcal{T}$ with largest estimators η_T until

$$\sum_{T \in \mathcal{M}} \eta_T^2 \geq \theta^2 \sum_{T \in \mathcal{T}} \eta_T^2.$$

Liao-Nochetto's Estimators (LNE).   In [LN03], Liao and Nochetto considered the equation $-\text{div}(\mathcal{A}(x)\nabla u) = f$ in 2D with homogeneous Dirichlet boundary conditions for a smooth coefficient matrix $\mathcal{A}(x)$, and proved that

$$\|u - U\|_{H^1(\Omega_0)}^2 \leq C_{LN} \sum_{T \in \mathcal{T}} \eta_{\text{LN},T}^2,$$

with

$$\eta_{\text{LN},T}^2 = \begin{cases} \eta_1^2(T) & \text{if } T \subset \Omega_1, \\ \frac{1}{d^2} \max_i |\log h_i| \eta_{0,-\beta}^2(T) & \text{if } T \not\subset \Omega_1, \end{cases}$$

where h_i denotes the meshsize at the i-th reentrant corner of Ω, $d = \text{dist}(\Omega \setminus \Omega_1, \Omega_0)$, $\Omega_1 \supset \Omega_0$ and

$$\eta_1^2(T) = h_T^2\|-\text{div}(\mathcal{A}\nabla U) - f\|_{L^2(T)}^2 + h_T\|[\mathcal{A}\nabla U]\|_{L^2(\partial T)}^2,$$
$$\eta_{0,-\beta}^2(T) = h_T^4\|(-\text{div}(\mathcal{A}\nabla U) - f)\sigma_{-\beta}\|_{L^2(T)}^2 + h_T^3\|[\mathcal{A}\nabla U]\sigma_{-\beta}\|_{L^2(\partial T)}^2.$$

Here $\sigma_{-\beta} : \Omega \to \mathbb{R}$ is a mesh-dependent weight defined as

$$\sigma_{-\beta}(x) = \min_i \left(r_i^2(x) + h(x)^2 \right)^{-\beta_i/2},$$

with r_i the distance to the i-th corner of Ω, $h(x) = h_T$ if $x \in T$, and $\beta_i = \max\{0, 1 - \pi/w_i\}$ where w_i is the size of the inner angle at the i-th corner of Ω.

Notice that the estimators depend on the choice of the set Ω_1, which is slightly larger than Ω_0, or in other words, on the parameter $d = \text{dist}(\Omega \setminus \Omega_1, \Omega_0)$.

They also propose to use Dörfler's strategy, i.e., in each step of the adaptive loop, the set of marked elements $\mathcal{M}$ is chosen to satisfy

$$\sum_{T \in \mathcal{M}} \eta_{\text{LN},T}^2 \geq \theta^2 \sum_{T \in \mathcal{T}} \eta_{\text{LN},T}^2,$$

for some $\theta \in (0, 1)$. We chose $\theta = 0.5$ in our experiments below.

Demlow's Estimators (DE)   In [Dem10], Demlow considered Poisson equation $-\Delta u = f$ and proved that

$$\|u - U\|_{H^1(\Omega_0)} \leq C_D \left[\left(\sum_{\substack{T \in \mathcal{T} \\ T \subset \Omega_1}} \eta_1^2(T) \right)^{1/2} + \frac{1}{d^{\frac{1}{2}+\frac{1}{p}}} \left(\sum_{T \in \mathcal{T}} \eta_{L^p}^p(T) \right)^{1/p} \right],$$

for $p = 2$ if Ω is convex and $4 < p < \infty$ otherwise, where $\eta_1(T)$ is defined as in the previous paragraph (with $\mathcal{A} = I$), and

$$\eta_{L^p}^p(T) = h_T^{2p} \| - \Delta U - f \|_{L^p(T)}^p + h_T^{p+1} \| [\nabla U] \|_{L^p(\partial T)}^p.$$

Demlow proved convergence of an AFEM with adaptive pollution control, which is based on the following marking strategy [Dem10, Section 4.3]: Given $\zeta > 0$,

