ISSN 2451-7100

IMAL preprints

http://www.imal.santafe-conicet.gov.ar/publicaciones/preprints/index.php

ON THE CALDERÓN-ZYGMUND STRUCTURE OF PETERMICHL'S KERNEL. WEIGHTED INEQUALITIES

By

Hugo Aimar and Ivana Gómez

IMAL PREPRINT # 2017-0041

Publication date: October 2, 2017

Editorial: Instituto de Matemática Aplicada del Litoral
IMAL (CCT CONICET Santa Fe – UNL)
http://www.imal.santafe-conicet.gov.ar

Director de Publicaciones: Dr. Rubén Spies
E-mail: rspies@santafe-conicet.gov.ar

IMAL PREPRINT # 2017-0041

ISSN 2451-7100
Publication date: October 2, 2017

ON THE CALDERÓN-ZYGMUND STRUCTURE OF PETERMICHL'S KERNEL. WEIGHTED INEQUALITIES

HUGO AIMAR AND IVANA GÓMEZ

Abstract. We show that Petermichl's dyadic operator $\mathcal{P}$ (S. Petermichl (2000), *Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol*) is a Calderón-Zygmund type operator on an adequate metric normal space of homogeneous type. As a consequence of a general result on spaces of homogeneous type, we get weighted boundedness of the maximal operator $\mathcal{P}^*$ of truncations of the singular integral. We show that dyadic A_p weights are the good weights for the maximal operator $\mathcal{P}^*$ of the scale truncations of $\mathcal{P}$.

1. Introduction

In [9], Stefanie Petermichl proves a remarkable identity that provides the Hilbert kernel $\frac{1}{x-y}$ in $\mathbb{R}$ as a mean value of dilations and translations of a basic kernel defined in terms of dyadic families on $\mathbb{R}$. The basic kernel for a fixed dyadic system $\mathcal{D}$ is described in terms of Haar wavelets. Assume that $\mathcal{D}$ is the standard dyadic family on $\mathbb{R}$, i.e. $\mathcal{D} = \cup_{j\in\mathbb{Z}}\mathcal{D}^j$ with $\mathcal{D}^j = \{I^j_k : k \in \mathbb{Z}\}$ and $I^j_k = [\frac{k}{2^j}, \frac{k+1}{2^j})$. Let $\mathscr{H}$ be the standard Haar system built on the dyadic intervals in $\mathcal{D}$. There is a natural bijection between $\mathscr{H}$ and $\mathcal{D}$. We shall use $\mathcal{D}$ as the index set and we shall write h_I to denote the function $h_I(x) = |I|^{-1/2}(\mathcal{X}_{I^-}(x) - \mathcal{X}_{I^+}(x))$ where I^- and I^+ are the respective left and right halves of I, $\mathcal{X}_E$ is, as usual, the indicator function of E and $|E|$ denote the Lebesgue measure of the measurable set E. With the above notation, the basic Petermichl's operator on $L^2(\mathbb{R})$ is given by

$$\mathcal{P}f(x) = \sum_{I\in\mathcal{D}} \langle f, h_I\rangle (h_{I^-}(x) - h_{I^+}(x)), \tag{1.1}$$

where, as usual, $\langle f, h_I\rangle = \int_{\mathbb{R}} f(y)h_I(y)dy$. Hence, at least formally, the operator $\mathcal{P}$ is defined by the nonconvolution nonsymmetric kernel

$$\begin{aligned} P(x,y) &= \sum_{h\in\mathcal{D}} h_I(y)(h_{I^-}(x) - h_{I^+}(x)) \\ &= P^+(x,y) + P^-(x,y); \end{aligned}$$

with

$$P^+(x,y) = \sum_{I\in\mathcal{D}^+} h_I(y)(h_{I^-}(x) - h_{I^+}(x)) \tag{1.2}$$

and $\mathcal{D}^+ = \{I^j_k \in \mathcal{D} : k \geq 0\}$.

Let us observe that for $x \geq 0$, $y \geq 0$ and $x \neq y$ the series $\sum_{I\in\mathcal{D}^+} h_I(y)[h_{I^-}(x) - h_{I^+}(x)]$ is absolute convergent. In fact

$$\sum_{I\in\mathcal{D}^+} |h_I(y)|\,|h_{I^-}(x) - h_{I^+}(x)| = \sum_{I\in\mathcal{D}^+, I\supseteq I(x,y)} \frac{1}{\sqrt{|I|}} |h_{I^-}(x) - h_{I^+}(x)|$$

2010 *Mathematics Subject Classification.* Primary 42B20, 42B25; Secondary 42C40.

This work was supported by the CONICET (grant PIP-112-2011010-0877, 2012); ANPCyT-MINCyT (grants PICT-2568,2012; PICT-3631,2015); and UNL (grant CAID-50120110100371LI,2013).

IMAL PREPRINT # 2017-0041 ISSN 2451-7100
Publication date: October 2, 2017

$$\leq \sum_{I\in\mathcal{D}^+, I\supseteq I(x,y)} \frac{2\sqrt{2}}{|I|} = \frac{4\sqrt{2}}{|I(x,y)|}$$

where $I(x,y)$ is the smallest dyadic interval in $\mathbb{R}$ containing x and y.

The aim of this paper is twofold. First we show that $\mathcal{P}^+$ (and $\mathcal{P}^-$) the operator induced by the kernel P^+ (resp. P^-) is of Calderón–Zygmund type in the normal space of homogeneous type $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) with the dyadic ultrametric $\delta(x,y) = \inf\{|I| : x,y \in I \text{ and } I \in \mathcal{D}\}$ and Lebesgue measure. Second, by an application of the known weighted norm inequalities for singular integrals in normal spaces of homogeneous type, we show that the operator $\mathcal{P}^* f(x) = \sup_{\{l,m\in\mathbb{Z}\}} \left|\sum_{\{I\in\mathcal{D}^+, 2^l\leq|I|<2^m\}} \langle f, h_I\rangle (h_{I^-}(x) - h_{I^+}(x))\right|$ is bounded on $L^p(\mathbb{R}^+, wdx)$ if and only if $w \in A_p^{dy}(\mathbb{R}^+)$ when $1 < p < \infty$.

In §2 we prove that $\mathcal{P}^+$ is of Calderón–Zygmund in an adequate space of homogeneous type. In Section 3 we give the characterization of the dyadic weights as those for which the maximal operator of the scale truncations of $\mathcal{P}^+$ is bounded in $L^p(\mathbb{R}^+, wdx)$ for $1 < p < \infty$.

