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By

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BOUNDEDNESS OF OPERATORS RELATED TO A DEGENERATE SCHRÖDINGER SEMIGROUP

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ABSTRACT. In this work we search for boundedness results for operators related to the semigroup generated by the degenerate Schrödinger operator $\mathcal{L}u = -\frac{1}{\omega}$ div $A \cdot \nabla u + Vu$, where ω is a weight, A is a matrix depending on x and satisfying $\lambda\, \omega(x)|\xi|^2 \le A(x)\xi \cdot \xi \le \Lambda\, \omega(x)|\xi|^2$ for some positive constants λ, Λ and all x, ξ in $\mathbb{R}^d$, assuming further suitable properties on the weight ω and on the non-negative potential V. In particular, we analyze the behaviour of T^*, the maximal semigroup operator, $\mathcal{L}^{-\alpha/2}$, the negative powers of $\mathcal{L}$, and the mixed operators $\mathcal{L}^{-\alpha/2}V^{\sigma/2}$ with $0 < \sigma \le \alpha$ on appropriate functions spaces measuring size and regularity. As in the non degenerate case, i.e. $\omega \equiv 1$, we achieve these results by first studying the case $V = 0$, obtaining also some boundedness properties in this context that we believe are new.

1. INTRODUCTION

In 1982, Fabes, Kenig and Serapioni (see [FKS]) studied the following second order degenerate elliptic differential operator in divergence form

$$\mathcal{L}_0\, u = -\frac{1}{\omega} \operatorname{div} A \cdot \nabla u,$$

where ω is a weight belonging to the Muckenhoupt class A_2, that is, ω satisfies

$$\left(\frac{1}{\omega(B)} \int_B \omega\right) \left(\frac{1}{\omega(B)} \int_B \omega^{-1}\right) \le C$$

for some fix constant C and any ball B.

Also, $A(x)$ is a $d \times d$ real and symmetric matrix such that for all $\xi \in \mathbb{R}^d$

$$\lambda\, \omega(x)|\xi|^2 \le A(x)\xi \cdot \xi \le \Lambda\, \omega(x)|\xi|^2, \quad \lambda > 0.$$

Under that assumption on the weight, the entries of $A(x)$ are functions that vanish at most on a set of measure zero. When $\omega \equiv 1$, we recover the case of an uniformly elliptic operator in divergence form.

Several years later, in [D], Dziubanski considered the associate Schrödinger operator, namely

$$\mathcal{L}u = \mathcal{L}_0 u + V\, u\,,$$

where the potential V is a non-negative locally integrable function with respect to the measure $d\mu = \omega dx$.

He also assumes additional conditions on ω and V. For a better understanding, let us remind that any A_∞ weight, so in particular an A_2 weight, satisfies a doubling and a reverse doubling condition. More precisely, there exist two numbers ν and

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γ, $0 < \nu \leq d \leq \gamma$, such that for and any ball $B(x, r)$ and $t > 1$, the following inequalities hold

$$c\, t^{\nu} \leq \frac{\omega(B(x, tr))}{\omega(B(x, r))} \leq C\, t^{\gamma}, \tag{1}$$

for some constants independent of the point x. Although for an A_2 weight over $\mathbb{R}^d$ the doubling index must be smaller than $2d$, there is not a lower bound for the reverse doubling coefficient ν. In fact, straightforward calculations show that $\omega(x) = |x|^{\alpha-d}$ is in A_2 when $0 < \alpha < 2d$ but with lower index α. When the inequality on the left is satisfied we say $\omega \in RD_\nu$, while, if the other holds, we write $\omega \in D_\gamma$.

With this in mind, following [D], we further assume that ω is reverse doubling for some index $\nu > 2$ and hence d and γ must also be greater than two. We often refer to this assumption saying that $\omega \in RD_\nu \cap D_\gamma$ with $\nu > 2$.