1. if $\left(\sum_{\substack{T \in \mathcal{T} \\ T \subset \Omega_1}} \eta_1^2(T) \right)^{1/2} > \zeta \left(\sum_{T \in \mathcal{T}} \eta_{L^p}^p(T) \right)^{1/p}$, then

   take $\mathcal{M} \subset \{T \in \mathcal{T} : T \subset \Omega_1\}$ such that $\sum_{T \in \mathcal{M}} \eta_1^2(T) \geq \theta^2 \sum_{\substack{T \in \mathcal{T} \\ T \subset \Omega_1}} \eta_1^2(T),$

2. otherwise,

   take $\mathcal{M} \subset \mathcal{T}$ such that $\sum_{T \in \mathcal{M}} \eta_{L^p}^p(T) \geq \theta^p \sum_{T \in \mathcal{T}} \eta_{L^p}^p(T).$

We chose $\theta = 0.5$ in our experiments below.

Notice that in this case, not only the parameter d has to be chosen, but also the power p and the coefficient ζ for the separate marking.

We present three examples. The first one with a known solution to Poisson's equation on an L-shaped domain with a point source, the second one with a known solution to a diffusion problem with discontinuous coefficient $\mathcal{A}$, piecewise constant on a *checkerboard* pattern. The third

one is a diffusion-advection-reaction problem, with variable coefficients, simulating a wiggling flow on a canal. The goal of the first two examples is to compare the three proposals stated above. We have run the algorithms considering different choices of the parameters a_1 and a_2 for the localized weighted estimators (LWE), the parameter d for Liao-Nochetto's estimators (LNE) and the parameters d and ζ for Demlow's estimators (DE), but we only report the results corresponding to the best performance of each proposal. The goal of the third example is to show the performance of the newly proposed algorithm in a similar to real life problem, with variable convection coefficient, which simulates the transport, decay and diffusion of a pollutant from a point source.

4.1   Point source on L-shaped domain

We consider the equation $-\Delta u = \delta_{x_0}$ in $\Omega = (-1, 1)^2 \setminus [0, 1)^2$ with Dirichlet boundary values on $\partial\Omega$ and $x_0 = (0.5, 0.5)$. The boundary values were taken such that the exact solution is $u(x) = -\frac{1}{2\pi} \log |x - x_0| + |x|^{2/3} \sin(2\phi/3)$, with ϕ the angle between x and the positive x_1-axis. This solution exhibits two different singularities, due to the reentrant corner and the point source, respectively. We considered Ω_0 a strip on the left side of the domain, namely the set $\Omega_0 = (-1, -0.5) \times (-1, 1)$. In order to compare our proposal LWE with those by [LN03, Dem10] (LNE and DE, respectively), we show in Figure 1 the local error decay (i.e., $\|u - U\|_{H^1(\Omega_0)}$) versus degrees of freedom (DOFs) obtained with the different strategies. We have considered $a_1 = 10^5$ and $a_2 = 10^{-4}$ in the weights ω_1 and ω_2 for LWE, respectively; $d = 0.25$ for LNE; and $d = 0.25$, $p = 5$ and $\zeta = \frac{1}{2}d^{-\frac{1}{2}-\frac{1}{p}}$ for DE. We have also considered the algorithm guided by the *global weighted estimators* ($H^1_\alpha(\Omega)$-estimators, with $\alpha = \frac{1}{2}$) from [AGM14], because

$$\|u - U\|_{H^1(\Omega_0)} \le C_{\Omega_0, x_0} \|u - U\|_{H^1_\alpha(\Omega)} \le C_{AGM} \left(\sum_{T \in \mathcal{T}} \eta^2_{AGM, T} \right)^{\frac{1}{2}}, \tag{13}$$

where $\| \cdot \|_{H^1_\alpha(\Omega)} := \| \cdot \|_{L^2(\Omega, d^{2\alpha}_{x_0})} + \|\nabla \cdot \|_{L^2(\Omega, d^{2\alpha}_{x_0})}$.

We notice that the new proposal (LWE) behaves a bit better than the one by Liao-Nochetto (LNE) and is very similar to the one by Demlow (DE).

The meshes obtained after 5, 10, 15, 20, 25 and 30 iterations of the adaptive algorithm guided by the $H^1(\Omega, \omega_1)$-estimators (LWE) are shown in Figure 2. It is worth noticing how the refinement concentrates around x_0, the reentrant corner and the region of interest Ω_0.