2. Petermichl's operator as a Calderón–Zygmund operator

Following [8], a linear and continuous operator $T : \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$, with $\mathscr{D}$ and $\mathscr{D}'$ the test functions and the distributions on $\mathbb{R}^n$, is a Calderón-Zygmund operator if there exists $K \in L^1_{loc}(\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta)$ where Δ is the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$ such that

(1) there exists $C_0 > 0$ with

$$|K(x,y)| \leq \frac{C_0}{|x-y|^n}, \quad x \neq y;$$

(2) there exist C_1 and $\gamma > 0$ such that

(2.a) $|K(x',y) - K(x,y)| \leq C_1 \dfrac{|x'-x|^\gamma}{|x-y|^{n+\gamma}}$ when $2|x'-x| \leq |x-y|$;

(2.b) $|K(x,y') - K(x,y)| \leq C_1 \dfrac{|y'-y|^\gamma}{|x-y|^{n+\gamma}}$ when $2|y'-y| \leq |x-y|$;

(3) T extends to $L^2(\mathbb{R}^n)$ as a continuous linear operator;

(4) for φ and $\psi \in \mathscr{D}(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$ we have

$$\langle T\varphi, \psi\rangle = \iint_{\mathbb{R}^n\times\mathbb{R}^n} K(x,y)\varphi(x)\psi(y)dxdy.$$

With a little effort the notions of Calderón-Zygmund operator and Calderón-Zygmund kernel K (i.e. satisfying (1) and (2)) can be extended to normal metric spaces of homogeneous type. Even when the formulation can be stated in quasi-metric spaces for our application it shall be enough the following context. Let (X,d) be a metric space. If there exists a Borel measure μ on X such that for some constants $0 < \alpha \leq \beta < \infty$ such that the inequalities $\alpha r \leq \mu(B(x,r)) \leq \beta r$ hold for every $r > 0$ and every $x \in X$, we shall say that (X,d,μ) is a normal space. As usual $B(x,r) = \{y \in X : d(x,y) < r\}$. In particular, (X,d,μ) is a space of homogeneous type in the sense of [4], [6], [5], [2], and many problems of harmonic analysis find there a natural place to be solved.

In this setting in [6] a fractional order inductive limit topology is given to the space of compactly supported Lipschitz γ functions ($0 < \gamma < 1$). We shall still write $\mathscr{D} = \mathscr{D}(X,d)$ to denote this test functions space. And $\mathscr{D}' = \mathscr{D}'(X,d)$ its dual, the space of distributions. So, the extension of the definition of Calderón-Zygmund operators to this setting becomes natural.

IMAL PREPRINT # 2017-0041
ISSN 2451-7100
Publication date: October 2, 2017

Definition 1. Let (X, d, μ) be a normal metric measure space such that continuous functions are dense in $L^1(X, \mu)$. We say that a linear and continuous operator $T : \mathscr{D} \to \mathscr{D}'$ is Calderón-Zygmund on (X, d, μ) if there exists $K \in L^1_{loc}(X \times X \setminus \Delta)$, where Δ is the diagonal in $X \times X$, such that

(i) there exists $C_0 > 0$ with

$$|K(x, y)| \leq \frac{C_0}{d(x, y)}, \quad x \neq y;$$

(ii) there exist $C_1 > 0$ and $\gamma > 0$ such that

(ii.a) $|K(x', y) - K(x, y)| \leq C_1 \dfrac{d(x', x)^\gamma}{d(x, y)^{1+\gamma}}$ when $2d(x', x) \leq d(x, y)$;

(ii.b) $|K(x, y') - K(x, y)| \leq C_1 \dfrac{d(y, y')^\gamma}{d(x, y)^{1+\gamma}}$ when $2d(y', y) \leq d(x, y)$;

(iii) T extends to $L^2(X, \mu)$ as a continuous linear operator;

(iv) for φ and $\psi \in \mathscr{D}$ with $d(\operatorname{supp}\varphi, \operatorname{supp}\psi) > 0$ we have

$$\langle T\varphi, \psi \rangle = \iint_{X \times X} K(x, y)\varphi(x)\psi(y) d(\mu \times \mu)(x, y).$$

Our first result shows that $\mathcal{P}^+$ and $\mathcal{P}^-$ are Calderón-Zygmund operators. In what follows we shall keep using P for P^+ and $\mathcal{P}$ for $\mathcal{P}^+$.

Theorem 2. *There exists a metric δ on $\mathbb{R}^+ = \{x : x \geq 0\}$ such that $(\mathbb{R}^+, \delta, |\cdot|)$ is a normal space where δ-continuous functions are dense in $L^1(\mathbb{R}^+, dx)$ and P can be written, for $x \neq y$ both in $\mathbb{R}^+$, as*

$$P(x, y) = \frac{\Omega(x, y)}{\delta(x, y)}, \tag{2.1}$$

where Ω is bounded and δ-smooth. Moreover, $\mathcal{P}$ is a Calderón-Zygmund operator on $(\mathbb{R}^+, \delta, |\cdot|)$.

Proof. For $x \neq y$ two points in $\mathbb{R}^+$, define $\delta(x, y) = \inf\{|I| : x, y \in I \in \mathcal{D}\}$. Define also $\delta(x, x) = 0$ for every $x \in \mathbb{R}^+$. It is easy to see that δ is an ultra-metric on $\mathbb{R}^+$. This means that the triangle inequality improves to $\delta(x, z) \leq \sup\{\delta(x, y), \delta(y, z)\}$ for every x, y and $z \in \mathbb{R}^+$. Notice that $|x - y| \leq \delta(x, y)$ but they are certainly not equivalent. Also, for $x \in \mathbb{R}^+$ and $r > 0$ given, taking $m \in \mathbb{Z}$ such that $2^{-m} < r \leq 2^{-m+1}$ we see that $B_\delta(x, r) = \{y \in \mathbb{R}^+ : \delta(x, y) < r\} = \{y \in \mathbb{R}^+ : \delta(x, y) \leq 2^{-m}\} = I^m_{k(x)}$, where $k(x)$ is the only index $k \in \mathbb{N} \cup \{0\}$ such that $x \in I^m_k$. Hence the Lebesgue measure of $B_\delta(x, r)$ is that of the interval $I^m_{k(x)}$. Precisely, $|B_\delta(x, r)| = 2^{-m}$. So that $\frac{r}{2} \leq |B_\delta(x, r)| < r$, for every $x \in \mathbb{R}^+$ and every $r > 0$. In terms of our above definitions $(\mathbb{R}^+, \delta, |\cdot|)$ is a normal metric space. The integrability properties of powers of δ resemble completely those, of the powers of x. In fact, for fixed $x \in \mathbb{R}^+$, the function of $y \in \mathbb{R}^+$ given by $1/\delta^\alpha(x, y)$ is integrable inside a δ-ball when $\alpha < 1$. It is integrable outside a δ-ball when $\alpha > 1$. In particular, $1/\delta(x, y)$ is neither locally nor globally integrable on $\mathbb{R}^+$.