Let us say one more word about that assumption. In [FJK], the authors show that the fundamental solution Γ_0 of the differential operator $\mathcal{L}_0$ in a ball behaves like

$$\Gamma_0(x, y) \simeq \int_{|x-y|}^{R} \frac{s^2}{\omega(B(x, s))} \frac{ds}{s}.$$

Therefore, if $\omega \in RD_\nu$ with $\nu > 2$, it follows that

$$\Gamma_0(x, y) \leq \frac{C|x-y|^{\nu}}{\omega(B(x, |x-y|))} \int_{|x-y|}^{R} s^{2-\nu} \frac{ds}{s} \leq \frac{C|x-y|^2}{\omega(B(x, |x-y|))}.$$

Since it is clear that the last quantity is also a lower bound for Γ_0 we get

$$\Gamma_0(x, y) \simeq \frac{C|x-y|^2}{\omega(B(x, |x-y|))}.$$

In this way, the assumption $\nu > 2$ allows to know the precise size of the singularity of Γ_0 at the diagonal.

As for the potential V, we additionally assume that it belongs to a reverse-Hölder class of order q, denoted as $RH_q(\omega)$, for some $q > \gamma/2$, where γ is such that $\omega \in D_\gamma$. We remind that $V \in RH_q(\omega)$ means that there is a constant such that

$$\left(\frac{1}{\omega(B)} \int_B V^q\, \omega \right)^{1/q} \leq C \frac{1}{\omega(B)} \int_B V\, \omega,$$

for any ball B in $\mathbb{R}^d$. Let us remark that, following [D], the above condition on V allows to define a critical radius function as it was done in the case $w \equiv 1$ (see [Sh] and [K]), namely

$$\rho(x) = sup \left\{ r : \frac{r^2}{\omega(B(x, r))} \int_{B(x,r)} V\omega \ \leq 1 \right\}. \tag{2}$$

It turns out that $0 < \rho(x) < \infty$ for any x and it further satisfies the following crucial property: there exist positive constants c_1 c_2 and N_0 such that

$$c_1 \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N_0} \rho(x) \leq \rho(y) \leq c_2\, \rho(x) \left(1 + \frac{|x-y|}{\rho(x)}\right)^{N_0/N_0+1}. \tag{3}$$

Both differential operators, $\mathcal{L}$ and $\mathcal{L}_0$, are non-negative and selfadjoint with respect to the measure $d\mu = \omega dx$ and they generate semigroups of selfadjoint linear

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operators on $L^2(d\mu)$, $\{S_t\}_{t>0}$ and $\{T_t\}_{t>0}$ with symmetric kernels denoted $h_t(x,y)$ and $k_t(x,y)$ respectively. Moreover, by a perturbation argument, they satisfy

$$0 \leq k_t(x,y) \leq h_t(x,y).$$

Further estimates on the size and regularity of these kernels have been proved as well as on their difference (see [D] and [HLL]). We state them all in the next section.

The aim of this paper is to obtain norm-inequalities for some operators related to both semigroups under the stated assumptions on ω and V. More precisely, we shall be concerned with negative powers of $\mathcal{L}$ and $\mathcal{L}_0$ and with the corresponding maximal operators of their generated semigroups. Besides, for the Schrödinger case, we shall also consider the operators $\mathcal{L}^{-\alpha/2}V^{\sigma/2}$ for $0 < \sigma \leq \alpha < \nu$. We shall study their behavior on suitable size spaces as well as on regularity spaces.

For the fractional operators $\mathcal{L}_0^{-\alpha/2}$ and $\mathcal{L}^{-\alpha/2}$ we will show that Lebesgue spaces $L^p(\omega)$ are, in general, non-appropriate. Rather, we shall introduce some kind of "mixed" Morrey spaces, combining the distance and the weight that, needless to say, coincide with Lebesgue spaces when $\omega \equiv 1$. Regarding smoothness spaces, and for the operators related to $\mathcal{L}_0$, we will prove boundedness over BMO and Lipschitz spaces corresponding to the space of homogeneous type $(\mathbb{R}^n, |.|, \omega dx)$ that, under mild assumptions on ω, are the same as the classical $BMO(dx)$ and $\Lambda^\beta(dx)$ (see for example [MW]). For the Schrödinger case we use appropriate subspaces, adding a condition on averages over critical balls, that is, balls of the type $B(x,\rho(x))$, similarly to the non-degenerate case.

The organization of the paper is as follows.

We start with a series of results involving some estimates for the kernels of the semigroups as well as other known inequalities that will be essential to our work.

