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## REGULARITY OF MAXIMAL FUNCTIONS ASSOCIATED TO A CRITICAL RADIUS FUNCTION

By

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# REGULARITY OF MAXIMAL FUNCTIONS ASSOCIATED TO A CRITICAL RADIUS FUNCTION

B. BONGIOANNI, A. CABRAL AND E. HARBOURE

ABSTRACT. This work deals with boundedness on $BMO$ and Lipschitz type spaces of maximal operators appearing in the context of a critical radius function.

## 1. Introduction and preliminaries

In this work we deal with the boundedness of some maximal operators acting on $BMO$ and Lipschitz type spaces that comes from the localized analysis considering a *critical radius function* $\rho$, i.e. a function that satisfies

$$c_\rho^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N_0} \leq \rho(y) \leq c_\rho\,\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{N_0}{N_0+1}}, \tag{1}$$

for all $x, y \in \mathbb{R}^d$ (see [5] and [1]).

This analysis appears in the context of the Schrödinger operator $\mathcal{L} = -\Delta + V$ in $\mathbb{R}^d$, $d \geq 3$ (see for example [14], [7] and references therein).

For $x \in \mathbb{R}^d$, a ball of the form $B(x, \rho(x))$ is called *critical* and a ball $B(x, r)$ with $r < \rho(x)$ will be called *sub-critical*. We denote by $\mathcal{B}_\rho$ the family of all *sub-critical* balls.

One of the operators we are interested in is the localized maximal operator $M_\rho$ defined for $f \in L^1_{\rm loc}$ as

$$M_\rho f(x) = \sup_{x\in B\in\mathcal{B}_\rho} \frac{1}{|B|}\int_B |f|.$$

In [4] the authors prove that $M_\rho$ is boundend on $L^p(w)$ for $1 < p < \infty$ where $w$ belongs to a suitable class larger than classical $A_p$ Mukenhoupt weights. In this work we deal with the boundedness of $M_\rho$ in a weighted $BMO$ type space that appears in [8] for $w = 1$, and in [2] with weighted versions.

We also deal with some type of maximal operator of a family of operators presented in Section 5 that is a model to deal with semi-groups appearing in the theory related to the Schrödinger operator $\mathcal{L}$. Some results concerning this operator in a more general context can be found in [15] and [16].

In the rest of this section we present some facts about the critical radius function. Section 2 is devoted to present the classes of weights involved in this work and some properties of them that will be useful. In Section 3 we state the type of spaces where we are interested in study the behavior of the operators and we prove some results about them in order to simplify the treatment of boundedness in the main results and we hope it will aim in other works that uses techniques of this type. In Section 4

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and Section 5 we state and prove the main results of this work finding the behavior of maximal operators we have already talked about, and finally we present some applications to the context of the Schrödinger operator in Section 6.

*Remark* 1. Inequality (1) implies that if $\sigma > 0$ and $x, y \in \sigma Q$, where $Q$ is a critical ball, then $\rho(x) \simeq \rho(y)$, with a constant that depends on $\sigma$. More precisely, from (1) and the fact that both belong to $\sigma Q$, we have

$$\rho(x) \leq c_\sigma \rho(y), \tag{2}$$

where $c_\sigma = c_\rho^2(1+\sigma)^{\frac{N_0^2+2N_0}{N_0+1}}$, and $c_\rho$ is the constant appearing in (1). If we change the role of $x$ and $y$ we obtain $\rho(x) \simeq \rho(y)$.

As a consequence of (1), we have the following result that can be found in [9] presenting a useful covering of $\mathbb{R}^d$ by critical balls.

**Proposition 1.** *There exists a sequence of points $x_j$, $j \geq 1$, in $\mathbb{R}^d$, so that the family $Q_j = B(x_j, \rho(x_j))$, $j \geq 1$, satisfies*

*i)* $\cup_j Q_j = \mathbb{R}^d$.
*ii) For every $\sigma \geq 1$ there exist constants $C$ and $N_1$ such that, $\sum_j \chi_{\sigma Q_j} \leq C\sigma^{N_1}$.*

Given ball $B$ we shall also need a particular covering by critical balls with centers inside $B$ as the following lemma shows.

**Lemma 1.** *Let $B = B(x_0, r)$ with $x_0 \in \mathbb{R}^d$ and $r \geq \rho(x_0)$. There exists a set of points $\{x_i\}_{i=1}^N \subset B$ such that $B \subset \cup_{i=1}^N B(x_i, \rho(x_i))$ and $\sum_{i=1}^N \chi_{B(x_i,\rho(x_i))} \leq C$ where $C$ depends only on the constants in* (1).

*Proof.* Consider the family of sets

$$\mathcal{F} = \{S \subset B : \; B(x, \gamma\rho(x)) \cap B(y, \gamma\rho(y)) = \emptyset, \quad \forall\, x, y \in S, \; x \neq y\}$$

with a constant $\gamma < 1/(c_1^2+1)$ where $c_1$ is the constant in (2). It is clear that $\mathcal{F} \neq \emptyset$ since $\{x_0\} \in \mathcal{F}$. Observe that if $\mathcal{C}$ be a chain in $\mathcal{F}$ endowed with the order of inclusion, then $V = \cup_{S \in \mathcal{C}} S$ is an upper bound of $\mathcal{C}$. Therefore, there exists a maximal element $S_{\max}$ in $\mathcal{F}$. The set $S_{\max}$ must be finite. In fact, due to (1),

$$\rho(x) \geq c_0^{-1}\left(1 + \frac{r}{\rho(x_0)}\right)^{-N_0} \rho(x_0) = \delta_0 > 0,$$

for all $x \in B$, and thus there is no more than $N$ balls in $S_{\max}$ with $N \geq \left(\frac{r+\gamma\delta_0}{\gamma\delta_0}\right)^d$. Denote $x_1, x_2, \ldots, x_N$ the elements of $S_{\max}$. We shall see that $B \subset \cup_{i=1}^N B(x_i, \rho(x_i))$ and the overlapping of the balls $B(x_i, \rho(x_i))$, $i = 1, \ldots, N$, is finite.

Suppose there exists $y \in B$ such that $y \notin \cup_{i=1}^N B(x_i, \rho(x_i))$, which means $|y - x_i| > \rho(x_i)$, $i = 1, \ldots, N$. Now see that $B(y, \gamma\rho(y)) \cap B(x_i, \gamma\rho(x_i))$ is empty. In fact, suppose $z \in B(y, \gamma\rho(y)) \cap B(x_i, \gamma\rho(x_i))$, then $|y - x_i| \leq |y - z| + |z - x_i| \leq \gamma(\rho(y) + \rho(x_i)) \leq \gamma(c_1^2+1)\rho(x_i)$ which is a contradiction by the choice of $\gamma$. So $S_{\max} \cup \{y\}$ belongs to $\mathcal{F}$ and this means the contradiction that $S_{\max}$ is not a maximal element of $\mathcal{F}$.

Now we see that the overlapping $\{B(x_i, \rho(x_i))\}_{i=1}^N$ is finite and depends only on the contants in (1).

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Suppose that $m$ is such that $\cap_{i=1}^m B(x_i, \rho(x_i)) \neq \emptyset$ for some points $x_i \in S$ with $S \in \mathcal{F}$. Since $\frac{1}{C}\rho(x_1) \leq \rho(x_i) \leq C\rho(x_1)$, $i = 1, \ldots, m$, with $C = c_\rho^2 2^{\frac{N_0^2+2N_0}{N_0+1}}$ (see inequality (2)) we have

$$\cup_{i=1}^m B(x_i, \gamma\rho(x_i)) \subset B(x_1, 3C\rho(x_1)).$$

Now we use the fact that the balls $\{B(x_i, \gamma\rho(x_i))\}_{i=1}^m$ are disjoints to conclude

$$\begin{aligned} m\left[\gamma\frac{\rho(x_1)}{C}\right]^d &\leq \sum_{i=1}^m |B(x_i, \gamma\rho(x_i))| \\ &= |\cup_{i=1}^m B(x_i, \gamma\rho(x_i))| \leq |B(x_1, 3C\rho(x_1))| \\ &= [3C\rho(x_1)]^d, \end{aligned}$$

thus $m \leq \frac{3^d C^{2d}}{\gamma^d}$. □

## 2. Weights

Following [4], for $1 < p < \infty$, we say that a weight $w$ belongs to the class $A_p^{\rho,\text{loc}}$ if there exists a constant $C$ such that

$$\left(\int_B w\right)\left(\int_B w^{-\frac{1}{p-1}}\right)^{p-1} \leq C|B|^p, \tag{3}$$

for every ball $B = B(x, r)$ with $x \in \mathbb{R}^d$ and $r \leq \rho(x)$.

In the case $p = 1$ we define the class $A_1^{\rho,\text{loc}}$ as those weights $w$ satisfying

$$w(B) \sup_B w^{-1} \leq C|B|, \tag{4}$$

for every ball $B = B(x, r)$ with $x \in \mathbb{R}^d$ and $r \leq \rho(x)$, for some constat $C$ independent of $B$. We denote $A_\infty^{\rho,\text{loc}} = \cup_{p\geq 1} A_p^{\rho,\text{loc}}$.

In the rest of this section we will state and prove some facts about weights in the classes defined above that will be useful in what follows and they are of interest on itself.

**Proposition 2** (see Corollary 1 in [4]). *If $1 \leq p < \infty$ and $c > 1$, then $A_p^{\rho,loc} = A_p^{c\rho,loc}$.*

**Lemma 2.** *If $w \in A_p^{\rho,\text{loc}}$ and $B = B(x, r)$, with $p \geq 1$, $x \in \mathbb{R}^d$ and $r \leq c\rho(x)$ for some constant $c > 1$, then there exists a constant $C$ such that for every measurable subset $E \subset B$, it holds*

$$w(B) \leq Cw(E)\left(\frac{|B|}{|E|}\right)^p. \tag{5}$$

*Proof.* In the case $p = 1$, since $w \in A_1^{\rho,\text{loc}} = A_1^{c\rho,\text{loc}}$ (see Proposition 2) and $E \subset B$, we have for some constant $C$,

$$w(B) \leq C|B| \inf_{x\in B} w(x) \leq C|B| \inf_{x\in E} w(x) \leq C_1 w(E)\frac{|B|}{|E|}.$$

For the case $p > 1$, using the condition $w \in A_p^{c\rho,\text{loc}}$ and Hölder's inequality we get for some constant $C$,

$$w(B) \leq \frac{C|B|^p}{[w^{-1/(p-1)}(B)]^{p-1}} \leq Cw(E)\left(\frac{|B|}{|E|}\right)^p.$$

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□

*Remark* 2. The constant $C$ in (5) is the constant appearing in (3) (or (4) when $p = 1$) for the critical radius function $c\rho$ instead of $\rho$.