Notice now that real valued simple functions built on the dyadic intervals are continuous as functions defined on $(\mathbb{R}^+, \delta)$. In fact, for $I \in \mathcal{D}$ we have that $|\mathcal{X}_I(x) - \mathcal{X}_I(y)|$ equals zero for x and y in I or for x and y outside I. Assume that $x \in I$ and $y \notin I$, then $\delta(x, y) \geq 2|I|$. So that $|\mathcal{X}_I(x) - \mathcal{X}_I(y)| \leq \delta(x, y)(2|I|)^{-1}$ for every x and $y \in \mathbb{R}^+$. In other words, for $I \in \mathcal{D}$, $\mathcal{X}_I$ is Lipschitz with respect to δ with constant $(2|I|)^{-1}$. Hence δ-continuous functions are dense in $L^1(\mathbb{R}^+, dx)$.

The operator $\mathcal{P}$ is actually defined as an operator in $L^2(\mathbb{R}^+, dx)$. For $f \in L^2(\mathbb{R}^+, dx)$,

$$\mathcal{P}f(x) = \sum_{I \in \mathcal{D}^+} \langle f, h_I \rangle (h_{I^-}(x) - h_{I^+}(x))$$

ISSN 2451-7100
IMAL PREPRINT # 2017-0041 Publication date: October 2, 2017

$$= \sum_{I\in\mathcal{D}^+} \langle f, h_I\rangle\, h_{I^-}(x) - \sum_{I\in\mathcal{D}^+} \langle f, h_I\rangle\, h_{I^+}(x).$$

Hence $\|\mathcal{P}f\|_2^2 \leq 2\sum_{I\in\mathcal{D}^+} |\langle f, h_I\rangle|^2 = 2\|f\|_2^2$, which proves (iii) in Definition 1. In particular, if φ is a simple function built on the dyadic intervals, we see that $\mathcal{P}\varphi \in L^2(\mathbb{R}^+, dx)$. So that when ψ is another simple function such that $\delta(\operatorname{supp}\varphi, \operatorname{supp}\psi) > 0$, the two variables function $F(x,y) = \varphi(x)\psi(y)$ is simple in $\mathbb{R}^+\times\mathbb{R}^+$ and for some $\varepsilon > 0$, $\operatorname{supp} F \cap \{\delta < \varepsilon\} = \emptyset$, we have that, since only a finite subset of $\mathcal{D}^+$ is actually involved,

$$\begin{aligned}\iint_{\mathbb{R}^+\times\mathbb{R}^+} &\left(\sum_{I\in\mathcal{D}^+} h_I(y)[h_{I^-}(x) - h_{I^+}(x)]\right)\varphi(y)\psi(x)dydx \\ &= \int_{x\in\mathbb{R}^+}\left(\int_{y\in\mathbb{R}^+} P(x,y)\varphi(y)\right)\psi(x)dx \\ &= \int_{x\in\mathbb{R}^+} \mathcal{P}\varphi(x)\psi(x)dx \\ &= \langle \mathcal{P}\varphi, \psi\rangle .\end{aligned}$$

Hence $P(x,y) = \sum_{I\in\mathcal{D}^+} h_I(y)[h_{I^-}(x) - h_{I^+}(x)]$ is the kernel for $\mathcal{P}$. Let us now show that $P(x,y) = \frac{\Omega(x,y)}{\delta(x,y)}$ for $x \neq y$. For $J \in \mathcal{D}^+$ define

$$\Omega_J(x,y) = \Theta_J^1(y)\Theta_J^2(x)$$

where

$$\begin{aligned}\Theta_J^1(y) &= \mathcal{X}_{J^-}(y) - \mathcal{X}_{J^+}(y)\\ \Theta_J^2(x) &= (\mathcal{X}_{J^{-+}}(x) + \mathcal{X}_{J^{+-}}(x)) - (\mathcal{X}_{J^{--}}(x) + \mathcal{X}_{J^{++}}(x)).\end{aligned}$$

Let us denote with $I(x,y)$ the smallest interval containing x and y, then we have

$$P(x,y) = \sum_{I\in\mathcal{D}^+} h_I(y)[h_{I^-}(x) - h_{I^+}(x)] = \sqrt{2}\sum_{I\in\mathcal{D}^+, I\supseteq I(x,y)} \frac{1}{|I|}\Omega_I(x,y).$$

Since $|I(x,y)| = \delta(x,y)$ and in the last series we are adding on all the dyadic ancestors of $I(x,y)$, including $I(x,y)$ itself,

$$P(x,y) = \frac{\sqrt{2}}{\delta(x,y)}\sum_{m=0}^{\infty}\frac{1}{2^m}\Omega_{I^{(m)}(x,y)}(x,y) = \frac{\Omega(x,y)}{\delta(x,y)}$$

with $I^{(m)}(x,y)$ the m-th ancestor of $I(x,y)$ and

$$\Omega(x,y) = \sqrt{2}\sum_{m=0}^{\infty} 2^{-m}\Omega_{I^{(m)}(x,y)}(x,y).$$

Hence (i) in Definition 1 holds with $C_0 = 2^{5/2}$.