Then, we analyze the behaviour of all mentioned operators when acting on size spaces. Even $L^p(\omega)$ are the natural spaces for the maximal semigroup operators, it is not longer the case for the negative powers of the differential operators. This issue is discussed in Section 3 and new and suitable spaces are introduced, noted as $M_p^\lambda(\omega)$, proving some relationships with weighted Lebesgue spaces. In particular we show that if $\omega \in RD_\nu$ and doubling, bounded functions with compact support are in $M_p^\nu(\omega)$, $1 \leq p \leq \infty$. As we noticed above, being our weight in A_2, it also belongs to RD_ν for some ν and hence we make sure that this particular space, $M_p^\nu(\omega)$, contains enough functions. Besides, since $\omega \in RD_\nu$ implies $\omega \in RD_\lambda$ for $\lambda < \nu$ we have that $M_p^\lambda(\omega)$ are non-trivial for all such values of λ.

Next, we study the behaviour of S^* and negative powers of $\mathcal{L}_0$ on BMO and regularity spaces. Besides, we prove that $\mathcal{L}_0^{-\alpha/2}$ can be extended to be a bounded operator from $M_1^{\alpha-\beta}(\omega)$ into BMO when $\beta = 0$ and into Lipschitz spaces of order β otherwise. When $\omega \equiv 1$, the space M_1^α contains $L^{d/\alpha}$ and even more its weaker version, so that we recover the classical result for fractional integrals associate to the Laplacian operator showing, somehow, that these new size spaces are suitable substitutes of Lebesgue spaces in this context.

We devote the next section to present a general theorem on boundedness of operators on $BMO_\beta^\rho(\omega)$, the regularity spaces adapted to the Schrödinger situation (see the precise definition at the beginning of Section 5). Such result is similar to Theorem 1.1 in [MSTZ] with some minor differences. With this tool at hand, we obtain regularity results for T^*, $\mathcal{L}^{-\alpha/2}$ and $\mathcal{L}^{-\alpha/2}V^{\sigma/2}$ with $\alpha \geq \sigma$.

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2. Preliminaries

As mentioned, the semigroups generated by $\mathcal{L}_0$ and $\mathcal{L}$, that we call $\{S_t\}_{t>0}$ and $\{T_t\}_{t>0}$ respectively, are, for each $t>0$, integral operators given by kernels, denoted as $h_t(x,y)$ and $k_t(x,y)$, in the sense that

$$S_t f(x) = \int_{\mathbb{R}^d} h_t(x,y) f(y)\omega(y)dy \quad \text{and} \quad T_t f(x) = \int_{\mathbb{R}^d} k_t(x,y) f(y)\omega(y)dy.$$

In the following Lemma we collect some known estimates for h_t and k_t, as well as for their difference $q_t = h_t - k_t$. We will give references for each of them.

Lemma 1. *Suppose $\omega \in RD_\nu \cap D_\gamma \cap A_2$, $2 < \nu \le \gamma$ and V satisfying a $RH_q(\omega)$ condition with $q > \gamma/2$. Then the following estimates hold:*

(a) *There exist constants c and C such that*

$$0 \le h_t(x,y) \le C \frac{e^{-\frac{|x-y|^2}{ct}}}{\omega(B(x,\sqrt{t}))}.$$

(b) *If $|x-z| \le |x-y|/4$, for some $0 < \eta \le 1$ we have*

$$|h_t(x,y) - h_t(z,y)| \le C \min\left\{1, \left(\frac{|x-z|}{\sqrt{t}}\right)^\eta\right\} \frac{e^{-\frac{|x-y|^2}{ct}}}{\omega(B(x,\sqrt{t}))}.$$

(c) *For each $N \ge 0$ there is a constant C_N such that*

$$0 \le k_t(x,y) \le C_N \frac{e^{-\frac{|x-y|^2}{ct}}}{\omega(B(x,\sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

(d) *There exists $\delta_0 > 0$ such that*

$$|q_t(x,y)| \le C \min\left\{1, \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0}\right\} \frac{e^{-\frac{|x-y|^2}{ct}}}{\omega(B(x,\sqrt{t}))}.$$

(e) *for any $\delta < \min\{\eta, \delta_0\}$, there exists a constant C such that*

$$|q_t(x,y) - q_t(z,y)| \le C \left(\frac{|x-z|}{\rho(x)}\right)^\delta \frac{e^{-\frac{|x-y|^2}{ct}}}{\omega(B(x,\sqrt{t}))},$$

provided $|x-z| < |x-y|/4$ and $|x-z| \le \rho(x)$.