4.2   Discontinuous coefficients

We now consider the following diffusion equation with discontinuous diffusion coefficient

$$\begin{cases} -\nabla \cdot (a\nabla u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

with $\Omega = (-1, 1)^2$, and $\Omega_0 = (-1, 1) \times (-1, -0.75)$, a band at the lower side of the square. We consider two situations, one with $a(x_1, x_2) = 25.2741423690882$ if $x_1 x_2 > 0$ and 1 otherwise, and another one with $a(x_1, x_2) = 161.447638797588$ if $x_1 x_2 > 0$ and 1 otherwise. The boundary values were taken so that the exact solution is, in polar coordinates, $u(r, \phi) = r^\gamma \mu(\phi)$, where

$$\mu(\phi) = \begin{cases} \cos((\pi/2 - \sigma)\gamma) \cdot \cos((\phi - \pi/2 + \rho)\gamma) & \text{if } 0 \le \phi \le \pi/2 \\ \cos(\rho\gamma) \cdot \cos((\phi - \pi + \sigma)\gamma) & \text{if } \pi/2 \le \phi \le \pi \\ \cos(\sigma\gamma) \cdot \cos((\phi - \pi - \rho)\gamma) & \text{if } \pi \le \phi < 3\pi/2 \\ \cos((\pi/2 - \rho)\gamma) \cdot \cos((\phi - 3\pi/2 - \sigma)\gamma) & \text{if } 3\pi/2 \le \phi \le 2\pi. \end{cases}$$