Let us check (ii.a). Let x, y and $x' \in \mathbb{R}^+$ be such that $\delta(x,x') \leq \frac{1}{2}\delta(x,y)$. Let $I(x,y)$ be the smallest dyadic interval containing x and y. Then $|I(x,y)| = \delta(x,y)$. In a similar way $|I(x,x')| = \delta(x,x')$ and $|I(x',y)| = \delta(x',y)$. Since those three intervals are all dyadic and since $|I(x,x')| \leq \frac{1}{2}|I(x,y)|$, we necessarily must have that x' belongs to the same half of $I(x,y)$ as x does. Hence $I(x',y) = I(x,y)$ and certainly also are the same all the ancestors $I^{(m)}(x',y) = I^{(m)}(x,y)$. Now,

$$\begin{aligned}\frac{1}{\sqrt{2}}|P(x',y) - P(x,y)| &= \left|\frac{\Omega(x',y)}{\delta(x',y)} - \frac{\Omega(x,y)}{\delta(x,y)}\right|\\ &\leq \frac{|\Omega(x',y) - \Omega(x,y)|}{\delta(x,y)} + |\Omega(x',y)|\left|\frac{1}{\delta(x',y)} - \frac{1}{\delta(x,y)}\right|\end{aligned}$$

IMAL PREPRINT # 2017-0041 ISSN 2451-7100
Publication date: October 2, 2017

$$= I + II.$$

In order to estimate I, let us first explore the δ-regularity of each Ω_J. Let us prove that

(a) for fixed $y \in \mathbb{R}^+$ we have that $|\Omega_J(x', y) - \Omega_J(x, y)| \leq \frac{8}{|J|}\delta(x, x')$; and

(b) for fixed $x \in \mathbb{R}^+$, $|\Omega_J(x, y') - \Omega_J(x, y)| \leq \frac{2}{|J|}\delta(y, y')$.

Let us check (a). The regularity in the second variable is similar. Since the indicator function of a dyadic interval I is δ-Lipschitz with constant $\frac{1}{2|I|}$, we have

$$\begin{aligned}
|\Omega_J(x', y) - \Omega_J(x, y)| &= \left|\Theta^1_J(y)(\Theta^2_J(x') - \Theta^2_J(x))\right| \\
&= \left|\Theta^2_J(x') - \Theta^2_J(x)\right| \\
&\leq |\mathcal{X}_{J^{-+}}(x') - \mathcal{X}_{J^{-+}}(x)| + |\mathcal{X}_{J^{+-}}(x') - \mathcal{X}_{J^{+-}}(x)| + \\
&\qquad + |\mathcal{X}_{J^{--}}(x') - \mathcal{X}_{J^{--}}(x)| + |\mathcal{X}_{J^{++}}(x') - \mathcal{X}_{J^{++}}(x)| \\
&\leq 4\frac{4}{2\,|J|}\delta(x, x').
\end{aligned}$$

Since the series defining Ω is absolutely convergent, from the above remarks, we have

$$\begin{aligned}
I &\leq \frac{1}{\delta(x, y)}\sum_{m=0}^{\infty} 2^{-m}\left|\Omega_{I^{(m)}(x',y)}(x', y) - \Omega_{I^{(m)}(x,y)}(x, y)\right| \\
&= \frac{1}{\delta(x, y)}\sum_{m=0}^{\infty} 2^{-m}\left|\Omega_{I^{(m)}(x,y)}(x', y) - \Omega_{I^{(m)}(x,y)}(x, y)\right| \\
&\leq \frac{8}{\delta(x, y)}\sum_{m=0}^{\infty} 2^{-m}\frac{\delta(x, x')}{\left|I^{(m)}(x, y)\right|} \\
&= 16\frac{\delta(x, x')}{\delta^2(x, y)}.
\end{aligned}$$

Let us estimate II. Since $|\Omega|$ is bounded above by 2 and δ is a metric on $\mathbb{R}^+$, we have

$$II \leq 2\frac{|\delta(x, y) - \delta(x', y)|}{\delta(x, y)\delta(x', y)} \leq 2\frac{\delta(x, x')}{\delta(x, y)\delta(x', y)}$$

as we already observed, under the current conditions, $\delta(x', y) = \delta(x, y)$. And we get the desired type estimate $II \leq 2\frac{\delta(x,x')}{\delta^(x,y)}$. Hence $|P(x', y) - P(x, y)| \leq \sqrt{2}\frac{14}{3}\frac{\delta(x,x')}{\delta^2(x,y)}$ when $\delta(x, x') \leq \frac{1}{2}\delta(x, y)$.

The analogous procedure, using (b) and a similar geometric consideration for x, y, y' with $\delta(y, y') \leq \frac{1}{2}\delta(x, y)$ gives

$$|P(x, y') - P(x, y)| \leq \sqrt{2}12\frac{\delta(y, y')}{\delta^2(x, y)}.$$

□

The next result contains some additional properties of P that shall be used in the next section in order to get weighted inequalities for the maximal operator of the truncations of $\mathcal{P}$.

As usual, for Calderón-Zygmund operators, the truncations of the kernel and the associated maximal operator play a central role in the analysis of the boundedness properties of the operator. For $0 < \varepsilon < R < \infty$ set

$$P_{\varepsilon,R}(x, y) = \mathcal{X}_{\{\varepsilon\leq\delta(x,y)<R\}}P(x, y) = \mathcal{X}_{\{\varepsilon\leq\delta(x,y)<R\}}\frac{\Omega(x, y)}{\delta(x, y)}.$$

Sometimes, for example when P acts on $L^p(\mathbb{R}^+, dx)$ with $p > 1$, only the local truncation about the diagonal is actually needed. For $\varepsilon > 0$, $P_{\varepsilon,\infty}(x, y) = \mathcal{X}_{\{\delta(x,y)\geq\varepsilon\}}(x, y)P(x, y)$. Since the original form of Petermichl's kernel is provided in terms of the Haar–Fourier analysis, a scale

ISSN 2451-7100
IMAL PREPRINT # 2017-0041 Publication date: October 2, 2017

truncation is still possible and natural. For $l < m$ both in $\mathbb{Z}$ we consider also the scale truncation of P between 2^l and 2^m. In other words,

$$P^{l,m}(x,y) = \sum_{\{I\in\mathcal{D}^+ : 2^l \le |I| < 2^m\}} h_I(y)[h_{I^-}(x) - h_{I^+}(x)].$$

Since δ takes only dyadic values, $P_{\varepsilon,R}$ can also be written as $P_{2^\lambda,2^\mu}$ for λ and $\mu \in \mathbb{Z}$. For simplicity we shall write $P_{\lambda,\mu}$ to denote $P_{2^\lambda,2^\mu}$. Hence in our notation the distinction between the two truncations is only positional: $P^{l,m}$ is scale truncation; $P_{l,m}$ is metric truncation. Let us compare these two kernels and the operators induced by them. The calligraphic versions $\mathcal{P}^{l,m}$ and $\mathcal{P}_{l,m}$ denote the operators induced by $P^{l,m}$ and $P_{l,m}$ respectively.

In the next statement we use two notations for the ancestrality of a dyadic interval. Given $I \in \mathcal{D}^+$, $I^{(n)}$ denotes, as before, the n-th ancestor of I. Instead $\widehat{I^j}$ denotes the only, if any, ancestor of I in the level $\mathcal{D}^j$ of the dyadic interval. For instance if $I = [\frac{3}{2}, 2)$, then $I^{(1)} = [1,2)$, $I^{(2)} = [0,2)$, $\widehat{I^0} = [1,2)$, $\widehat{I^3} = [0,8)$.