Given $\theta \geq 0$ and $p > 1$ we introduce the class $A_p^{\rho,\theta}$ as those weights $w$ such that

$$\left(\int_B w\right)\left(\int_B w^{-\frac{1}{p-1}}\right)^{p-1} \leq C|B|\left(1+\frac{r}{\rho(x)}\right)^{p\theta}, \tag{6}$$

for every ball $B = B(x,r)$. For $p = 1$ we define $A_1^{\rho,\theta}$ as the set of weights $w$ such that

$$\int_B w \leq C|B|\left(1+\frac{r}{\rho(x)}\right)^{\theta} \inf_{x\in B} w. \tag{7}$$

holds for all balls $B = B(x,r)$. We denote $A_p^\rho = \cup_{\theta\geq 0} A_p^{\rho,\theta}$.

*Remark* 3. It follows easily from their definitions that $w \in A_p^{\rho,\theta}$ implies $w \in A_p^{\rho,\mathrm{loc}}$, for every $\theta \geq 0$.

**Lemma 3.** *Let $w \in A_p^{\rho,\theta}$, with $p \geq 1$, $\theta > 0$, $x \in \mathbb{R}^d$ and $r \leq R$. Then there exits a constant $C$ such that*

$$w(B(x,R)) \leq C w(B(x,r))\left(\frac{R}{r}\right)^{dp}\left(1+\frac{R}{\rho(x)}\right)^{p\theta}.$$

*Proof.* The proof follows the same lines of Lemma 2 with the corresponding modifications.

□

## 3. Weighted BMO type spaces

Let $\beta \geq 0$, a weight $w$, $f \in L^1_{\mathrm{loc}}$ and call $f_B = \frac{1}{|B|}\int_B f$. Following [2] we say that $f$ belongs to the space $BMO_\rho^\beta(w)$ if

$$\int_B |f - f_B| \leq C\, w(B)\, |B|^{\beta/d}, \quad \text{for all } B \in \mathcal{B}_\rho, \tag{8}$$

and

$$\int_B |f| \leq C\, w(B)\, |B|^{\beta/d}, \quad \text{for all } B \notin \mathcal{B}_\rho, \tag{9}$$

where $\mathcal{B}_\rho$ is the family of *sub-critical* balls defined in the introduction.

Let us note that if (9) is true for some ball $B$ then (8) holds for the same ball, so it equivalent to ask (8) for all balls as was presented in [2].

We can give a norm in $BMO_\rho^\beta(w)$ as the less constant that satisfies (8) and (9) and we denote it by $\|f\|_{BMO_\rho^\beta(w)}$. It is not difficult to see that $BMO_\rho^\beta(w) \subset BMO^\beta(w)$ where $BMO^\beta(w)$ is the Lipschitz space appearing in [12] in the classical context. On the other hand, if $Q$ a fixed ball in $\mathbb{R}^d$, we call $BMO_Q^\beta(w)$ the space of locally integrable functions on $Q$ that satisfy condition (8) for all ball $B \subset Q$. From its definition, it is easy to see that

$$BMO_\rho^\beta(w) \subset BMO^\beta(w) \subset BMO_Q^\beta(w)$$

and

$$\|f\|_{BMO_Q^\beta(w)} \leq \|f\|_{BMO^\beta(w)} \leq 2\|f\|_{BMO_\rho^\beta(w)}. \tag{10}$$

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**Proposition 3.** *If $w \in A_\infty^{\rho,\mathrm{loc}}$ and $\beta \geq 0$ then $BMO_\rho^\beta(w) = BMO_{\gamma\rho}^\beta(w)$ for all $\gamma > 0$, with equivalent norms.*

*Proof.* If $\gamma > 0$, let us observe that $\gamma\rho$ is also a critical radius function.

Without loss of generality, we may suppose $\gamma > 1$ (elsewhere, we can start with $\gamma\rho$ and then we multiply by $1/\gamma > 1$).

Let us start with the inclusion $BMO_\rho^\beta(w) \subset BMO_{\gamma\rho}^\beta(w)$. Give $f \in BMO_\rho^\beta(w)$, we know $f \in BMO(w)$ and also from (10) we have

$$\|f\|_{BMO^\beta(w)} \leq 2\|f\|_{BMO_\rho^\beta(w)}.$$

In particular,

$$\frac{1}{w(B)}\int_B |f - f_B| \leq 2\|f\|_{BMO_\rho^\beta(w)}\,|B|^{\beta/d}, \quad \text{for all } B \in \mathcal{B}_{\gamma\rho}.$$

On the other hand, since $\mathcal{B}_\rho \subset \mathcal{B}_{\gamma\rho}$, if $B \notin \mathcal{B}_{\gamma\rho}$ then $B \notin \mathcal{B}_\rho$ and therefore,

$$\frac{1}{w(B)}\int_B |f| \leq \|f\|_{BMO_\rho^\beta(w)}\,|B|^{\beta/d}.$$

Thus, $f \in BMO_{\gamma\rho}^\beta(w)$ and

$$\|f\|_{BMO_{\gamma\rho}^\beta(w)} \leq 2\|f\|_{BMO_\rho^\beta(w)}.$$

Now, we will see the inclusion $BMO_{\gamma\rho}^\beta(w) \subset BMO_\rho^\beta(w)$. Let $f \in BMO_{\gamma\rho}^\beta(w)$. From the fact that $\mathcal{B}_\rho \subset \mathcal{B}_{\gamma\rho}$ it follows

$$\frac{1}{w(B)}\int_B |f - f_B| \leq \|f\|_{BMO_{\gamma\rho}^\beta(w)}\,|B|^{\beta/d}, \quad \text{for all } B \in \mathcal{B}_\rho,$$

Therefore, it last to see

$$\frac{1}{w(B)}\int_B |f(x)|dx \leq C\|f\|_{BMO_{\gamma\rho}^\beta(w)}\,|B|^{\beta/d}, \quad \text{for all } B \notin \mathcal{B}_\rho$$

If $B = B(x,r) \notin \mathcal{B}_{\gamma\rho}$ there is nothing to prove. On the other hand, if $B \in \mathcal{B}_{\gamma\rho}$ and $B \notin \mathcal{B}_\rho$, we have $\rho(x) \leq r < \gamma\rho(x)$. Since $w \in A_\infty^{\rho,\mathrm{loc}}$, it must exists $p \geq 1$ such that $w \in A_p^{\rho,\mathrm{loc}}$. Therefore, from Lemma 2, we get for some constant $C$,

$$\begin{aligned}
\frac{1}{w(B)}\int_B |f(x)|dx &\leq \frac{w(B(x,\gamma\rho(x)))}{w(B)}\frac{1}{w(B(x,\gamma\rho(x)))}\int_{B(x,\gamma\rho(x))}|f(x)|dx \\
&\leq C\frac{|B(x,\gamma\rho(x))|^p}{|B|^p}\|f\|_{BMO_{\gamma\rho}^\beta(w)}|B(x,\gamma\rho(x))|^{\beta/d} \\
&\leq C\left(\frac{|B(x,\gamma\rho(x))|}{|B|}\right)^{p+\beta/d}\|f\|_{BMO_{\gamma\rho}^\beta(w)}|B|^{\beta/p} \\
&\leq C\gamma^{\beta+dp}\|f\|_{BMO_{\gamma\rho}^\beta(w)}|B|^{\beta/p},
\end{aligned}$$

and the proof is finished. □

**Proposition 4.** *Let $w \in A_p^{\rho,\mathrm{loc}}$, for some $1 \leq p < \infty$ and $f \in L^1_{loc}$. If*

$$A = \sup_{x\in\mathbb{R}^d}\frac{1}{w(B(x,\rho(x))|B(x,\rho(x)|^{\beta/d}}\int_{B(x,\rho(x))}|f| < \infty, \tag{11}$$

*then there exists a constant $C$ such that*

$$\sup_{x\in\mathbb{R}^d, r\geq\rho(x)}\frac{1}{w(B(x,r))|B(x,r)|^{\beta/d}}\int_{B(x,r)}|f| < CA.$$

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*Proof.* Suppose (11) holds and consider a ball $B = B(x,r)$, $x \in \mathbb{R}^d$ and $r \geq \rho(x)$. In the case that there exists $y \in B$ such that $\rho(y) > 2r$ then $B \subset B(y, \rho(y))$. Therefore, by Lemma 2,

$$\begin{aligned}
\frac{1}{w(B)|B|^{\beta/d}} \int_B |f| &\lesssim \left(\frac{|B(y,\rho(y))|}{|B|}\right)^{\frac{\beta}{d}+1} \frac{1}{w(B(y,\rho(y)))|B(y,\rho(y))|^{\beta/d}} \int_{B(y,\rho(y))} |f| \\
&\lesssim A \left(\frac{\rho(y)}{r}\right)^{\beta+d} \\
&\lesssim A \left(\frac{\rho(y)}{\rho(x)}\right)^{\beta+d}.
\end{aligned}$$

Since $x \in B(y,\rho(y))$ then $\rho(y) \simeq \rho(x)$, and thus the last quantity is constant.

Suppose now that for all $y \in B$, $\rho(y) \leq 2r$. From Lemma 1, there exist $N$ balls $B_i = B(x_i, \rho(x_i))$, $i = 1, \ldots, N$ such that $B \subset \cup_{i=1}^N B_i$ and $\sum_{i=1}^N \chi_{B(x_i,\rho(x_i))} \leq C$ where $N$ and $C$ depends only on the constants in (1) and the dimension $d$. Now for each $i = 1, \ldots, N$ consider the ball $P_i = B(z_i, \rho(x_i)/4)$, with $z_i = \frac{\rho(x_i)}{4|x-x_i|}(x - x_i) + x_i$, that satisfies $P_i \subset B \cap B_i$ and $|B_i|/|P_i| = 4^d$.