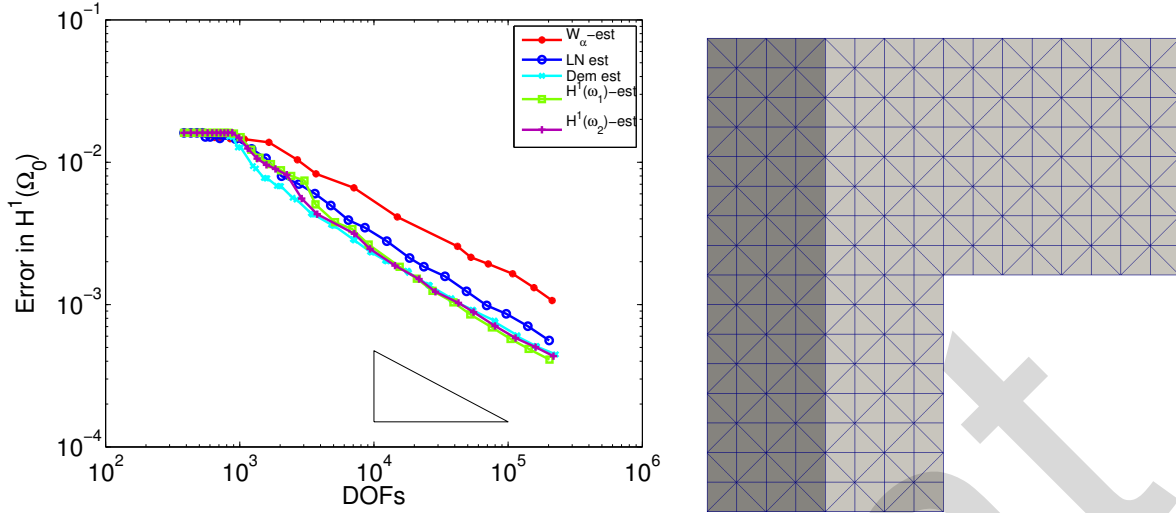Figure 1: Error decay and initial mesh with Ω_0 shaded (225 DOFs). We plot the $H^1(\Omega_0)$-error versus the number of degrees of freedom (DOFs), obtained with an adaptive procedure guided by the newly proposed estimators taking $\alpha = \frac{1}{2}$; and $a_1 = 10^5$ and $a_2 = 10^{-4}$ in the weights ω_1 and ω_2 for LWE, respectively; and also by the estimators proposed by Liao-Nochetto (LNE) with $d = 0.25$, by Demlow (DE) with $d = 0.25$, $p = 5$ and $\zeta = \frac{1}{2}d^{-\frac{1}{2}-\frac{1}{p}}$, and the (global) $H^1_\alpha(\Omega)$-estimators from (13), with $\alpha = \frac{1}{2}$. The new proposal behaves a bit better than the one by Liao-Nochetto and is very similar to the one by Demlow.
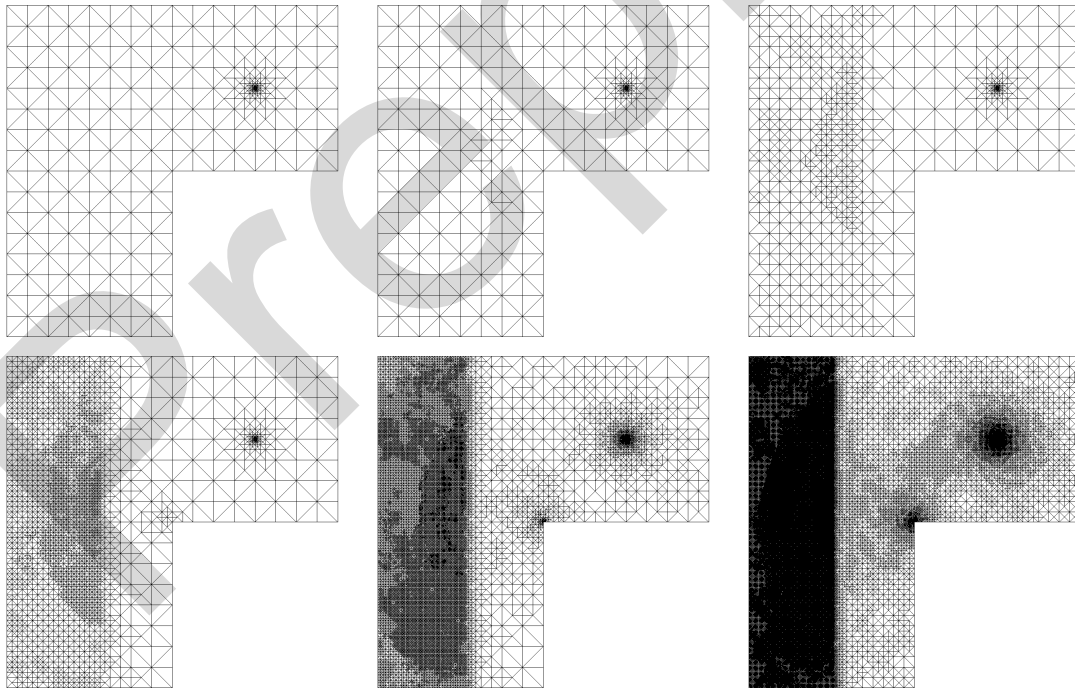
Figure 2:   Meshes after 5, 10, 15, 20, 25 and 30 iterations of the adaptive algorithm guided by the $H^1(\Omega, \omega_1)$-estimators (LWE). They have 345, 486, 1052, 3523, 19715 and 71403 degrees of freedom, respectively. It is worth noticing how the refinement concentrates around x_0, the reentrant corner and the region of interest Ω_0.
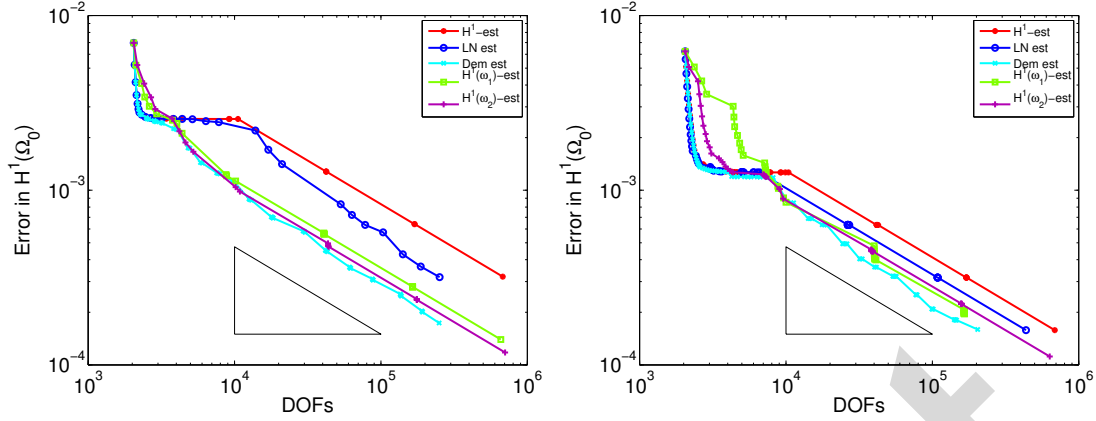
11