Lemma 3. *Let l and m in $\mathbb{Z}$ with $l < m$. Then*

(1) $P^{l,m}(x,y) = P_{l,m}(x,y) + Q_{l,m}(x,y)$, where

$$Q_{l,m}(x,y) = \begin{cases} 0, & \text{for } \delta(x,y) \ge 2^m;\\ \sqrt{2}\sum_{j=l}^{m-1} 2^{-j}\Omega_{\widehat{I^j}(x,y)}(x,y), & \text{for } 0 < \delta(x,y) < 2^l;\\ -\frac{\sqrt{2}}{\delta(x,y)} \sum_{n=\log_2 \frac{2}{\delta(x,y)}}^{\infty} 2^{-n}\Omega_{I^{(n)}(x,y)}(x,y), & \text{when } 2^l \le \delta(x,y) < 2^m. \end{cases}$$

(2) $P^{l,m}$ belongs to $L^1(\mathbb{R}^+, dx)$ in each variable when the other variable remains fixed. Moreover

$$\int_{y\in\mathbb{R}^+} P^{l,m}(x,y)dx = \int_{y\in\mathbb{R}^+} P^{l,m}(x,y)dy = 0.$$

(3) $\left|Q_{l,m}(x,y)\right| \le 2\sqrt{2}\left(2^{-l}\mathcal{X}_{\{\delta(x,y)<2^l\}}(x,y) + 2^{-m}\mathcal{X}_{\{\delta(x,y)<2^m\}}\right)$.

(4) The inequality $\left|\int_{y\in\mathbb{R}^+} Q_{l,m}(x,y)dy\right| \le 2\sqrt{2}$ holds for every l,m in $\mathbb{Z}$ and every $x \in \mathbb{R}^+$.

(5) The sequence $\int_{y\in\mathbb{R}^+} Q_{l,0}(x,y)dy$ converges uniformly in $x \in \mathbb{R}^+$ for l tends to $-\infty$.

Proof. Let us rewrite together the two truncations of P for the same values of l and m with $l < m$,

$$\begin{aligned} P^{l,m}(x,y) &= \sum_{I\in\mathcal{D}^+, 2^l\le|I|<2^m} h_I(y)[h_{I^-}(x) - h_{I^+}(x)];\\ P_{l,m}(x,y) &= \mathcal{X}_{\{2^l\le\delta(x,y)<2^m\}}(x,y)\frac{\Omega(x,y)}{\delta(x,y)} \end{aligned}$$

with $\Omega(x,y) = \sqrt{2}\sum_{n=0}^{\infty} 2^{-n}\Omega_{I^{(n)}(x,y)}(x,y)$. Let us compute $P^{l,m}(x,y)$ for the three bands around the diagonal Δ of $\mathbb{R}^+\times\mathbb{R}^+$ determined by 2^l and 2^m. First, assume that $0 < \delta(x,y) < 2^l$. Then

$$P^{l,m}(x,y) = \sqrt{2}\sum_{\substack{I\in\mathcal{D}^+\\ 2^l\le|I|<2^m}} \frac{1}{|I|}\Omega_I(x,y).$$

Since $\operatorname{supp}\Omega_I \subset I\times I$, once (x,y) is given, with $\delta(x,y) < 2^l$, the sum above is performed only on those dyadic intervals I for which $2^l \le |I| < 2^m$ that contain $I(x,y)$; the smallest dyadic interval

ISSN 2451-7100
Publication date: October 2, 2017

containing both x and y. Hence

$$P^{l,m}(x,y) = \sqrt{2}\sum_{j=l}^{m-1}\frac{1}{2^j}\Omega_{\widehat{I^j}(x,y)}(x,y) = Q_{l,m}(x,y) = Q_{l,m}(x,y) + P_{l,m}(x,y)$$

in the δ-strip $\{(x,y) : \mathbb{R}^+ \times \mathbb{R}^+ : \delta(x,y) < 2^l\}$. Second, assume that $\delta(x,y) \geq 2^m$. Then no dyadic interval I containing both x and y has a measure less than 2^m. So that $P^{l,m}$ vanishes when $\delta(x,y) \geq 2^m$ and again $P^{l,m} = Q_{l,m} + P_{l,m}$. The third and last case to be considered is when $2^l \leq \delta(x,y) < 2^m$. Again the non-vanishing condition for $\Omega_I(x,y)$ requires $I \supseteq I(x,y)$, hence

$$P^{l,m}(x,y) = \sqrt{2}\sum_{\substack{I\in\mathcal{D}\\ |I|<2^m\\ I\supseteq I(x,y)}}\frac{1}{|I|}\Omega_I(x,y).$$

Since $I \supseteq I(x,y)$ then, in the above sum, I has to be an ancestor of $I(x,y)$. Hence $|I| = 2^n\,|I(x,y)| = 2^n\delta(x,y)$ for some $n = 0,1,2,\ldots$ The upper restriction on the measure of I, $|I| < 2^m$, provides an upper bound for n. In fact, since $2^m > |I| = 2^n\delta(x,y)$, $n \leq (\log_2 2^m\delta^{-1}(x,y)) - 1$. Notice that $2^m\delta^{-1}(x,y)$ is an integral power of 2, so that $\log_2 2^m\delta^{-1}(x,y) \in \mathbb{Z}$. Hence

$$\begin{aligned}
P^{l,m} &= \frac{\sqrt{2}}{\delta(x,y)}\sum_{n=0}^{\log_2\frac{2^m}{\delta(x,y)}-1}\frac{1}{2^n}\Omega_{I^{(n)}(x,y)}(x,y)\\
&= \frac{\sqrt{2}}{\delta(x,y)}\left(\Omega(x,y) - \sum_{n=\log_2\frac{2^m}{\delta(x,y)}}^{\infty}\frac{1}{2^n}\Omega_{I^{(n)}(x,y)}(x,y)\right)\\
&= P_{l,m}(x,y) + Q_{l,m}(x,y),
\end{aligned}$$

and (1) is proved.

In order to prove (2), notice that for x fixed $P^{l,m}(x,\cdot)$ is a finite linear combination of Haar functions in the variable y. Hence $P^{l,m}(x,\cdot)$ is an $L^1(\mathbb{R}^+, dx)$ function and its integral in y vanishes, since each Haar function has mean value zero. An analogous argument hold for y fixed and $P^{l,m}(\cdot,y)$.