Therefore,

$$\begin{aligned}
\int_B |f| \;&\leq\; \sum_{i=1}^N \int_{B_i} |f| \;\leq\; A \sum_{i=1}^N w(B_i)|B_i|^{\beta/d} \;=\; A \sum_{i=1}^N w(P_i) \frac{w(B_i)}{w(P_i)} |B_i|^{\beta/d} \\
&\leq\; AC4^{dp+\beta} \sum_{i=1}^N w(P_i)|P_i|^{\beta/d} \;\leq\; C|B|^{\beta/d} w(\cup_{i=1}^N P_i) \;\leq\; C|B|^{\beta/d} w(B),
\end{aligned}$$

where $C$ is the constant of Lemma 2 and we also have used the bounded overlapping property of the balls $B_i$ (see Lemma 1).

$\square$

Following the previous proof, Corollary 1 in [2] may be improved. Actually, instead of $w \in A_p^{\rho,\text{loc}}$ we only need to ask a doubling condition for the weight $w$ on sub-critical balls.

In [3] it was proved the followig result for $w$ in the Muckenhoupt class $A_p$. Here, we shall prove an extention of that for $w \in A_p^{\rho,\text{loc}}$.

**Lemma 4.** *Let $0 \leq \beta < 1$ $w \in A_p^{\rho,\text{loc}}$, $1 < s \leq p'$ and $f \in BMO_\rho^\beta(w)$. Then,*

$$\left(\int_B |f|^s w^{1-s}\right)^{1/s} \lesssim w(B)^{1/s}|B|^{\beta/d}\|f\|_{BMO_\rho^\beta(w)}, \tag{12}$$

*for every ball $B = B(x,r)$ with $r \geq \rho(x)$, and*

$$\left(\int_B |f - f_B|^s w^{1-s}\right)^{1/s} \lesssim w(B)^{1/s}|B|^{\beta/d}\|f\|_{BMO_\rho^\beta(w)}, \tag{13}$$

*for every ball $B = B(x,r)$ with $r \leq \rho(x)$.*

*Proof.* First, we will prove that (13) holds. Let us consider the covering $\{Q_k\}$ of critical balls given by Proposition 1 and a ball $B = B(x,r)$ with $r \leq \rho(x)$. Then, there exists $Q_k$ such that $x \in Q_k = B(x_k, \rho(x_k))$, and by (2) we have

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$B \subset \tilde{Q}_k = B(x_k, C\rho(x_k))$ for a constant $C$ independent of $x$ and $r$. If we have a cube $Q$ and we call $BMO_Q^{\beta,s}(w)$ to the space of functions $f$ such that

$$\|f\|_{BMO^{\beta,s}(w)} = \sup_{B\subset Q} \frac{1}{|B|^{\beta/d}} \left( \frac{1}{w(B)} \int_B |f - f_B|^s w^{1-s} \right)^{1/s} < \infty, \tag{14}$$

and $BMO^{\beta,s}(w)$ to the space of functions when the supremum (14) is considred for all balls $B \subset \mathbb{R}^d$, according to [3], it follows $BMO^{\beta,s}(w) \equiv BMO^{\beta}(w)$ with $\|f\|_{BMO^{\beta,s}(w)} \cong \|f\|_{BMO^{\beta}(w)}$ and also $BMO^{\beta,s}(w) \subset BMO_Q^{\beta,s}(w)$ with

$$\|f\|_{BMO_Q^{\beta,s}(w)} \le \|f\|_{BMO^{\beta,s}(w)}.$$

Therefore, since $B \subset \tilde{Q}_k$, we get

$$\frac{1}{|B|^{\beta/d}} \left( \frac{1}{w(B)} \int_B |f - f_B|^s w^{1-s} \right)^{1/s} \le \|f\|_{BMO^{\beta,s}(w)} \lesssim \|f\|_{BMO^{\beta}(w)},$$

and thus (13) is a consequence of inequality (10).

From Proposition 4, it is enough to check (12) over a critical ball $\tilde{B} = B(x, \rho(x))$ with $x \in \mathbb{R}$. Observe that

$$\left( \int_{\tilde{B}} |f|^s w^{1-s} \right)^{1/s} \lesssim \left( \int_{\tilde{B}} |f - f_{\tilde{B}}|^s w^{1-s} \right)^{1/s} + |f_{\tilde{B}}| \left( w^{1-s}(\tilde{B}) \right)^{1/s}.$$

The first term of the right side is bounded following the same argument as before. For the second term, observe that $w^{1-s} \in A_s^{\rho,\mathrm{loc}}$, since $w \in A_p^{\rho,\mathrm{loc}}$ and $p \le s'$. Then

$$\begin{aligned} \left( w^{1-s}(\tilde{B}) \right)^{1/s} |f_{\tilde{B}}| &\lesssim \frac{|\tilde{B}|}{w^{1/s'}(\tilde{B})} |f_{\tilde{B}}| = \frac{1}{w^{1/s'}(\tilde{B})} \int_{\tilde{B}} |f| \\ &\lesssim \frac{w(\tilde{B})}{w^{1/s'}(\tilde{B})} |\tilde{B}|^{\beta/d} \|f\|_{BMO_\rho^{\beta}(w)} \\ &= w^{1/s}(\tilde{B}) |\tilde{B}|^{\beta/d} \|f\|_{BMO_\rho^{\beta}(w)}. \end{aligned}$$

□

## 4. The localized maximal operator associated to $\rho$

In [4] (see Theorem 1 therein) the behavior of $M_\rho$ is studied, and it is proved that $M_\rho$ is bounded on weighted Lebesgue spaces for localized weights, as is stated in the following theorem.

**Proposition 5.** *The operator $M_\rho$ is bounded on $L^p(w)$, $1 < p < \infty$, for $w \in A_p^{\rho,loc}$, and it is of weak type $(1,1)$ for $w \in A_1^{\rho,loc}$.*

Now we present one of the main results of this work that tell us about the behavior of $M_\rho$ in the extreme $BMO_\rho(w)$.

**Theorem 1.** *Let $w \in A_1^{\rho,\mathrm{loc}}$. There exists a constant $C$ such that*

$$\|M_\rho f\|_{BMO_\rho(w)} \le C \|f\|_{BMO_\rho(w)},$$

*for every $f \in BMO_\rho(w)$.*

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*Proof.* Let $f \in BMO_\rho(w)$. We start proving condition (8) for $M_\rho f$. For $B \in \mathcal{B}_\rho$, with $B = B(x_0, r)$, as it is well known it shall be enough to see

$$\frac{1}{w(B)} \int_B |M_\rho f(x) - c|dx \leq C\|f\|_{BMO_\rho(w)},$$

for some constant $c$ that depends only on $f$ and $B$. Before start, observe that if $z \in B$ is given, and $P$ is a ball such that $z \in P$ and $P \in \mathcal{B}_\rho$, it follows from (1) that $P \subset \tilde{B} = B(x_0, c_0\rho(x_0))$, with $c_0 = 1 + c_\rho^2 2^{N_0+1+\frac{N_0}{N_0+1}}$. Therefore, for $x \in B$, we have

$$M_\rho f(x) = M_\rho(f\chi_{\tilde{B}})(x). \tag{15}$$

On the other hand, there exits a constant $\tilde{C}$ and a ball $Q_0 = B(y_0, \rho(y_0))$ of the covering given by Proposition 1, such that $x_0 \in Q_0$ and $\tilde{B} \subset \tilde{Q}_0 = \tilde{C}Q_0$, with $\tilde{C} = 1 + c_\rho 2^{\frac{N_0}{N_0+1}} c_0$.

Therefore, for $x \in B$,

$$M_\rho f(x) \leq M_{\tilde{Q}_0} f(x),$$

where the maximal operator $M_{\tilde{Q}_0}$ is defined as

$$M_{\tilde{Q}_0} f(x) = \sup_{x \in B \subset \tilde{Q}} \frac{1}{|B|} \int_B |f|.$$

Thus, for any constant $c$,

$$\begin{aligned}\frac{1}{w(B)} \int_B |M_\rho f(x) - c|dx &\leq \frac{1}{w(B)} \int_B |M_\rho f(x) - M_{\tilde{Q}_0} f(x)|dx \\ &\quad + \frac{1}{w(B)} \int_B |M_{\tilde{Q}_0} f(x) - c|dx = I + II.\end{aligned}$$

Since for every $x \in B$ we have

$$M_\rho f(x) \leq M_{\tilde{Q}_0} f(x) \leq M_\rho f(x) + \tilde{M}_\rho f(x),$$

where

$$\tilde{M}_\rho f(x) = \sup_{\substack{x \in P \subset \tilde{Q}_0 \\ P \notin \mathcal{B}_\rho}} \frac{1}{|P|} \int_P |f(y)|\, dy,$$

then

$$I \leq \frac{1}{w(B)} \int_B \tilde{M}_\rho f(x)\, dx.$$

It is not difficult to deduce from (1) that if $P = B(x_P, r_P)$, such that $P \subset \tilde{Q}_0$ and $P \notin \mathcal{B}_\rho$, then $r_P \simeq \rho(y_0)$. In fact, $r_P \leq \tilde{C}\rho(y_0)$ and also, $r_P \geq \rho(x_P)$ and $\rho(x_P) \simeq \rho(y_0)$ (since $x_P \in \tilde{Q}_0 = B(y_0, \tilde{C}\rho(y_0))$). Therefore, for every $x \in B$ we obtain

$$\tilde{M}_\rho f(x) \leq C \frac{1}{|\tilde{Q}_0|} \int_{\tilde{Q}_0} |f(y)|\, dy \leq C\|f\|_{BMO_\rho(w)} \frac{w(\tilde{Q}_0)}{|\tilde{Q}_0|},$$

Thus,

$$I \leq C\|f\|_{BMO_\rho(w)} \frac{|B|}{w(B)} \frac{w(\tilde{Q}_0)}{|\tilde{Q}_0|}. \tag{16}$$

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As $w \in A_1^{\rho,\mathrm{loc}}$, from Lemma 2 and the fact $B \subset \tilde{Q}_0$ we have

$$w(\tilde{Q}_0) \leq C(w) \frac{|\tilde{Q}_0|}{|B|} w(B). \tag{17}$$

With (16) and (17) we can conclude that $I \lesssim \|f\|_{BMO_\rho(w)}$.

In order to deal with $II$, we will use the local boundedness of $M_{\tilde{Q}_0} f$ on $BMO(\tilde{Q}_0)$, a result that appears in [6] (see Theorem 2.3). Since $f \in BMO(w)$ we have $M_{\tilde{Q}_0} f < \infty$ almost everywhere. On the other hand, since $w \in A_1^{\rho,\mathrm{loc}}$ from Lemma 2, it follows $w \in A_1^{\tilde{C}\rho,\mathrm{loc}}$, and thus there exists a constant $C$ depending on $w$ such that

$$w(B) \leq C|B| \inf_B w,$$

whenever $B = B(x, r)$ with $r \leq \tilde{C}\rho(x)$. It is clear then that $w \in A_1(\tilde{Q}_0)$. Therefore, if we choose $c = (M_{\tilde{Q}_0} f)_B$, by using Theorem 2.3 in [6] applied to the cube $\tilde{Q}_0$, we obtain $II \leq C\|f\|_{BMO_{\tilde{Q}_0}(w)} \leq C\|f\|_{BMO_\rho(w)}$.