Figure 3:   Local error versus DOFS for the discontinuous coefficient examples. The situation is similar to the one depicted for the L-shaped domain. Our approach is a bit better than the one by Liao and Nochetto (LNE) and very similar to the one by Demlow (DE). We recall that our estimators are indeed an upper bound for the error, but there is no rigorous proof (yet) that the others bound the local error, due to the lack of a corresponding regularity result for the duality approach.

The constants take the values $\rho = 0.785398163397448$, $\gamma = 0.25$, $\sigma = -5.49778714378214$ in the first case and $\rho = 0.785398163397448$, $\gamma = 0.1$, $\sigma = -14.9225651045515$ in the second. These solutions have a singularity like $|x|^{0.25}$ and $|x|^{0.1}$, respectively, around the origin.

The estimators from [LN03, Dem10], which need a duality argument for the lower order error, do not carry over immediately to this situation due to the lack of corresponding regularity results with discontinuous coefficients. In order to make a comparison, we assume that the upper bound for the Liao-Nochetto estimators (LNE) holds with

$$\sigma_{-\beta}(x) = \left(|x|^2 + h(x)^2 \right)^{-\beta/2},$$

where $\beta = 1 - \gamma$, and for the Demlow estimator (DE), we assume the upper bound holds for $p = 9$ in the first case and $p = 21$ in the second case ($p = \lceil \frac{2}{\gamma} \rceil$).

We believe this is reasonable under the assumption that the precise singularity $|x|^\gamma$ is the worst one for such coefficients. The chosen values of p satisfy that the exact solution u belongs to $W^{2,p'}(\Omega)$ with $1/p + 1/p' = 1$. However, we emphasize that the equivalences between error and estimator for the Liao-Nochetto and Demlow estimators have not been rigorously proved, we infer the possible form of the estimators by analogy after looking at the worst singularity.

In order to compare the behavior of our method with those of Liao-Nochetto and Demlow we plot in Figure 3 the local error (i.e., $\|u - U\|_{H^1(\Omega_0)}$) versus the number of DOFs. For the first case (left), we have taken $a_1 = 10^4$ and $a_2 = 10^{-4}$ in the weights ω_1 and ω_2 for LWE, respectively; $d = 0.25$ for LNE; and $d = 0.125$, $p = 9$ and $\zeta = \frac{1}{4}d^{-\frac{1}{2}-\frac{1}{p}}$ for DE. For the second case (right), we have considered $a_1 = 10^5$ and $a_2 = 10^{-4}$ in the weights ω_1 and ω_2 for LWE, respectively; $d = 0.75$ for LNE; and $d = 0.125$, $p = 21$ and $\zeta = \frac{1}{4}d^{-\frac{1}{2}-\frac{1}{p}}$ for DE. We have also considered the algorithm guided by the standard global $H^1(\Omega)$-estimators.

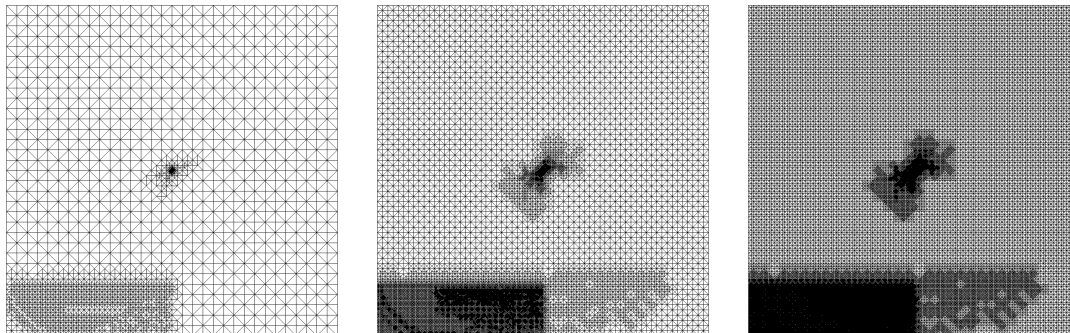A sequence of meshes for the more singular case is shown in Figure 4.