Let us get the bound in (3). We only have to check it in the bands $\{\delta(x,y) < 2^l\}$ and $\{2^l \leq \delta(x,y) < 2^m\}$. Let us first take $\delta(x,y) < 2^l$. Then

$$\left|Q_{l,m}(x,y)\right| = \sqrt{2}\left|\sum_{j=l}^{m-1}2^{-j}\Omega_{\widehat{I^j}(x,y)}(x,y)\right| \leq \sqrt{2}\sum_{j=l}^{m}2^{-j} \leq 2\sqrt{2}2^{-l},$$

as desired. Assume now that $2^l \leq \delta(x,y) < 2^m$. Then

$$\left|Q_{l,m}(x,y)\right| \leq \sqrt{2}\frac{1}{\delta(x,y)}\sum_{n=\log_2\frac{2^m}{\delta(x,y)}}^{\infty}2^{-n} = 2\sqrt{2}\frac{1}{\delta(x,y)}\frac{\delta(x,y)}{2^m} = 2\sqrt{2}2^{-m}.$$

For the proof of (4) notice that from (3) we have that, for fixed x and fixed l and m, as a function of y, $Q_{l,m}(x,y)$, and hence $P_{l,m}(x,y)$, is integrable. Then

$$\left|\int_{y\in\mathbb{R}^+} Q_{l,m}(x,y)dy\right| \leq 2\sqrt{2}\int_{y\in\mathbb{R}^+}\left\{2^{-l}\mathcal{X}_{\{\delta(x,y)<2^l\}}(x,y) + 2^{-m}\mathcal{X}_{\{\delta(x,y)<2^m\}}(x,y)\right\}dy = 2\sqrt{2}.$$

ISSN 2451-7100
IMAL PREPRINT # 2017-0041 Publication date: October 2, 2017

Let us prove (5). From the expression in (1) for $Q_{l,0}$, we have

$$\int_{y\in\mathbb{R}^+} Q_{l,0}(x,y)dy = \sqrt{2}\int_{B_\delta(x,2^l)}\left(\sum_{j=l}^{-1}2^{-j}\Omega_{\widehat{I^j}(x,y)}(x,y)\right)dy+$$

$$-\sqrt{2}\int_{B_\delta(x,1)\setminus B_\delta(x,2^l)}\frac{1}{\delta(x,y)}\left(\sum_{n=\log_2\frac{1}{\delta(x,y)}}^{\infty}\frac{1}{2^n}\Omega_{I^{(n)}(x,y)}(x,y)\right)dy$$

$$= \sqrt{2}\sum_{j=l}^{-1}2^{-j}\int_{B_\delta(x,2^l)}\Omega_{\widehat{I^j}(x,y)}(x,y)dy - \sqrt{2}\sum_{i=l}^{-1}2^{-i}\int_{\{y:\delta(x,y)=2^i\}}\left(\sum_{n=-i}^{\infty}\frac{1}{2^n}\Omega_{I^{(n)}(x,y)}(x,y)\right)dy$$

$$= \sqrt{2}\left(\sum_{j=l}^{-1}2^{-j}2^l\widehat{\sigma}_{l,j}(x) - \tfrac{1}{2}\sum_{i=l}^{-1}2^{-i}\sum_{n=-i}^{\infty}2^{-n}2^i\sigma_{n,i}(x)\right),$$

where $\widehat{\sigma}_{l,j}(x) = \fint_{B_\delta(x,2^l)}\Omega_{\widehat{I^j}(x,y)}(x,y)dy$ and $\sigma_{n,i}(x) = \fint_{\{\delta(x,y)=2^i\}}\Omega_{I^{(n)}(x,y)}(x,y)dy$ and $\fint_E f$ denotes the mean value of f on E. So that

$$\int_{y\in\mathbb{R}^+} Q_{l,0}(x,y)dy = \sqrt{2}\sum_{i=0}^{-l-1}2^{-i}\widehat{\sigma}_{l,i+l}(x) - \frac{\sqrt{2}}{2}\left(\sum_{n=1}^{-l}2^{-n}\sum_{i=-n}^{-1}\sigma_{n,i}(x) + \sum_{n=-l+1}^{\infty}2^{-n}\sum_{i=l}^{-1}\sigma_{n,i}(x)\right).$$

Since in the definitions of $\widehat{\sigma}$ and σ we are taking mean values of functions with L^∞-norm equal to 1, we certainly have that $\left|\widehat{\sigma}\right| \leq 1$ and $|\sigma| \leq 1$. Hence $\left|\sum_{i=-n}^{-1}\sigma_{n,i}(x)\right| \leq n$, and $\left|\sum_{i=l}^{-1}\sigma_{n,i}(x)\right| \leq |l| = -l$. So the first term in the expression for the integral is dominated by the geometric series $\sum_{i\geq 0}2^{-i}$, the second term is dominated by the convergent series $\sum_{n=1}^{\infty}n2^{-n}$ and the third term is bounded by $|l|\sum_{n=-l+1}^{\infty}2^{-n}$ which tends to zero as $|l|$ tends to infinity. □

Let us notice that (4) and (5) in the above lemma hold also integrating in the variable x.

One more remark is in order; P is dyadicaly homogeneous of degree -1 and Ω of degree zero. In other words $P(2^jx,2^jy) = 2^{-j}P(x,y)$ and $\Omega(2^jx,2^jy) = \Omega(x,y)$.

From the above lemma, we conclude that with

$$\mathcal{P}^*f(x) = \sup_{\substack{l<m\\ l,m\in\mathbb{Z}}}\left|\int_{\mathbb{R}^+}P^{l,m}(x,y)f(y)dy\right|, \text{ and}$$

$$\mathcal{P}_*f(x) = \sup_{\substack{l<m\\ l,m\in\mathbb{Z}}}\left|\mathcal{P}_{l,m}(x,y)\right|$$

we have

$$\mathcal{P}_*f(x) \leq 4\sqrt{2}M_{dy}f(x) + \mathcal{P}^*f(x), \text{ and}$$

$$\mathcal{P}^*f(x) \leq 4\sqrt{2}M_{dy}f(x) + \mathcal{P}_*f(x), \tag{2.2}$$

where

$$M_{dy}f(x) = \sup_{x\in I\in\mathcal{D}^+}\frac{1}{|I|}\int_I|f(y)|\,dy$$

the dyadic maximal operator.