(f) *For any given $\delta < \min\{\delta_0, \eta\}$, there is a constant C such that*

$$|k_t(x,y) - k_t(z,y)| \le C_N \left(\frac{|x-z|}{\sqrt{t}}\right)^\delta \frac{e^{-\frac{|x-y|^2}{ct}}}{\omega(B(y,\sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N},$$

provided $|x-z| \le |x-y|/8$.

All of the above estimates are already known. Items a) and b) are referred in [D] and are consequence of Theorems 2.7, 2.3, 2.4 and Corollary 3.4 of [HS-C]. Estimates given in c) and d) can be found in [D]. Finally, proofs of e) and f) are provided in [HLL] (see Propositions 3.1 and 3.2 therein).

Remark 1. We observe that since both the kernels are symmetric we can change x by y on any of the above estimates. Also, using the size condition given in d), it is very simple to check that e) holds for $|x-z| < 2\rho(x)$.

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3. Boundedness results on $L^p(\omega)$ and $M^p_\lambda(\omega)$

Boundedness on $L^p(\omega)$ of the semigroup maximal operators $S^* = \sup_{t>0} S_t$ and $T^* = \sup_{t>0} T_t$ are known. In fact they are an easy consequence of Lemma 1 a) and c) since both are controlled by

$$U^* f(x) = \sup_{t>0} \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-y|^2}{t}}}{\omega(B(x,\sqrt{t}))} f(y)\,\omega(y)\,dy.$$

If we divide the integral as $B(x,\sqrt{t}) \cup \left(\bigcup_{k\geq 0} B(x, 2^{k+1}\sqrt{t}) \setminus B(x, 2^k\sqrt{t})\right)$, it is easy to see that $U^* f(x)$ is bounded by $\mathcal{M}_\omega f(x)$, the maximal function with respect to the measure ωdx, that is,

$$\mathcal{M}_\omega f(x) = \sup_{B \ni x} \frac{1}{\omega(B)} \int_B |f(y)|\,\omega(y)\,dy.$$

As it is well known, $\mathcal{M}_\omega$ is bounded on $L^p(\omega)$ if $1 < p \leq \infty$ and of weak type $(1,1)$ with respect to ωdx, provided ω is doubling.

Therefore, the following result holds.

Theorem 1. *Let ω be a A_2-weight such that $\omega \in RD_\nu \cap D_\gamma$ with $\nu > 2$. Then we have:*

(a) S^* is bounded on $L^p(\omega)$ if $1 < p \leq \infty$ and of weak type $(1,1)$ with respect to ωdx.

(b) If $V \in RH_q(\omega)$ for $q > \gamma/2$ the above properties also hold for T^*.

Remark 2. Let us point out that when $1 < p \leq \infty$ the above result is a consequence of the maximal theorem proved by Stein in a general frame of semigroups. We refer the reader to [St].

Regarding fractional integral operators, they can be written in terms of the semigroup as follows

$$\mathcal{L}_0^{-\alpha/2} f(x) = \int_0^\infty S_t f(x)\, t^{\alpha/2} \frac{dt}{t},$$

and in a similar way for $\mathcal{L}^{-\alpha/2}$. Thus, their kernels are given by

$$H_\alpha(x,y) = \int_0^\infty h_t(x,y)\, t^{\alpha/2} \frac{dt}{t} \tag{4}$$

and

$$K_\alpha(x,y) = \int_0^\infty k_t(x,y)\, t^{\alpha/2} \frac{dt}{t}, \tag{5}$$

respectively. Again, by Lemma 1 a) and c), analyzing the behaviour over size spaces of the following fractional operator

$$J_\alpha f(x) = \int_{\mathbb{R}^n} \frac{|x-y|^\alpha}{\omega(B(x,|x-y)|)} f(y)\,\omega(y) dy,$$

will give us information for the negative powers of $\mathcal{L}_0^{-\alpha/2}$ and $\mathcal{L}^{-\alpha/2}$. In fact the following estimates hold. We point out that the second estimate below was also proved in [L] for $0 < \alpha \leq 2$.