Now we are going to prove (9) for $M_\rho f$. From Proposition 4, it is enough to check the condition over a critical ball $B_0 = B(x_0, \rho(x_0))$ with $x_0 \in \mathbb{R}$.

Let, $f = f_1 + f_2$ where $f_1 = f\chi_{B_0^*}$ with $B_0^* = B(x_0, \alpha\rho(x_0))$ and $\alpha = 2^{2N_0} c_\rho^2 + 2$.

We first consider $M_\rho f_1$. By Hölder's inequality

$$\begin{aligned} \frac{1}{w(B_0)} \int_{B_0} |M_\rho f_1(x)|\, dx &= \frac{1}{w(B_0)} \int_{B_0} |M_\rho f_1(x)| w^{-1/2}(x)\, w^{1/2}(x)\, dx \\ &\leq \left( \frac{1}{w(B_0)} \int_{B_0} |M_\rho f_1(x)|^2 w^{-1}(x)\, dx \right)^{1/2}. \end{aligned} \tag{18}$$

Since $w \in A_1^{\rho,\mathrm{loc}} \subset A_2^{\rho,\mathrm{loc}}$ it follows $w^{-1} \in A_2^{\rho,\mathrm{loc}}$. From Theorem 1 in [4], we know that $M_\rho$ is bounded on $L^2(v)$ with $v = w^{-1}$. Therefore,

$$\begin{aligned} \frac{1}{w(B_0)} \int_{B_0} |M_\rho f_1(x)|\, dx &\lesssim \left( \frac{1}{w(B_0)} \int_{\mathbb{R}^d} |f_1(x)|^2 w^{-1}(x)\, dx \right)^{1/2} \\ &= \left( \frac{1}{w(B_0)} \int_{B_0^*} |f_1(x)|^2 w^{-1}(x)\, dx \right)^{1/2}. \end{aligned}$$

Since $|B_0^*| = \alpha^d |B_0|$, by Lemma 2 we have $w(B_0^*) \leq Cw(B_0)$ and then

$$\frac{1}{w(B_0)} \int_{B_0} |M_\rho f_1(x)|\, dx \lesssim \left( \frac{1}{w(B_0^*)} \int_{B_0^*} |f(x)|^2 w^{-1}(x)\, dx \right)^{1/2}.$$

In this way, considering that $B_0^* \notin \mathcal{B}_\rho$ and Lemma 4 it follows that the left hand side of (18) is bounded by a constant times $\|f\|_{BMO_\rho(w)}$.

Now, for $x \in B_0$ we will deal with $M_\rho f_2(x)$. It follows from the definition of $f_2$ that it is enough to take the supremum of the averages over those balls $B \in \mathcal{B}_\rho$ such that $x \in B$ and $B \cap (B_0^*)^c \neq \emptyset$. Let $B = B(x_B, r_B)$ one of those balls. From (1), it follows easily that $\rho(x_0) \simeq \rho(x) \simeq \rho(x_B)$. More precisely, we have $\rho(x_B) \leq 2^{2N_0} c_\rho^2 \rho(x_0)$.

Then,

$$|x_0 - x_B| \leq |x_0 - x| + |x - x_B| < \rho(x_0) + r_B \leq \rho(x_0) + \rho(x_B) \leq (2^{2N_0} c_\rho^2 + 1)\rho(x_0)$$

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On the other hand, since $B \cap (B_0^*)^c \neq \emptyset$, there exists a point $z$ such that $z \in B$ and $z \notin B_0^*$, then

$$r_B \geq |z - x_B| \geq |z - x_0| - |x_0 - x_B| \geq \alpha\rho(x_0) - (2^{2N_0}c_\rho^2 + 1)\rho(x_0) = \rho(x_0)$$

If we denote $B_0^{**} = 2B_0^*$, it is clear that $B_0^* \subset B_0^{**}$. Moreover, $B \subset B_0^{**}$. In fact, given $y \in B$, it follows

$$\begin{aligned}
|y - x_0| &\leq |y - x_B| + |x_B - x_0| \\
&\leq \rho(x_B) + (2^{2N_0}c_\rho^2 + 1)\rho(x_0) \\
&\leq (2^{2N_0+1}c_\rho^2 + 1)\rho(x_0) \\
&< 2\alpha\rho(x_0).
\end{aligned}$$

Therefore, for all $x \in B_0$ we have

$$\begin{aligned}
M_\rho f_2(x) &= \sup_{\substack{x \in B \in \mathcal{B}_\rho \\ B \cap (B_0^*)^c \neq \emptyset}} \frac{1}{|B|} \int_B |f(y)|\, dy \\
&\leq \frac{C}{|B_0^{**}|} \int_{B_0^{**}} |f(y)|\, dy \qquad (19) \\
&\leq C\|f\|_{BMO_\rho(w)} \frac{w(B_0^{**})}{|B_0^{**}|}
\end{aligned}$$

From the bound of $M_\rho f_2(x)$, for every $x \in B_0$ given by (19) and Lemma 2 we get

$$\frac{1}{w(B_0)} \int_{B_0} |M_\rho f_2(x)|\, dx \leq C\|f\|_{BMO_\rho(w)} \frac{|B_0|}{w(B_0)} \frac{w(B_0^{**})}{|B_0^{**}|} \leq C\|f\|_{BMO_\rho(w)},$$

and this complete the proof.

□

## 5. The maximal operator of a family of operators

Let $\{T_t\}_{t>0}$ a family of bounded integral operators on $L^2(\mathbb{R}^d)$ with integrable kernels $\{T_t(x,y)\}_{t>0}$. Suppose also that there exist constants $C, \gamma, \gamma', \delta, \sigma, \sigma'$ y $\epsilon$ such that for all $t > 0$ and $x, x_0, y \in \mathbb{R}^d$ with $|x - x_0| \leq t/2$ it holds

$$|T_t(x,y)| \leq C \frac{1}{t^d + |x-y|^d} \left(\frac{t}{t + |x-y|}\right)^\gamma \left(\frac{\rho(x)}{t + \rho(x)}\right)^\sigma, \tag{20}$$

(21)

$$|T_t(x,y) - T_t(x_0,y)| \;\leq\; \frac{C}{t^d + |x-y|^d} \left(\frac{t}{t + |x-y|}\right)^{\gamma'} \left(\frac{|x - x_0|}{t}\right)^\delta \left(\frac{\rho(x)}{t + \rho(x)}\right)^{\sigma'},$$

and

$$|1 - T_t(1)(x)| \;\leq\; C \left(\frac{t}{t + \rho(x)}\right)^\epsilon. \tag{22}$$

For that family of operators we define the maximal operator $T^* = \sup_{t>0} |T_t|$.

We present the following technical lemmas that will be used in the proof of Theorem 2.

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**Lemma 5.** *Let $B = B(x_0, r)$ with $r < \rho(x_0)$ and $f \in BMO_\rho^\beta(w)$ with $w \in A_p^{\rho,\theta}$, where $\beta > 0$, $p > 1$ and $\theta > 0$. Then*

$$|f_B| \le 2^\eta C \|f\|_{BMO_\beta^\rho(w)} \frac{w(B)}{|B|} |B|^{\beta/d} \left(\frac{\rho(x_0)}{r}\right)^{d(p-1)+\beta},$$

*where $\eta = p(d + 2\theta) + \beta + 1$ and $C$ is the constant appearing in (6).*

*Proof.* Let $f \in BMO_\rho^\beta(w)$ and $j_0 \in \mathbb{N}$ such that $2^{j_0-1}r < \rho(x) \le 2^{j_0}r$. Then

$$\begin{aligned}
|f_B| &\le \frac{1}{|B|}\int_B |f - f_B| + \sum_{j=1}^{j_0-1} |f_{2^{j-1}B} - f_{2^jB}| + |f_{2^{j_0-1}B}| \\
&\le \frac{1}{|B|}\int_B |f - f_B| + \sum_{j=1}^{j_0-1} \frac{2^d}{|2^jB|}\int_{2^jB} |f - f_{2^jB}| + \frac{2^d}{|2^{j_0}B|}\int_{2^{j_0}B} |f| \\
&\le \sum_{j=0}^{j_0-1} \frac{2^d}{|2^jB|}\int_{2^jB} |f - f_{2^jB}| + \frac{2^d}{|2^{j_0}B|}\int_{2^{j_0}B} |f| \\
&\le 2^d \|f\|_{BMO_\rho^\beta(w)} \sum_{j=0}^{j_0} \frac{w(2^jB)}{|2^jB|} |2^jB|^{\beta/d},
\end{aligned}$$

where in the last inequality we have used (8) and (9) since $2^{j_0-1}r < \rho(x) \le 2^{j_0}r$. From Lemma 3, we get

$$\begin{aligned}
|f_B| &\le 2^{d+2p\theta} C \|f\|_{BMO_\rho^\beta(w)} w(B)|B|^{\beta/d-1} \sum_{j=0}^{j_0} 2^{j(pd-d+\beta)} \\
&\le 2^{d+2p\theta+1} 2^{j_0(d(p-1)+\beta)} C \|f\|_{BMO_\rho^\beta(w)} w(B)|B|^{\beta/d-1} \\
&\le 2^{p(d+2\theta)+\beta+1} C \|f\|_{BMO_\beta^\rho(w)} \frac{w(B)}{|B|} |B|^{\beta/d} \left(\frac{\rho(x_0)}{r}\right)^{d(p-1)+\beta}.
\end{aligned}$$

□

**Lemma 6.** *Let $z \in \mathbb{R}^d$, $0 < r < R$ and $f \in BMO_\rho^\beta(w)$ with $w \in A_p^{\rho,\theta}$, where $p > 1$, $\beta > 0$ and $\theta > 0$. Then*

$$\int_{B(z,R)} |f - f_{B(z,r)}| \lesssim C \|f\|_{BMO_\beta^\rho(w)} w(B(z,r)) |B(z,r)|^{\beta/d} \left(\frac{R}{r}\right)^{pd+\beta} \left(1 + \frac{R}{\rho(z)}\right)^{p\theta},$$

*where $C$ is the constant appearing in (6).*

*Proof.* Let $j_0 \in \mathbb{N}$ such that $2^{j_0-1}r < R \le 2^{j_0}r$. For simplicity, let us denote by $B_t = B(z, t)$ for any $t > 0$.