3. Weighted norm inequalities for the Petermichl's operator

We shall see in this section that $\mathcal{P}$ satisfies all the conditions in [1] in order to show the $L^p(\mathbb{R}^+, wdx)$ boundedness for $w \in A_p(\mathbb{R}^+,\delta,dx)$ which coincides with the dyadic Muckenhoupt

IMAL PREPRINT # 2017-0041 ISSN 2451-7100
Publication date: October 2, 2017

weights in $\mathbb{R}^+$. For the sake of completeness we proceed to provide the statement of the main result in [1] on normal spaces of homogeneous type for general Calderón-Zygmund operators.

Let X be a set. A quasi-distance on X is a nonnegative and symmetric function d on $X \times X$, vanishing only on the diagonal of $X \times X$ such that for some $\kappa > 0$ the inequality $d(x,z) \leq \kappa(d(x,y) + d(y,z))$ holds for every x, y and $z \in X$. The main results on the structure of quasi-metric spaces are contained in [6]. The Borel sets in X are those in the σ-algebra generated by the topology induced in X by the neighborhoods defined by the d-balls. If the d-balls are Borel sets and μ is a positive Borel measure such that for some constant A the inequalities

$$0 < \mu(B(x,2r)) \leq A\mu(B(x,r)) < \infty$$

hold for every $x \in X$ and every $r > 0$, where $B(x,r) = \{y \in X : d(x,y) < r\}$, we say the (X,d,μ) is a space of homogeneous type.

Let (X,d,μ) be a space of homogeneous type such that continuous functions are dense in $L^1(X,\mu)$. Let $1 < p < \infty$, a nonnegative and locally integrable function w defined on X is said to satisfy the Muckenhoupt A_p condition, or $w \in A_p(X,d,\mu)$, if there exists a constant C such that

$$\left(\fint_B w d\mu\right)\left(\fint_B w^{-\frac{1}{p-1}} d\mu\right)^{p-1} \leq C$$

for every d-ball B. As before, $\fint_E w d\mu = \mu(E)^{-1}\int_E w(x)d\mu(x)$. A weight w is said to belong to A_∞ if there exist two constants C and $\eta > 0$ such that the inequality

$$\frac{w(E)}{w(B)} \leq C\left(\frac{\mu(E)}{\mu(B)}\right)^{\eta}$$

holds for every ball B and every measurable subset E of B. The Hardy-Littlewood maximal function in this setting is, naturally, given by

$$Mf(x) = \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f|\, d\mu.$$

The results in [7] show the reverse Hölder inequality for A_p weights and, as a consequence, the boundedness of the Hardy–Littlewood maximal in $L^p(X, wd\mu)$ when $w \in A_p$.

Theorem 4 ([7], [3])**.** *Let (X,d,μ) be a space of homogeneous type and $1 < p < \infty$. Then $w \in A_p$ if and only if for some constant C we have*

$$\int_X (Mf(x))^p w(x)dx \leq C \int_X |f(x)|^p\, w(x)d\mu(x)$$

for every measurable function f.

For singular integrals, the detection of the correct integral singularity of the space is attained after normalization of the space (X,d,μ) ([6]). We shall assume here that (X,d,μ) is a normal space in the sense that there exist two constants $0 < \alpha \leq \beta < \infty$ such that $\alpha r \leq \mu(B(x,r)) \leq \beta r$. Let us only recall two particular instances of this situation. The first, $X = \mathbb{R}^n$, $d(x,y) = |x-y|^n$ and μ Lebesgue measure. The second, $X = \mathbb{R}^+$, $d(x,y) = \delta(x,y) = |I(x,y)|$, where $I(x,y)$ is the smallest dyadic interval containing x and y. In this case μ is one dimensional Lebesgue measure.

The next statement collects the boundedness results for singular integrals in [1].

Theorem 5 ([1])**.** *Let (X,d,μ) be a normal space such that continuous functions are dense in L^1. Assume that for every $r > 0$ and every $x_0 \in X$ we have that $\mu(B(x,r) \vartriangle B(x_0,r)) \to 0$ when $d(x,x_0) \to 0$, where $E\triangle F$ denotes the symmetric difference of E and F. Let T be a Calderón-Zygmund operator on (X,d,μ) in the sense of Definition 1 in §2. Let $K(x,y)$ be the kernel of T. Assume that the kernel K satisfies also,*

ISSN 2451-7100
IMAL PREPRINT # 2017-0041 Publication date: October 2, 2017

(iii) *for every $R > r > 0$, we have*
(iii.a) $\left|\int_{r\leq d(x,y)<R} K(x,y)d\mu(y)\right|$ *is bounded uniformly in r, R and x.*
Moreover, $\int_{r\leq d(x,y)<1} K(x,y)d\mu(y)$ *converges uniformly in x when r tends to zero.*
(iii.b) $\left|\int_{r\leq d(x,y)<R} K(x,y)d\mu(x)\right|$ *is bounded uniformly in r, R and y.*
Moreover, $\int_{r\leq d(x,y)<1} K(x,y)d\mu(x)$ *converges uniformly in y when r tends to zero.*

Then, with $T_{R,r}f(x) = \int_{y\in X} K_{R,r}(x,y)f(y)d\mu(y)$, $K_{R,r} = \mathcal{X}_{r\leq d<R}K$ *and* $T_*f(x) = \sup_{\varepsilon>0}\left|T_{\infty,\varepsilon}f(x)\right|$, *we have*

(1) for $1 < p < \infty$ there exists the $L^p(X,\mu)$ limit Tf of $T_{R,r}f$ when $R \to +\infty$ and $r \to 0$;
(2) for $f \in L^p(X,\mu)$ and $1 < p < \infty$ we have Cotlar's inequality

$$T_*f(x) \leq CM(Tf(x)) + CMf(x);$$

(3) the maximal operator T_* is of weak type (1,1). In other words, for some constant $C > 0$ we have

$$\mu\left(\{T_*f > \lambda\}\right) \leq \frac{C}{\lambda}\|f\|_{L^1};$$

(4) for $w \in A_\infty(X,\mu)$

$$\int_X [T_*f(x)]^p w(x)d\mu(x) \leq C\int_X [Mf(x)]^p w(x)d\mu(x);$$

(5) for $w \in A_p(X,\mu)$ we have

$$\int_X [T_*f(x)]^p w(x)d\mu(x) \leq C\int_X |f(x)|^p w(x)d\mu(x).$$

As a consequence of the above result and of the results in Section 2, we get the weighted boundedness of the maximal operators associated to Petermichl's kernel. We say that w defined on $\mathbb{R}^+$ is in $A_p^{dy}(\mathbb{R}^+,dx)$ if the inequality $\left(\fint_I w d\mu\right)\left(\fint_I w^{-1/(p-1)}d\mu\right)^{p-1} \leq C$ holds for every $I \in \mathcal{D}^+$.