Figure 4:   Meshes after 10, 20 and 30 iterations of the adaptive algorithm guided by the $H^1(\Omega, \omega_1)$-estimators (LWE) for the more singular case.

4.3   Diffusion-advection-reaction with a point source

We end this section by showing how the adaptive method behaves on a diffusion-advection-reaction equation. This is another case where the duality theory fails and our approach provides a simple a posteriori estimator for the local error, by just using an appropriate weight. We consider the equation

$$
\begin{cases}
-0.02\Delta u + \begin{bmatrix} 2 \\ \sin(5x_1) \end{bmatrix} \cdot \nabla u + 0.1u = \delta_{(0.2,0.4)} & \text{in } \Omega = (0,3) \times (0,1), \\
\qquad\qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega \cap \{x_1 < 3\}, \\
\qquad\qquad\qquad\qquad\qquad \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \cap \{x_1 = 3\},
\end{cases}
$$

and the region of interest is $\Omega_0 = (0,3) \times (0,0.25)$. An approximate solution obtained with an adaptive method tailored to the W_α error has been presented in [AGM14]. We now present the sequence of meshes (Figure 5) obtained by the new localized weighted estimators (LWE) taking $\alpha = \frac{1}{2}$ and $a_1 = 10^5$ in the definition of the weight ω_1. Notice that the refinement focuses on the region of interest Ω_0 which is now a narrow band at the bottom of the domain. This emulates the situation where there is a pollutant discharge in a river or canal, and we are interested in the amount of pollutant at the coast.

Acknowledgements

References

[AGM14] J.P. Agnelli, E.M. Garau and P. Morin, *A posteriori error estimates for elliptic problems with Dirac measure terms in weighted spaces*, ESAIM Math. Model. Numer. Anal. **48** (2014), no. 6, 1557–1581.