Lemma 2. *Let ω be a A_2-weight such that $\omega \in RD_\nu \cap D_\gamma$ with $\nu > 2$. Then, for $0 < \alpha < \nu$, we have:*

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(a)

$$0 \le H_\alpha(x,y) \le C\frac{|x-y|^\alpha}{\omega(B(x,|x-y)|)}. \tag{6}$$

(b) Further, if $V \in RH_q$ with $q > \gamma/2$,

$$0 \le K_\alpha(x,y) \le C_N \frac{|x-y|^\alpha}{\omega(B(x,|x-y|))}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N},$$

for any positive N.

.

Proof. To prove the estimate for H_α, according to expression 1, it will be enough to check

$$\int_0^\infty \frac{e^{-\frac{|x-y|^2}{ct}}}{\omega(B(x,\sqrt{t}))}t^{\alpha/2}\frac{dt}{t} \le C\frac{|x-y|^\alpha}{\omega(B(x,|x-y)|)}.$$

To see the above inequality, we make the change of variable $t = |x-y|^2 s$ arriving at the bound

$$|x-y|^\alpha \int_0^\infty \frac{s^\alpha e^{-1/s^2}}{\omega(B(x,\sqrt{s}|x-y|))}\frac{ds}{s}.$$

Now, dividing the integration from 0 to 1 and from 1 to ∞, and applying the D_γ and RD_ν conditions in each case we are led to

$$\frac{|x-y|^\alpha}{\omega(B(x,|x-y)|)}\left(\int_0^1 s^{(\alpha-\gamma)/2}e^{-1/s^2}\frac{ds}{s} + \int_1^\infty s^{(\alpha-\nu)/2}\frac{ds}{s}\right),$$

and both integrals are convergent since $\alpha < \nu$.

Next, to prove the estimate for K_α we use expression (5) and Lemma 1 item (b). Proceeding as above, after changing variables, we get

$$|x-y|^\alpha \int_0^\infty \frac{s^\alpha e^{-1/s^2}}{\omega(B(x,\sqrt{s}|x-y|))}\left(1+\frac{\sqrt{s}|x-y|}{\rho(x)}\right)^{-N}\frac{ds}{s}.$$

Now we use that

$$1+\frac{\sqrt{s}|x-y|}{\rho(x)} \ge c\left(1+\frac{|x-y|}{\rho(x)}\right)\min\left\{1,\sqrt{s}\right\},$$

which follows easily considering $s < 1$ and $s \ge 1$. Now, using the same estimates as before we obtain the bound

$$\frac{|x-y|^\alpha}{\omega(B(x,|x-y)|)}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}\left(\int_0^1 s^{(\alpha-\gamma-N)/2}e^{-1/s^2}\frac{ds}{s} + \int_1^\infty s^{(\alpha-\nu)/2}\frac{ds}{s}\right),$$

that gives the stated inequality since $\alpha < \nu$. □

Te next proposition shows that Lebesgue spaces may not be the appropriate spaces for J_α.

Proposition 1. *Suppose ω is doubling and J_α maps continuously $L^p(\omega)$ into $L^s(\omega)$ with $s > p$, then the measure ωdx is lower-Ahlfors, i.e., there exists a positive number λ such that*

$$Cr^\lambda \le \omega(B(x,r)) \tag{7}$$

for some constant C independent of x.

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Proof. Fix a ball $B(x,r)$ and take as f the characteristic of the ring $B(x,3r) \setminus B(x,2r)$. Since ω is doubling, $\omega(B(x,3r) \setminus B(x,2r)) \simeq \omega(B(x,r)$. Therefore, for $z \in B(x,r)$ we have

$$J_\alpha f(z) \geq C r^\alpha.$$

Thus, by the continuity assumption

$$C\left(\omega(B(x,r))\right)^{1/s} r^\alpha \leq \|J_\alpha f\|_s \leq C'\left(\omega(B(x,r))\right)^{1/p},$$

and hence

$$r^\alpha \leq C\left(\omega(B(x,r))\right)^{1/p-1/s},$$

which gives the desired result since $s > p$. □

Remark 3. The converse statement is also true. More precisely, if ω is doubling and satisfies (7), then J_α maps continuously $L^p(\omega)$ into $L^s(\omega)$ with $1/s = 1/p - \alpha/\lambda$. In fact, it is a consequence of the results in [BS] since, under the assumption (7), J_α is clearly dominated by the fractional integral in the space of homogeneous type $(\mathbb{R}^d, |.|, \omega dx)$ appearing there.