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$$
\begin{aligned}
\int_{B_R} |f - f_{B_r}|
&\le \int_{B_R} |f - f_{B_R}| + |B_R| \sum_{j=0}^{j_0-1} |f_{B_{R/2^j}} - f_{B_{R/2^{j+1}}}| + |B_R||f_{B_{R/2^{j_0}}} - f_{B_r}| \\
&\le \int_{B_R} |f - f_{B_R}| + \sum_{j=0}^{j_0-1} 2^{d(j+1)} \int_{B_{R/2^j}} |f - f_{B_{R/2^j}}| + 2^{j_0 d} \int_{B_r} |f - f_{B_r}| \\
&\le 2 \sum_{j=0}^{j_0-1} 2^{d(j+1)} \int_{B_{R/2^j}} |f - f_{B_{R/2^j}}| + 2^{j_0 d} \int_{B_r} |f - f_{B_r}| \\
&\le 2\|f\|_{BMO_\rho^\beta(w)} \sum_{j=0}^{j_0-1} 2^{d(j+1)} |B_{R/2^j}|^{\beta/d} w(B_{R/2^j}) \\
&\qquad + 2^{j_0 d} \|f\|_{BMO_\rho^\beta(w)} |B_r|^{\beta/d} w(B_r).
\end{aligned}
$$

Again, applying Lemma 3, we obtain

$$
\begin{aligned}
\int_{B_R} |f - f_{B_r}|
&\le 2C\|f\|_{BMO_\rho^\beta(w)} \sum_{j=0}^{j_0-1} 2^{d(j+1)} |B_{R/2^j}|^{\beta/d} w(B_r) \left(\frac{R/2^j}{r}\right)^{dp} \left(1 + \frac{R/2^j}{\rho(z)}\right)^{p\theta} \\
&\qquad + 2^d \left(\frac{R}{r}\right)^d \|f\|_{BMO_\rho^\beta(w)} |B_r|^{\beta/d} w(B_r) \\
&\le C\|f\|_{BMO_\rho^\beta(w)} w(B_r) |B_r|^{\beta/d} \left(\frac{R}{r}\right)^{dp+\beta} \left(1 + \frac{R}{\rho(z)}\right)^{p\theta} \\
&\qquad \times \left(2^{(d+1)} \sum_{j=0}^{j_0} 2^{-j[d(p-1)+\beta]} + 2^d\right) \\
&\lesssim C\|f\|_{BMO_\rho^\beta(w)} w(B_r) |B_r|^{\beta/d} \left(\frac{R}{r}\right)^{dp+\beta} \left(1 + \frac{R}{\rho(z)}\right)^{p\theta}.
\end{aligned}
$$

□

Now we state the main result of this section.

**Theorem 2.** *Let* $w \in A_p^{\rho,\theta}$, $\beta \ge 0$ *and* $\{T_t\}_{t>0}$ *a family of operators satisfying* (20), (21) *and* (22) *with* $\sigma, \gamma, \gamma' \ge \beta + p\theta + d(p-1)$, $\sigma' > p\theta$ *and* $\frac{\epsilon\delta}{\delta+\epsilon} \ge d(p-1) + \beta$.
*Then, there exists a constant* $C$ *such that*

$$\|T^* f\|_{BMO_\rho^\beta(w)} \le C\|f\|_{BMO_\rho^\beta(w)},$$

*for every* $f \in BMO_\rho^\beta(w)$.

*Proof.* Let $f \in BMO_\rho^\beta(w)$. We start proving that condition (11) is satisfied by $T^* f$. To this end, we shall use the hypothesis on the exponents $\sigma \ge \beta + p\theta + d(p-1)$ and $\gamma > \beta + p\theta + d(p+1)$.

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If $B_0 = B(x_0, \rho(x_0))$ then

$$\int_{B_0} \sup_{t>0} |T_t f(x)| dx \leq \int_{B_0} \sup_{t \geq \rho(x)} |T_t f(x)| dx \; + \; \int_{B_0} \sup_{t<\rho(x)} |T_t f(x) - T_t^0 f(x)| dx$$
$$+ \; \int_{B_0} \sup_{t<\rho(x)} |T_t^0 f(x)| dx = \mathbf{I} + \mathbf{II} + \mathbf{III},$$

where, for $x \in \mathbb{R}$,

$$T_t^0 f(x) = \int_{|x-y|<\rho(x)} T_t(x,y) f(y) \; dy.$$

Let us start with **III**. If $x \in B_0$ and $0 < t < \rho(x)$, then

$$|T_t^0 f(x)| \leq \int_{|x-y|<t} |T_t(x,y)||f(y)| \; dy \; + \; \int_{t<|x-y|<\rho(x)} |T_t(x,y)||f(y)| \; dy. \tag{23}$$

From (20) and the definition of $M_\rho$ it follows easily that

$$\int_{|x-y|<t} |T_t(x,y)||f(y)| \; dy \; \lesssim \frac{1}{t^d} \int_{|x-y|<t} |f(y)| \; dy \lesssim M_\rho f(x).$$

For the second term of (23), if $k_0 \in \mathbb{N}_0$ is such that $2^{k_0} t \leq \rho(x) < 2^{k_0+1} t$ and we call $B_k = B(x, 2^k t)$, we get

$$\begin{aligned}
\int_{t<|x-y|<\rho(x)} & |T_t(x,y)||f(y)| \; dy \\
& \lesssim t^\gamma \int_{t<|x-y|<\rho(x)} \frac{|f(y)|}{|x-y|^{d+\gamma}} \; dy \\
& \lesssim t^\gamma \sum_{k=0}^{k_0-1} \int_{B_{k+1} \setminus B_k} \frac{|f(y)|}{|x-y|^{d+\gamma}} \; dy \\
& \qquad + \; t^\gamma \int_{2^{k_0} t<|x-y|<\rho(x)} \frac{|f(y)|}{|x-y|^{d+\gamma}} \; dy \\
& \lesssim \sum_{k=1}^{k_0} \frac{2^{-k\gamma}}{|B_k|} \int_{B_k} |f(y)| \; dy \\
& \qquad + \left(\frac{\rho(x)}{2^{k_0} t}\right)^d \frac{2^{-k_0 \gamma}}{|B(x,\rho(x))|} \int_{B(x,\rho(x))} |f(y)| \; dy \\
& \lesssim M_\rho f(x) \sum_{k=1}^{k_0} 2^{-k\gamma}.
\end{aligned}$$

In this way, we have $\sup_{t<\rho(x)} |T_t^0 f(x)|$ uniformly bounded in $B_0$ by a constant times $M_\rho f(x)$. Since $w \in A_p^{\rho,\theta}$ by Remark 3 also belongs to $A_p^{\rho,\mathrm{loc}}$. Now, if $1 < s < p'$, it follows $w \in A_{s'}^{\rho,\mathrm{loc}}$ and then the operator $M_\rho$ is bounded on $L^p(w^{1-s})$ (see Proposition 5). Therefore, if $\tilde{B}_0 = c_0 B_0$ con $c_0$ como en (15), from Hölder's

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inequality, Lemma 4 and Lemma 2, we get

$$\begin{aligned}\mathbf{III} &\lesssim \int_{B_0} M_\rho f(x)\,dx = \int_{B_0} M_\rho(f\chi_{\tilde B_0})(x)\,dx\\ &\le \left(\int_{B_0} M_\rho(f\chi_{\tilde B_0})^s w^{1-s}\,dx\right)^{1/s} w(B_0)^{1/s'}\\ &\lesssim w(\tilde B_0)|\tilde B_0|^{\beta/d}\|f\|_{BMO^\beta_\rho(w)}\\ &\lesssim w(B_0)|B_0|^{\beta/d}\|f\|_{BMO^\beta_\rho(w)}.\end{aligned}$$

Now, we deal with **I**. Consider $x \in B_0$ and $t \ge \rho(x)$. Then,

$$|T_t f(x)| \le \int_{|x-y|<t} |T_t(x,y)||f(y)|\,dy \;+\; \int_{|x-y|\ge t} |T_t(x,y)||f(y)|\,dy$$

Bearing in mind that $B(x,t) \notin \mathcal{B}_\rho$ and $B(x,\rho(x)) \subset B(x_0, c_1\rho(x_0))$ (with $c_1 = 1 + c_\rho 2^{\frac{N_0}{N_0+1}}$), from (20), Lemma 3, we have

$$\begin{aligned}&\int_{|x-y|<t} |T_t(x,y)||f(y)|\,dy\\ &\qquad\lesssim \left(\frac{\rho(x)}{t}\right)^\sigma \frac{1}{t^d}\int_{|x-y|<t}|f(y)|\,dy\\ &\qquad\lesssim \left(\frac{\rho(x)}{t}\right)^\sigma \frac{w(B(x,t))}{|B(x,t)|}|B(x,t)|^{\beta/d}\|f\|_{BMO^\beta_\rho(w)}\\ &\qquad\lesssim \left(\frac{\rho(x)}{t}\right)^{\sigma-\beta-p\theta-d(p-1)} w(B(x,\rho(x)))\,\rho(x)^{\beta-d}\|f\|_{BMO^\beta_\rho(w)}\\ &\qquad\lesssim w(B(x_0,c_1\rho(x_0)))\,\rho(x)^{\beta-d}\|f\|_{BMO^\beta_\rho(w)},\end{aligned}$$

where in the last inequality we have used $\sigma \ge \beta + p\theta + d(p-1)$.

Now, from Lemma 2 and the fact that $\rho(x) \simeq \rho(x_0)$ (see Remark 1), we obtain that the last expression is bounded by $w(B_0)|B_0|^{\beta/d-1}\|f\|_{BMO^\beta_\rho(w)}$.