[ABR07] R. Araya, E. Behrens and R. Rodríguez, *An adaptive stabilized finite element scheme for a water quality model*, Comput. Methods Appl. Mech. Engrg. **196** (2007), 2800-2812.
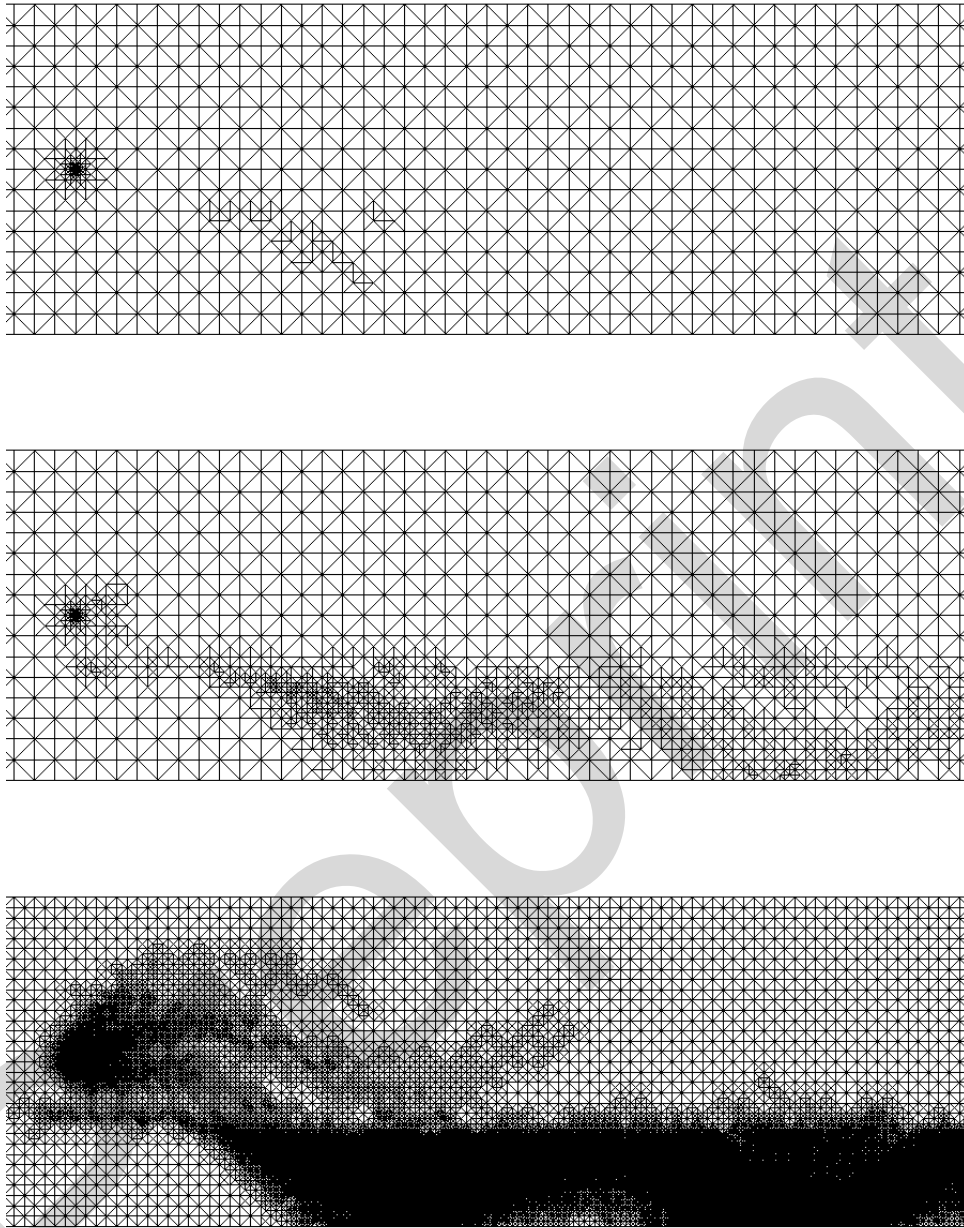
Figure 5: Sequence of meshes for the diffusion-advection-reaction experiment. Meshes obtained after 15 (2338 elements/1234 DOFs), 20 (4799 elements/2473 DOFs) and 25 (251744 elements/126181 DOFs) iterations of the adaptive algorithm guided by the localized weighted estimators (LWE). During the first 14 adaptive steps the refinement was focused on solving the singularity due to the point source. Starting at step 15, the meshes get refined at the region of interest, and a combination of refinement around the singularity and the region of interest occurs after step 20.

[C86] E. Casas, *Control of an elliptic problem with pointwise state constraints*, SIAM J. Control. Optim. **24** (1986), 1309–1318.

[CW85] S. Chanillo and R.L. Wheeden, *Weighted Poincar and Sobolev inequalities and estimates for weighted Peano maximal functions*, Amer. J. Math. **107** (1985), no. 5, 1191-1226.

[DQ08] C. D'Angelo and A. Quarteroni, *On the coupling of 1D and 3D diffusion-reaction equations. Application to tissue perfusion problems*, Math. Models Methods Appl. Sci. **18** (2008), no. 8, 1481–1504.

[Dem07] A. Demlow, *Local a posteriori estimates for pointwise gradient errors in finite element methods for elliptic problems*, Math. Comp. **76** (2007), no. 257, 19–42.

[Dem10] A. Demlow, *Convergence of an adaptive finite element method for controlling local energy errors*, SIAM J. Numer. Anal. **48** (2010), no. 2, 470–497.

[FKS82] E. Fabes, C. Kenig, and R. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116.

[J75] J.D. Jackson, *Classical electrodynamics*, John Wiley & Sons, Inc., New York-London-Sydney, second edition, 1975.

[KMR97] V.A. Kozlov, V.G. Maz'ya, J. Rossmann, *Elliptic boundary value problems in domains with point singularities*, Mathematical Surveys and Monographs, 52, American Mathematical Society, Providence, RI, 1997.

[LN03] X. Liao and R.H. Nochetto, *Local a posteriori error estimates and adaptive control of pollution effects*, Numer. Methods Partial Differential Equations **19** (2003), no. 4, 421–442.

[NOS16] R.H. Nochetto, E. Otárola, A.J. Salgado, *Piecewise polynomial interpolation in Muckenhoupt weighted Sobolev spaces and applications*, Numer. Math. **132** (2016), no. 1, 85–130.