Let us introduce the following size spaces: given a doubling weight ω, $1 \leq p < \infty$, and $\lambda > 0$ we say that a measurable function belongs to $M_p^\lambda(\omega)$ if

$$\|f\|_{M_p^\lambda(\omega)} = sup_{B=B(x,r)} \left(\frac{r^\lambda}{\omega(B)} \int_B |f|^p \omega\right)^{1/p} < \infty. \tag{8}$$

When $p = \infty$ the limiting space is that of a.e. bounded functions, so we define $M_\infty^\lambda(\omega) = L^\infty$

It is clear that the above quantity is a norm and it coincides with the standard L^p-norm when $\omega \equiv 1$ and $\lambda = d$. More generally, if the weight is Ahlfors of order λ, that is, $r^\lambda \simeq \omega(B(x,r))$, we have $M_p^\lambda(\omega) = L^p(\omega)$.

We also notice that if the weight is just lower-Ahlfors of order λ, only $M_p^\lambda(\omega) \supset L^p(\omega)$ holds. Nevertheless, not any weight that satisfies $\omega \in A_2 \cap RD_\nu$ for some $\nu > 2$ is necessarily lower Ahlfors, for example $\omega(x) = |x|^{d-\beta}$ with $d < \beta < 2d$. To check that is not lower Ahlfors we just observe that for $|x| > 2$ we have $\omega(B(x,1)) \simeq |x|^{d-\beta}$, which goes to zero when $|x|$ tends to infinity. Since we do not want to strengthen our assumptions on ω, the following question arises: are there non-trivial functions in the above spaces?. We shall see that, in fact, $M_p^\nu(\omega)$ contains many functions if we just assume that $\omega \in RD_\nu$ and is doubling.

Proposition 2. *Bounded functions with compact support belong to $M_p^\nu(\omega)$ as long as $\omega \in RD_\nu$ and is doubling.*

Proof. We may assume that $suppf \subset B(0,R)$ with $R \geq 1$. Let $B = B(x,r)$ be a ball. To check that (8) holds we consider different cases.

Case 1: If $|x| > 2R$ and $r < |x|/2$, $B(x,r) \cap suppf = \emptyset$ and $\int_B |f|^p\omega = 0$.

Case 2 : If $|x| > 2R$ and $r \geq |x|/2$ we have $B(x,r) \subset B(0,3r) \subset B(x.5r)$ and hence $\omega((B(x,r)) \simeq \omega(B(0,r)) \geq Cr^\nu \omega(B(0,1))$, where we used that ω is doubling and in RD_ν. In this way

$$\frac{r^\nu}{\omega(B(x,r))} \int_{B(x,r)} |f|^p \omega \leq C \int_{B(0,3r)} |f|^p \omega \leq C\| \; f\|_{L^p(\omega)}^p.$$

Case 3: If $|x| \leq 2R$ and $r \leq R$ we just use that f is bounded.

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Case 4: If $|x| \leq 2R$ and $r > R \geq 1$ we have $\omega(B(x,r)) \geq Cr^\nu\, \omega(B(x,1))$, and then

$$\frac{r^\nu}{\omega(B(x,r))}\int_{B(x,r)} |f|^p\ \omega\ \leq \frac{C}{\omega(B(x,1))}||f||^p_{L^p(\omega)}.$$

However, being $\omega(B(x,1))$ a positive and continuous function of x, we have that,for some constant $c>0$, $\omega(B(x,1)) \geq c$ over the compact set $\{|x| \leq 2R\}$.

□

Remark 4. Let us notice that when $\omega = 1$ and $p = 1$ the above spaces for $0 < \lambda < d$ turn out to be the Morrey spaces as appearing in [A] (see also [M]) and it is just L^1 for $\lambda = d$. If we do not have any information on the weight probably many of the spaces might be trivial. The above result shows that when $\omega \in RD_\nu$ that is not the case for the specific spaces $M_p^\nu(\omega)$. Moreover, since $\omega \in RD_\nu$ implies $\omega \in RD_\lambda$ for $0 < \lambda < \nu$, we will have that bounded functions with compact support also belong to $M_p^\lambda(\omega)$. Nevertheless we remind that there may be not an optimal exponent for the reverse doubling condition.