On the other hand, if we denote $B_k = B(x, 2^k t)$, then $B_k \notin \mathcal{B}_\rho$, for any $k \in \mathbb{N}$. Hence, from (20) and the definition of $BMO^\beta_\rho(w)$ we obtain

$$\begin{aligned}&\int_{|x-y|\ge t} |T_t(x,y)||f(y)|\,dy\\ &\qquad\lesssim \left(\frac{\rho(x)}{t}\right)^\sigma \int_{|x-y|>t}\frac{1}{|x-y|^d}\left(\frac{t}{|x-y|}\right)^\gamma |f(y)|\,dy\\ &\qquad\lesssim \left(\frac{\rho(x)}{t}\right)^\sigma \sum_{k\ge1} t^\gamma \int_{|x-y|\simeq 2^k t}\frac{|f(y)|}{|x-y|^{d+\gamma}}\,dy\\ &\qquad\lesssim \left(\frac{\rho(x)}{t}\right)^\sigma \sum_{k\ge1}\frac{2^{-k\gamma}}{|B_k|}\int_{B_k}|f(y)|\,dy\\ &\qquad\lesssim \|f\|_{BMO^\beta_\rho(w)}\left(\frac{\rho(x)}{t}\right)^\sigma \sum_{k\ge1} 2^{-k\gamma} w(B_k)|B_k|^{\beta/d-1}.\end{aligned}\tag{24}$$

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Moreover, taking into account that $\rho(x_0) \le c_\rho 2^{N_0}\rho(x)$ and $2^k t \ge \rho(x)$ follows that $B_k \subset B(x_0, c_2 2^k t)$ with $c_2 = 1 + 2^{N_0} c_\rho$. Then, applying Lemma 3, we get

$$\begin{aligned}
&\left(\frac{\rho(x)}{t}\right)^{\sigma} \sum_{k\ge 1} 2^{-k\gamma} w(B_k)|B_k|^{\beta/d-1} \\
&\qquad \lesssim \left(\frac{\rho(x)}{t}\right)^{\sigma} \sum_{k\ge 1} 2^{-k\gamma} w(B(x_0, c_2 2^k t))|B_k|^{\beta/d-1} \\
&\qquad \lesssim \left(\frac{\rho(x)}{t}\right)^{\sigma} w(B_0) \sum_{k\ge 1} 2^{-k\gamma} \left(\frac{2^k t c_2}{\rho(x_0)}\right)^{dp} \left(1 + \frac{c_2 2^k t}{\rho(x_0)}\right)^{p\theta} (2^k t)^{\beta-d} \qquad (25)\\
&\qquad \lesssim \left(\frac{\rho(x_0)}{t}\right)^{\sigma-\beta-p\theta-d(p-1)} w(B_0)\rho(x_0)^{\beta-d} \sum_{k\ge 1} 2^{-k[\gamma-\beta-p\theta-d(p-1)]} \\
&\qquad \lesssim \frac{w(B_0)}{|B_0|}|B_0|^{\beta/d},
\end{aligned}$$

where in the last inequality we have use the hypothesis $\sigma \ge \beta + p\theta + d(p-1)$ and $\gamma > \beta + p\theta + d(p-1)$. Therefore, from (24) and (25) we get

$$\int_{|x-y|\ge t} |T_t(x,y)||f(y)|\, dy \lesssim \|f\|_{BMO^{\beta}_{\rho}(w)} \frac{w(B_0)}{|B_0|}|B_0|^{\beta/d},$$

and thus we have

$$\mathbf{I} \lesssim \int_{B_0} \sup_{t\ge \rho(x)} |T_t f(x)|\, dx \lesssim \|f\|_{BMO^{\beta}_{\rho}(w)} w(B_0)|B_0|^{\beta/d}.$$

In order to finish this part, let us see **II**. Observe that

$$\sup_{t<\rho(x)} |T_t f(x) - T^0_t f(x)| \le \sup_{t<\rho(x)} \int_{|x-y|>\rho(x)} |T_t(x,y)||f(y)|\, dy.$$

If $t < \rho(x)$ and $\tilde{B}_k = B(x, 2^k\rho(x))$, $k \in \mathbb{N}$, then from (20)

$$\begin{aligned}
\int_{|x-y|>\rho(x)} |T_t(x,y)||f(y)|\, dy &\lesssim \int_{|x-y|>\rho(x)} \frac{1}{|x-y|^d} \left(\frac{t}{|x-y|}\right)^{\gamma} |f(y)|\, dy \\
&\lesssim \sum_{k\ge 1} \int_{|x-y|\simeq 2\rho(x)} \frac{1}{|x-y|^d} \left(\frac{\rho(x)}{|x-y|}\right)^{\gamma} |f(y)|\, dy \\
&\lesssim \sum_{k\ge 1} \frac{2^{-k\gamma}}{|\tilde{B}_k|} \int_{\tilde{B}_k} |f(y)|\, dy \\
&\lesssim \|f\|_{BMO^{\beta}_{\rho}(w)} \sum_{k\ge 1} 2^{-k\gamma} w(\tilde{B}_k)|\tilde{B}_k|^{\beta/d-1}.
\end{aligned}$$

From here, we can proceed as in (24) and (25), replacing $B_k$ and $t$ by $\tilde{B}_k$ and $\rho(x)$, respectively. Therefore,

$$\int_{|x-y|>\rho(x)} |T_t(x,y)||f(y)|\, dy \lesssim \|f\|_{BMO^{\beta}_{\rho}(w)} \frac{w(B_0)}{|B_0|}|B_0|^{\beta/d},$$

whenever $\gamma > \beta + p\theta + d(p-1)$.

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Thus

$$\mathbf{II} = \int_{B_0} \sup_{t<\rho(x)} |T_t f(x) - T_t^0 f(x)| \ dx \lesssim \|f\|_{BMO_\rho^\beta(w)} w(B_0)|B_0|^{\beta/d},$$

and this finish the proof that $T^*f$ satisfies (11) and then condition (9) (see Proposition 4).

To estimate the oscillation of $T^*f$ we consider a ball $B = B(x_0, r)$ with $0 < r < \rho(x_0)$. We decompose $f = f_1+f_2+f_3$, where $f_1 = (f-f_B)\chi_{2B}$, $f_2 = (f-f_B)\chi_{(2B)^c}$ and $f_3 = f_B$ to deal with each one separately.

We start with $f_1$. In this case (it is enough) we will estimate the average $\sup_{t>0} |T_t f_1|$. For $x \in B$, we have

$$\sup_{t>0} |T_t f_1(x)| \leq \sup_{0<t<r} |T_t f_1(x)| + \sup_{t>r} |T_t f_1(x)|.$$

If $t < r$, since $f_1$ is supported on $2B$ and considering (20), it follows

$$\begin{aligned}
|T_t f_1(x)| &\leq \int_{|x_0-y|<2r} |T_t(x,y)||f_1(y)| \, dy \\
&\leq \int_{|x-y|<3r} |T_t(x,y)||f_1(y)| \, dy \\
&\lesssim \frac{1}{t^d} \int_{|x-y|<t} |f_1(y)| \, dy \ + \ t^\gamma \int_{t\leq|x-y|<3r} \frac{1}{|x-y|^{d+\gamma}} |f_1(y)| \, dy \\
&\lesssim M_{\rho'} f_1(x) \ + \ \sum_{j=1}^{j_0} 2^{-j\gamma} \frac{1}{(2^j t)^d} \int_{|x-y|<2^j t} |f_1(y)| \, dy,
\end{aligned}$$

where $\rho'(x) = 2^{N_0} c_\rho \rho(x)$ (see inequality (1)) and $j_0 \in \mathbb{N}$ is such that $2^{j_0-1}t < 3r \leq 2^{j_0}t$. In this way, since $x \in B$, we have $2^j t \leq 6\rho(x_0) \leq 6\,2^{N_0} c_\rho \rho(x)$, for all $0 < j \leq j_0$. Now, if we denote $\tilde{\rho}(x) = 6\,2^{N_0} c_\rho \rho(x)$, the second term is bounded by a constant times $M_{\tilde{\rho}} f_1(x)$. From the fact that $M_{\rho'} \leq M_{\tilde{\rho}}$ and applying Hölder's inequality with exponent $s > 1$, we obtain

$$\int_B \sup_{t<r} |T_t f_1(x)| \, dx \lesssim \int_B M_{\tilde{\rho}} f_1(x) \ dx \leq \left( \int_B M_{\tilde{\rho}} f_1(x)^s w^{1-s} \ dx \right)^{1/s} w(B)^{1/s'}.$$

As $A_p^\rho \subset A_p^{\rho,\mathrm{loc}}$, implies $w^{1-p'} \in A_{p'}^{\rho,\mathrm{loc}} = A_{p'}^{\tilde{\rho},\mathrm{loc}}$, setting $s = p'$ in the last expression and using the Proposition 5 we have

$$\begin{aligned}
\int_B \sup_{t<r} |T_t f_1(x)| \, dx &\lesssim \left( \int_{2B} |f_1(x)|^{p'} w^{1-p'} \ dx \right)^{1/p'} w(B)^{1/p} \\
&= \left( \int_{2B} |f(x) - f_B|^{p'} w^{1-p'} \ dx \right)^{1/p'} w(B)^{1/p} \\
&\lesssim \left( \int_{2B} |f(x) - f_{2B}|^{p'} w^{1-p'} \ dx \right)^{1/p'} w(B)^{1/p} \\
&\quad + |f_{2B} - f_B| \left( \int_{2B} w^{1-p'} \ dx \right)^{1/p'} w(B)^{1/p}.
\end{aligned} \tag{26}$$

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For the first term we use the Lemma 4 (having in mind that $2B \in \mathcal{B}_{\tilde{\rho}}$ and $BMO_{\rho}^{\beta}(w) = BMO_{\tilde{\rho}}^{\beta}(w)$) and we obtain

$$\left(\int_{2B} |f(x) - f_{2B}|^{p'} w^{1-p'} \, dx\right)^{1/p'} w(B)^{1/p} \lesssim \|f\|_{BMO_{\beta}^{\rho}(w)} \, w(B) \, |B|^{\beta/d}. \tag{27}$$

On the other hand, from the fact that $|f_{2B} - f_B|$ is bounded by a constant times $\|f\|_{BMO_{\tilde{\rho}}^{\beta}(w)} \, w(2B) \, |2B|^{\beta/d-1}$ and the condition $A_p^{\tilde{\rho},\mathrm{loc}}$, we have

(28)

$$\begin{aligned}
|f_{2B} - f_B| \left(\int_{2B} w^{1-p'} \, dx\right)^{1/p'} w(B)^{1/p} &\lesssim \|f\|_{BMO_{\tilde{\rho}}^{\beta}(w)} \, w(2B) \, |2B|^{\beta/d-1} |2B| \\
&\lesssim \|f\|_{BMO_{\tilde{\rho}}^{\beta}(w)} \, w(2B) \, |2B|^{\beta/d} \\
&\lesssim \|f\|_{BMO_{\rho}^{\beta}(w)} \, w(B) \, |B|^{\beta/d},
\end{aligned}$$

where in the last inequality we have use 10 and Lemma 2.