Theorem 6. *For $1 < p < \infty$ and $w \in A_p^{dy}(\mathbb{R}^+,dx)$ we have that $\mathcal{P}_*$ is bounded in $L^p(\mathbb{R}^+,wdx)$.*

Proof. Let us check that we are in the hypothesis of Theorem 5. As we already proved $X = \mathbb{R}^+$, $d = \delta$ and μ =Lebesgue measure, provide a normal space in which δ-Lipschitz functions are dense in $L^1(\mathbb{R}^+,dx)$. In order to prove that $|B_\delta(x,r) \vartriangle B_\delta(x_0,r)|$ tends to zero when x tends to x_0 for fixed positive r, just notice that when $\delta(x,x_0) < r/2$, $B_\delta(x,r)$ and $B_\delta(x_0,r)$ coincide. From Theorem 2 we have the kernel $P(x,y)$ satisfies (i) and (ii) in the Definition of Calderón–Zygmund operator. On the other hand, since $P^{l,m} = P_{l,m} + Q_{l,m}$ from (2), (4) and (5) in Lemma 3, we get (iii) in Theorem 5. Then we can apply Theorem 5 to obtain the boundedness properties of $\mathcal{P}_*$ in particular the weighted boundedness contained in (5). It only remains to observe that $A_p(\mathbb{R}^+,\delta,dx) = A_p^{dy}(\mathbb{R}^+,dx)$ □

Theorem 7. *Let $1 < p < \infty$. Then $\mathcal{P}^*$ is bounded in $L^p(\mathbb{R}^+,wdx)$ if and only if $w \in A_p^{dy}(\mathbb{R}^+,dx)$.*

Proof. The sufficiency of $w \in A_p^{dy}(\mathbb{R}^+,dx)$ for the boundedness of $\mathcal{P}^*$ in $L^p(\mathbb{R}^+,wdx)$, $1 < p < \infty$, follows from (2.2), Theorem 6 and Theorem 4, since Mf in $(\mathbb{R}^+,\delta,dx)$ is the dyadic Hardy-Littlewood maximal function $M_{dy}f$. Let us finally show that $A_p^{dy}(\mathbb{R}^+,dx)$ is necessary for the $L^p(\mathbb{R}^+,wdx)$. Assume that w is a weight in X such that $\mathcal{P}^*$ is bounded as an operator on $L^p(X,wd\mu)$. Since $\mathcal{P}^*f(x) \geq \left|\sum_{I\in\mathcal{D},|I|=|I_0|}\langle f,h_I\rangle\left(h_{I^-}(x) - h_{I^+}(x)\right)\right|$ for any $I_0 \in \mathcal{D}$, taking $f = h_{I_0}w^{-1/(p-1)}$ we get

$$\mathcal{P}^*f(x) \geq \left\langle w^{-\frac{1}{p-1}}h_{I_0}, h_{I_0}\right\rangle\left|h_{I_0^-}(x) - h_{I_0^+}(x)\right| = \frac{1}{|I_0|}\left(\int_{I_0} w^{-\frac{1}{p-1}}d\mu\right)\frac{\sqrt{2}}{\sqrt{|I_0|}}\mathcal{X}_{I_0}(x).$$

ISSN 2451-7100
Publication date: October 2, 2017

Hence, from the inequality $\int_X (\mathcal{P}^* f)^p w d\mu \le C \int_X |f|^p w d\mu$ that we are assuming, taking $f = h_{I_0} w^{-1/(p-1)}$ we get

$$\frac{2^{p/2}}{|I_0|^{3p/2}} \left(\int_{I_0} w^{-\frac{1}{p-1}} d\mu \right)^p w(I_0) \le C \frac{1}{|I_0|^{p/2}} \int_{I_0} w^{-\frac{1}{p-1}} w d\mu$$

which implies that $w \in A_p^{dy}(\mathbb{R}^+, d\mu)$. □

As a final remark, let us observe that from the representation of the Hilbert kernel given in [9] and our result, we can get the well known weighted norm inequalities for the Hilbert transform.

References

[1] Hugo Aimar, *Integrales singulares y aproximaciones de la identidad en espacios de tipo homogéneo*, Doctoral thesis, Universidad Nacional de Buenos Aires, PEMA–INTEC. Available in http://www.imal.santafe-conicet.gov.ar/TesisIMAL/tesisAimarH.pdf, 1983.

[2] ______, *Singular integrals and approximate identities on spaces of homogeneous type*, Trans. Amer. Math. Soc. **292** (1985), no. 1, 135–153. MR 805957

[3] Hugo Aimar and Roberto A. Macías, *Weighted norm inequalities for the Hardy-Littlewood maximal operator on spaces of homogeneous type*, Proc. Amer. Math. Soc. **91** (1984), no. 2, 213–216. MR 740173

[4] Ronald R. Coifman and Guido Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Mathematics, Vol. 242, Springer-Verlag, Berlin-New York, 1971, Étude de certaines intégrales singulières. MR 0499948

[5] Roberto A. Macías and Carlos Segovia, *A decomposition into atoms of distributions on spaces of homogeneous type*, Adv. in Math. **33** (1979), no. 3, 271–309. MR 546296

[6] ______, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math. **33** (1979), no. 3, 257–270. MR 546295

[7] Roberto A. Macías and Carlos A. Segovia, *A well-behaved quasi-distance for spaces of homogeneous type*, Trabajos de Matemática, Serie I, vol. 32, IAM–CONICET, 1981.

[8] Yves Meyer and Ronald Coifman, *Wavelets*, Cambridge Studies in Advanced Mathematics, vol. 48, Cambridge University Press, Cambridge, 1997, Calderón-Zygmund and multilinear operators, Translated from the 1990 and 1991 French originals by David Salinger. MR 1456993

[9] Stefanie Petermichl, *Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 6, 455–460. MR 1756958

Instituto de Matemática Aplicada del Litoral, UNL, CONICET, FIQ.

CCT CONICET Santa Fe, Predio "Alberto Cassano", Colectora Ruta Nac. 168 km 0, Paraje El Pozo, S3007ABA Santa Fe, Argentina.

E-mail address: haimar@santafe-conicet.gov.ar

E-mail address: ivanagomez@santafe-conicet.gov.ar