Now we turn our attention to the boundedness of J_α. The result we state could be derived as a consequence of continuity properties of a larger family of operators over more general spaces given in [SGN]. Nevertheless, in our case those results are quite simple to obtain, so for the sake of completeness, we sketch their proof here.

The first step, interesting in itself, is to show that the maximal function $\mathcal{M}_\omega$ also preserves $M_p^\lambda(\omega)$, $1 < p \leq \infty$. We state such result and outline its proof.

Proposition 3. *Let ω be a doubling weight. Then for any $\lambda > 0$ and $1 < p \leq \infty$, the operator $\mathcal{M}_\omega$ is bounded on $M_p^\lambda(\omega)$.*

Proof. To check (8) for $\mathcal{M}_\omega f$, pick a ball $B = B(x,r)$ and divide f as $f_0 = f\chi_{B(x,2r)}$ and $f_k = f\chi_{B(x,2^{k+1}r)\setminus B(x,2^k r)}$ for $k \geq 1$. Let us bound each term.

For $\mathcal{M}_\omega f_0$ we just use the L^p- boundedness of $\mathcal{M}_\omega$.

Regarding any $\mathcal{M}_\omega f_k$ we use that for any $y \in B(x,r)$ is almost a constant, namely,

$$\begin{aligned}\mathcal{M}_\omega f_k(y) &\simeq \frac{1}{\omega(B(x,2^{k+4}r))}\int_{B(x,2^{k+4}r)} |f|\ \omega \\ &\leq \left(\frac{1}{\omega(B(x,2^{k+4}r))}\int_{B(x,2^{k+4}r)} |f|^p\ \omega\right)^{1/p} \\ &\leq C\left(2^k r\right)^{-\lambda/p}\|f\|_{M_p^\lambda(\omega)}.\end{aligned}$$

Therefore

$$\frac{r^\lambda}{\omega(B)}\int_B |\mathcal{M}_\omega f_k|^p\omega \leq C2^{-k\lambda}\|f\|^p_{M_p^\lambda(\omega)}.$$

Adding together all the estimates we arrive at the desired conclusion for $p < \infty$. The case $p = \infty$ is obvious since the space is just L^∞ by definition. □

As an immediate consequence of the above proposition and the pointwise inequalities $T^*f(x) \leq S^*f(x) \leq \mathcal{M}f_\omega(x)$, we obtain more boundedness results for the maximal operators.

Corollary 1. *Under the same assumptions made in Theorem 1, the operators S^* and T^* are bounded on $M_p^\lambda(\omega)$ for any $\lambda > 0$ and $1 < p \leq \infty$.*

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Now we state and prove the corresponding boundedness results for J_α.

Theorem 2. *Let ω be a doubling weight. Given $\alpha > 0$ and $\lambda > \alpha$, the fractional operator J_α is bounded from $M_p^\lambda(\omega)$ into $M_s^\lambda(\omega)$ for $1 < p < \lambda/\alpha$ and $1/s = 1/p - \alpha/\lambda$.*

Proof. The idea is to get a kind of Hedberg's inequality involving the $M_p^\lambda(\omega)$-norm as in [H]. More precisely,

$$|J_\alpha f(x)| \leq C \, \|f\|_{M_p^\lambda(\omega)}^{\alpha p/\lambda} \, (\mathcal{M}_\omega f(x))^{1-\alpha p/\lambda} ,$$

for almost all x.

The proof of the above inequality follows the same steps than the classical one, splitting the integral into $B(x,R)$ and its complement and dividing each region into rings of thickness $2^{-k}R$ and $2^k R$ respectively. In this manner, the first piece gets bounded by $R^\alpha \mathcal{M}_\omega f(x)$ and the second by $R^{\alpha-\lambda/p}\|f\|_{M_p^\lambda(\omega)}$. Now it is just a matter of minimizing with respect to R to obtain the stated inequality.

With this tool at hand we easily get

$$\|J_\alpha f\|_{M_s^\lambda(\omega)} \leq C \|f\|_{M_p^\lambda(\omega)}^{\alpha p/\lambda} \|(\mathcal{M}_\omega f)^{1-\alpha p/\lambda}\|_{M_s^\lambda(\omega)}.$$

Observe that if ϵ is such that $\epsilon s \geq 1$, $\|g^\epsilon\|_{M_s^\lambda(\omega)} = \|g\|_{M_{\epsilon s}^\lambda}^\epsilon$. Therefore the last factor above becomes $\|(\mathcal{M}_\omega f)\|_{M_p^\lambda(\omega)}$, since $s(1 - \alpha p/\lambda) = p$. Now, Proposition 3 gives the desired result. □

As a consequence we get the following boundedness results for the negative powers of $\mathcal{L}_0$ and $\mathcal{L}$. Notice that we have to ask more assumptions on the weight to make sure that the kernels of these operators are bounded by the kernel of J_α but, at the same time, such assumptions guarantee that $M_p^\lambda(\omega)$ are non trivial spaces.