Suppose now $t > r$. From the definition of the space $BMO_{\tilde{\rho}}^{\beta}(w)$, using (20), and Lemma 2, it follows

$$\begin{aligned}
|T_t f_1(x)| &\leq \int_{|x_0 - y| < 2r} |T_t(x, y)| |f_1(y)| \, dy \\
&\leq \frac{1}{t^d} \int_{2B} |f_1(y)| \, dy \\
&\lesssim \frac{1}{r^d} \int_{2B} |f(y) - f_B| \, dy \\
&\lesssim \frac{1}{r^d} \|f\|_{BMO_{\tilde{\rho}}^{\beta}(w)} w(2B) |2B|^{\beta/d} \\
&\lesssim \frac{1}{|B|} \|f\|_{BMO_{\rho}^{\beta}(w)} w(B) |B|^{\beta/d}.
\end{aligned} \tag{29}$$

From (26), (27), (28) and (29) we conclude

$$\int_B \sup_{t>0} |T_t f_1(x)| \, dx \lesssim \|f\|_{BMO_{\beta}^{\rho}(w)} \, w(B) \, |B|^{\beta/d}.$$

To deal with the term with $f_2$, if $c_B = T^* f_2(x_0)$, then

$$\int_B |T^* f_2(x) - T^* f_2(x_0)| dx \leq \int_B \sup_{t>0} |T_t f_2(x) - T_t f_2(x_0)| dx. \tag{30}$$

Now, for $x \in B$ and $t > 0$ we have

$$|T_t f_2(x) - T_t f_2(x_0)| \leq \int_{|x_0 - y| > 2r} |T_t(x, y) - T_t(x_0, y)| |f_2(y)| \, dy. \tag{31}$$

Suppose first that $r \leq t/2$. In this case, we have $|x - x_0| < t/2$. We now divide the integral (31) in two parts $|x - y| < t$ and $|x - y| > t$. When $|x - y| < t$, we denote by $k_1$ to the first integer such that $2^{k_1 - 1} r \leq 2t < 2^{k_1} r$. Having in mind

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condition (21), and $\rho(x) \simeq \rho(x_0)$ we obtain

$$\begin{aligned}
&\int_{\substack{|x_0-y|>2r\\|x-y|<t}} |T_t(x,y)-T_t(x_0,y)||f_2(y)|\,dy\\
&\qquad\lesssim \left(1+\frac{t}{\rho(x_0)}\right)^{-\sigma'}\left(\frac{r}{t}\right)^{\delta}\frac{1}{t^d}\int_{2r<|x_0-y|<2t}|f_2(y)|\,dy\\
&\qquad\lesssim \left(1+\frac{t}{\rho(x_0)}\right)^{-\sigma'}\left(\frac{r}{t}\right)^{\delta}\frac{1}{t^d}\sum_{k=2}^{k_1}\int_{2^kB}|f(y)-f_B|\,dy.
\end{aligned}$$

Applying Lemma 6, since $2^{k_1}r \simeq t$ we have

$$\begin{aligned}
\sum_{k=2}^{k_1}\int_{2^kB}|f(y)-f_B|\,dy &\lesssim \|f\|_{BMO^\beta_\rho(w)}w(B)r^\beta\sum_{k=2}^{k_1}2^{k(pd+\beta)}\left(1+\frac{2^kr}{\rho(x_0)}\right)^{p\theta}\\
&\lesssim \|f\|_{BMO^\beta_\rho(w)}w(B)r^\beta\left(1+\frac{2^{k_1}r}{\rho(x_0)}\right)^{p\theta}\sum_{k=1}^{k_1}2^{k(pd+\beta)}\\
&\lesssim \|f\|_{BMO^\beta_\rho(w)}w(B)r^\beta\left(1+\frac{t}{\rho(x_0)}\right)^{p\theta}\left(\frac{t}{r}\right)^{pd+\beta}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\substack{|x_0-y|>2r\\|x-y|<t}} |T_t(x,y)-T_t(x_0,y)||f_2(y)|\,dy\\
&\qquad\lesssim \left(1+\frac{t}{\rho(x_0)}\right)^{-\sigma'+p\theta}\left(\frac{r}{t}\right)^{\delta-d(p-1)-\beta}\frac{w(B)}{|B|}r^\beta\|f\|_{BMO^\beta_\rho(w)}\\
&\qquad\lesssim \frac{w(B)}{|B|}r^\beta\|f\|_{BMO^\beta_\rho(w)},
\end{aligned}\tag{32}$$

whenever $\sigma' > p\theta$ and $\delta \geq d(p-1)+\beta$.

In the part $|x-y|>t$, we use again estimate (21) to get

$$\begin{aligned}
&\int_{\substack{|x_0-y|>2r\\|x-y|>t}} |T_t(x,y)-T_t(x_0,y)||f_2(y)|\,dy\\
&\qquad\lesssim \left(1+\frac{t}{\rho(x_0)}\right)^{-\sigma'}\left(\frac{r}{t}\right)^{\delta}t^{\gamma'}\int_{|x-y|>t}\frac{|f_2(y)|}{|x-y|^{d+\gamma'}}\,dy\\
&\qquad\lesssim \left(1+\frac{t}{\rho(x_0)}\right)^{-\sigma'}\left(\frac{r}{t}\right)^{\delta}\frac{1}{t^d}\sum_{k\geq1}2^{-k(d+\gamma')}\int_{B(x,2^kt)}|f(y)-f_B|\,dy\\
&\qquad\lesssim \left(1+\frac{t}{\rho(x_0)}\right)^{-\sigma'}\left(\frac{r}{t}\right)^{\delta}\frac{1}{t^d}\sum_{k\geq1}2^{-k(d+\gamma')}\int_{2^kB}|f(y)-f_B|\,dy.
\end{aligned}\tag{33}$$

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Applying Lemma 6 with $R = 2^k t$, the sum in the last expression can be bounded by a constant times

$$\begin{aligned}
&\|f\|_{BMO^{\beta}_{\rho}(w)} w(B) r^{\beta} \sum_{k\geq 0} 2^{-k(d+\gamma')} \left(\frac{2^k t}{r}\right)^{pd+\beta} \left(1+\frac{2^k t}{\rho(x_0)}\right)^{p\theta} \\
&\quad \lesssim \|f\|_{BMO^{\beta}_{\rho}(w)} w(B) r^{\beta} \left(\frac{t}{r}\right)^{pd+\beta} \left(1+\frac{t}{\rho(x_0)}\right)^{p\theta} \left(\sum_{k\geq 0} 2^{-k(d+\gamma'-pd-\beta-p\theta)}\right) \\
&\quad \lesssim \|f\|_{BMO^{\beta}_{\rho}(w)} w(B) r^{\beta} \left(1+\frac{t}{\rho(x_0)}\right)^{p\theta} \left(\frac{t}{r}\right)^{pd+\beta},
\end{aligned}$$

whenever $\gamma' > \beta + d(p-1) + p\theta$.

Coming back to (33) it follows as before that

$$\begin{aligned}
&\int_{\substack{|x_0-y|>2r \\ |x-y|>t}} |T_t(x,y) - T_t(x_0,y)||f_2(y)|\,dy \\
&\qquad \lesssim \left(1+\frac{t}{\rho(x_0)}\right)^{-\sigma'+p\theta} \left(\frac{r}{t}\right)^{\delta-d(p-1)-\beta} \frac{w(B)}{|B|} r^{\beta} \|f\|_{BMO^{\beta}_{\rho}(w)} \\
&\qquad \lesssim \frac{w(B)}{|B|} r^{\beta} \|f\|_{BMO^{\beta}_{\rho}(w)}.
\end{aligned} \tag{34}$$

whenever $\sigma' > p\theta$ and $\delta \geq d(p-1) + \beta$.

Let us see the case $r \geq t/2$. In this case, we estimate the difference by the sum as follows

$$\begin{aligned}
\int_{|x_0-y|>2r} |T_t(x,y) - T_t(x_0,y)| f_2(y)\,dy &\leq \int_{|x_0-y|>2r} |T_t(x,y)||f_2(y)|\,dy \\
&\quad + \int_{|x_0-y|>2r} |T_t(x_0,y)||f_2(y)|\,dy \\
&= \mathbf{A} + \mathbf{B}.
\end{aligned}$$

We only deal with $\mathbf{A}$. The term $\mathbf{B}$ can be estimate analogously.

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Since in the domain of integration we have $|x_0 - y| \geq 2r \geq t$ and also $|x - y| \geq |x_0 - y| - |x - x_0| > r \geq t/2$, using condition (20) and Lemma 6 we obtain

(35)

$$\begin{aligned}
\int_{|x_0-y|>2r} |T_t(x,y)||f_2(y)|\,dy
&\lesssim t^\gamma \int_{|x-y|>r} \frac{|f_2(y)|}{|x-y|^{d+\gamma}}\,dy \\
&\lesssim \frac{t^\gamma}{r^{d+\gamma}} \sum_{k\geq 1} 2^{-k(d+\gamma)} \int_{B(x,2^k r)} |f(y)-f_B|\,dy \\
&\lesssim \frac{1}{r^d} \sum_{k\geq 1} 2^{-k(d+\gamma)} \int_{2^k B} |f(y)-f_B|\,dy \\
&\lesssim \frac{w(B)}{r^d} r^\beta \|f\|_{BMO^\beta_\rho(w)} \sum_{k\geq 1} 2^{-k(d+\gamma-pd-\beta)} \left(1+\frac{2^k r}{\rho(x_0)}\right)^{p\theta} \\
&\lesssim \frac{w(B)}{r^d} r^\beta \|f\|_{BMO^\beta_\rho(w)} \sum_{k\geq 1} 2^{-k(d+\gamma-pd-\beta-p\theta)} \\
&\lesssim \frac{w(B)}{r^d} r^\beta \|f\|_{BMO^\beta_\rho(w)},
\end{aligned}$$

whenever $\gamma > \beta + d(p-1) + p\theta$.

Therefore, from (32), (34) and (35), we obtain for $x \in B$

$$\sup_{t>0} |T_t f_2(x) - T_t f_2(x_0)| \lesssim \frac{w(B)}{r^d} r^\beta \|f\|_{BMO^\rho_\beta}.$$

Hence, it follows

$$\int_B |T^* f_2(x) - T^* f_2(x_0)|dx \lesssim w(B) r^\beta \|f\|_{BMO^\rho_\beta},$$

and this finishes the term with $f_2$.

To deal with the term with $f_3$, we shall find a bound for $T^* f_3 = T^* f_B = |f_B|\, T^* 1$. We will estimate the oscillation of $T^* f_3$ over $B$ subtracting the constant $c_B = |f_B| T^* 1(x_0)$.

Observe that,

$$\int_B |T^* f_3(x) - T^* f_3(x_0)| \leq |f_B| \int_B \sup_{t>0} |T_t 1(x) - T_t 1(x_0)|\,dx,$$

and

$$|T_t 1(x) - T_t 1(x_0)| \leq \int_{\mathbb{R}^d} |T_t(x,y) - T_t(x_0,y)|\,dy.$$

As before, we consider separately the cases $t \geq 2r$ and $t < 2r$. We start assuming $t \geq 2r$ and then $|x - x_0| \leq t/2$, that allows us to use condition (21).

We also divide the domain as before as

$$\int_{\mathbb{R}^d} |T_t(x,y) - T_t(x_0,y)|\,dy \leq \int_{|x-y|<t} + \int_{|x-y|>t} = \mathbf{C} + \mathbf{D}.$$

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Thus condition (21), implies

$$\mathbf{C} \lesssim \left(1+\frac{t}{\rho(x_0)}\right)^{-\sigma'} \left(\frac{r}{t}\right)^{\delta} \frac{1}{t^d}\int_{|x-y|<t} dy \lesssim \left(\frac{r}{t}\right)^{\delta},$$

and also

$$\mathbf{D} \lesssim \left(1+\frac{t}{\rho(x_0)}\right)^{-\sigma'} \left(\frac{r}{t}\right)^{\delta} \int_{|x-y|>t} \frac{t^{\gamma'}}{|x-y|^{d+\gamma'}} dy \lesssim \left(\frac{r}{t}\right)^{\delta},$$

whenever $\sigma' > 0$.

On the other hand, considering the inequality

$$|T_t 1(x) - T_t 1(x_0)| \leq |T_t 1(x) - 1| \; + \; |T_t 1(x_0) - 1|, \tag{36}$$

from condition (22), it is clear that

$$|T_t 1(x) - 1| \lesssim \left(\frac{t}{t+\rho(x)}\right)^{\epsilon} \lesssim \left(\frac{t}{t+\rho(x_0)}\right)^{\epsilon} \leq \left(\frac{t}{\rho(x_0)}\right)^{\epsilon}, \tag{37}$$

where we have used the fact that $\rho(x) \simeq \rho(x_0)$. The same estimate is valid for the second term.

Therefore, we may bound a convex combination of the previous estimates to get

$$|T_t 1(x) - T_t 1(x_0)| \lesssim \left(\frac{r}{t}\right)^{\delta(1-a)} \left(\frac{t}{\rho(x_0)}\right)^{\epsilon a}.$$

In this way, denoting $a = \delta/(\delta+\epsilon)$, we have $\epsilon a = \delta(1-a)$.

Then, for all $t \geq 2r$ and $x \in B$ we obtain

$$|T_t 1(x) - T_t 1(x_0)| \lesssim \left(\frac{r}{\rho(x_0)}\right)^{\epsilon a}. \tag{38}$$

In the case $t < 2r$, proceeding in the same way as in (36) and (37) follows

$$|T_t 1(x) - T_t 1(x_0)| \lesssim \left(\frac{r}{\rho(x_0)}\right)^{\epsilon}. \tag{39}$$

Having in mind that $a < 1$ and $r/\rho(x_0) \leq 1$ we obtain from (38) and (39) that

$$\sup_{t>0} |T_t 1(x) - T_t 1(x_0)| \lesssim \left(\frac{r}{\rho(x_0)}\right)^{\epsilon a}.$$

Finally, from Lemma 5 it follows

$$\begin{aligned}
\int_B |T^* f_3(x) - T^* f_3(x_0)| &\leq |f_B| \int_B \sup_{t>0} |T_t 1(x) - T_t 1(x_0)|\, dx \\
&\lesssim \|f\|_{BMO_\rho^\beta(w)} w(B) r^\beta \left(\frac{\rho(x_0)}{r}\right)^{d(p-1)+\beta} \left(\frac{r}{\rho(x_0)}\right)^{\epsilon a} \\
&\lesssim \|f\|_{BMO_\rho^\beta(w)} w(B) r^\beta,
\end{aligned}$$

whenever $\epsilon \frac{\delta}{\delta+\epsilon} \geq d(p-1)+\beta$.

An this finishes the proof of the theorem.

□

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## 6. Applications

In this section we consider a Schrödinger operator in $\mathbb{R}^d$ with $d \geq 3$,

$$\mathcal{L} = -\triangle + V,$$

where $V \geq 0$, not identically zero, is a function that satisfies for $q > d/2$, the reverse Hölder inequality

$$\left( \frac{1}{|B|} \int_B V(y)^q \, dy \right)^{1/q} \leq \frac{C}{|B|} \int_B V(y) \, dy, \tag{40}$$

for every ball $B \subset \mathbb{R}^d$. The set of functions with the last property is usually denoted by $RH_q$.

For a given potential $V \in RH_q$, with $q > d/2$, as in [14], we consider the auxiliary function $\rho$ defined for $x \in \mathbb{R}^d$ as

$$\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V \leq 1 \right\}.$$

Under the above conditions on $V$ we have $0 < \rho(x) < \infty$. Furthermore, according to [14, Lemma 1.4], if $V \in RH_{q/2}$ the associated function $\rho$ verifies (1).

Let $k_t$ be the kernel of $e^{-t\mathcal{L}}$, $t > 0$, where $\{e^{-t\mathcal{L}}\}_{t>0}$ is called the heat semigroup associated to $\mathcal{L}$. There are known (see [13] and [10]) the following estimates for $k_t$.

**Proposition 6.** *Let $V \in RH_q$, $q > d/2$, $N > 0$ and $0 < \lambda < \min\left\{1, 2 - \frac{d}{q}\right\}$. Then there exist positive constants $C$, $\tilde{C}$ and $C_N$ such that for all $t > 0$ and $x, y, x_0 \in \mathbb{R}^d$ with $|x - x_0| < \sqrt{t}$ we have,*

$$|k_t(x,y)| \leq C_N \, t^{-d/2} \, e^{-\frac{|x-y|^2}{C\,t}} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}, \tag{41}$$

(42)
$$|k_t(x,y) - k_t(x_0,y)| \leq C_N \left( \frac{|x - x_0|}{\sqrt{t}} \right)^{\lambda} t^{-d/2} \, e^{-\frac{|x-y|^2}{C\,t}} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N},$$

*and*

$$|k_t(x,y) - \tilde{k}_t(x,y)| \leq \tilde{C} t^{-d/2} \, e^{-\frac{|x-y|^2}{C\,t}} \left( 1 + \frac{\rho(x)}{\sqrt{t}} \right)^{\frac{d}{q} - 2}, \tag{43}$$

*where $\tilde{k}_t$ denotes the kernel of $e^{-t\Delta}$, $t > 0$.*

**Theorem 3.** *Let $V \in RH_q$ for some $q > d/2$, $\epsilon = 2 - \frac{d}{q}$, $0 < \delta < \min\{1, \epsilon\}$ and $w \in A_p^{\rho,\theta?}$. If $1 < p < 1 + \frac{\kappa}{d}$ and $0 \leq \beta < \kappa - d(p-1)$, with $\kappa = \frac{\epsilon\delta}{\epsilon+\delta}$, then there exists a constant $C$ such that*

$$\|T^* f\|_{BMO_\rho^\beta(w)} \;\leq\; C \|f\|_{BMO_\rho^\beta(w)},$$

*for every $f \in BMO_\rho^\beta(w)$.*

*Proof.* It is enough to prove that the family $\{e^{-t^2\mathcal{L}}\}_{t>0}$ satisfies the hypothesis of Theorem 2. Let us start proving that from (41) we can get (20). In fact, given $C > 0$ and $M > 0$, there exists $C_M > 0$ such that

$$e^{-\frac{|x-y|^2}{C\,t^2}} \leq C_M \left( \frac{t^2}{t^2 + |x-y|^2} \right)^M \leq 4^M C_M \left( \frac{t}{t + |x-y|} \right)^{2M}.$$

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Therefore, if we choose $M > d/2$, from (41) with $t^2$ instead of $t$, we have

$$\begin{aligned} k_{t^2}(x,y) &\lesssim t^{-d}\left(\frac{t}{t+|x-y|}\right)^{2M}\left(\frac{\rho(x)}{t+\rho(x)}\right)^{N} \\ &\lesssim \frac{1}{t^d+|x-y|^d}\left(\frac{t}{t+|x-y|}\right)^{2M-d}\left(\frac{\rho(x)}{t+\rho(x)}\right)^{N}, \end{aligned}$$

which is (20) with $\gamma = 2M - d$ and $\sigma = N$.

In the same way we can obtain (21) from (42) with $\gamma' = 2M - d$, $\sigma' = N$ and $\delta = \lambda$.

Now we will see that (43) implies (22) with $\epsilon = 2 - d/q$. It is known (see [11] or [10] for example), that $\tilde{k}_{t^2}(1) = 1$ for every $t > 0$, and thus

$$\begin{aligned} |1 - k_{t^2}(1)(x)| &\le |1 - \tilde{k}_{t^2}(1)(x)| + |\tilde{k}_{t^2}(1)(x) - k_{t^2}(1)(x)| \\ &= |\tilde{k}_{t^2}(1)(x) - k_{t^2}(1)(x)|. \end{aligned}$$

Therefore from (43) we obtain

$$\begin{aligned} |\tilde{k}_{t^2}(1)(x) - k_{t^2}(1)(x)| &\le \int_{\mathbb{R}^d} |k_{t^2}(x,y) - \tilde{k}_{t^2}(x,y)|\, dy \\ &\lesssim \int_{\mathbb{R}^d} t^{-d} e^{-\frac{|x-y|^2}{C\,t^2}} \left(1 + \frac{\rho(x)}{t}\right)^{\frac{d}{q}-2} dy \\ &\lesssim t^{-d}\left(\frac{t}{t+\rho(x)}\right)^{2-\frac{d}{q}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{C\,t^2}}\, dy \\ &\lesssim \left(\frac{t}{t+\rho(x)}\right)^{\epsilon}. \end{aligned}$$

□

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