Theorem 3. *Let ω be a weight in A_2 such that $\omega \in RD_\nu \cap D_\gamma$ for some $\nu > 2$. Then, given $\alpha > 0$, for any λ such that $\alpha < \lambda \leq \nu$ we have:*

- *(a) $\mathcal{L}_0^{-\alpha/2}$ maps continuously $M_p^\lambda(\omega)$ into $M_s^\lambda(\omega)$ provided $1 < p < \lambda/\alpha$ and $1/s = 1/p - \alpha/\lambda$.*
- *(b) Further, for $V \in RH_q$ with $q > \gamma/2$, $\mathcal{L}^{-\alpha/2}$ maps continuously $M_p^\lambda(\omega)$ into $M_s^\lambda(\omega)$ for $1 < p < \lambda/\alpha$ and $1/s = 1/p - \alpha/\lambda$.*

Remark 5. Although we proved that the kernel K_α has a stronger decay than H_α, at this point we do not obtain any better results. Anyway, we will go back to this matter since this feature will be needed in what follows, for working with the mixed operators, and also in the last section, when we look at the behaviour of fractional integrals on regularity spaces.

Before ending this section we take a look to the mixed operators $\mathcal{L}^{-\alpha/2}V^{-\sigma/2}$. We will work out our results for the case $V \in RH_\infty$, that is, replacing the q-average by the supremum, more precisely, if there is a constant C such that for any ball B

$$\sup_B \omega \leq \frac{C}{\omega(B)} \int_B V \, \omega \, .$$

Clearly we are requiring a stronger condition, but with that assumption we can deal, at this instance, with those operators as well as their adjoints.

As in the non-degenerate Schrödinger case, the mixed operators $\mathcal{L}^{-\alpha/2}V^{-\sigma/2}$ can be bounded by an appropriate fractional maximal operator when $\alpha > \sigma$ or just

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by the Hardy-Littlewood maximal function when $\alpha = \sigma$. We refer the reader to [S]. Anyway we are going to pursue a slightly different path, taking advantage of the results we already proved.

Theorem 4. *Let ω be a weight in A_2 such that $\omega \in RD_\nu \cap D_\gamma$ for some $\nu > 2$ and V belonging to RH_∞. Let α and σ such that $0 < \sigma \leq \alpha < \nu$. Then, for any λ with $\alpha - \sigma < \lambda \leq \nu$, the operators $\mathcal{L}^{-\alpha/2}V^{\sigma/2}$ and $V^{\sigma/2}\mathcal{L}^{-\alpha/2}$ map continuously:*

- *(a) $M_p^\lambda(\omega)$ into $M_s^\lambda(\omega)$ for $1 < p < \lambda/(\alpha - \sigma)$ and $1/s = 1/p - (\alpha - \sigma)/\lambda$.*
- *(b) $M^\lambda_{\lambda/(\alpha-\sigma)}(\omega)$ into $M^\lambda_\infty(\omega) = L^\infty$.*
- *(c) $L^p(\omega)$ into itself when $\alpha = \sigma$ and $1 < p \leq \infty$.*

Proof. By Lemma 2 part b), it is immediate that for $0 < \alpha < \nu$ the kernel $K_{\alpha,\sigma}$ of $\mathcal{L}^{-\alpha/2}V^{\sigma/2}$ is bounded by

$$K_{\alpha,\sigma}(x, y) \leq C_N \frac{|x - y|^\alpha}{\omega(B(x, |x - y)|)} \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N} V^{\sigma/2}(y).$$

But, since $V \in RH_\infty$, taking the ball $B(y, \rho(y))$ in the definition, we easily get $V(y) \leq \rho^{-2}(y)$. Moreover, from the left inequality in (3), we have

$$\rho^{-1}(y) \leq C\rho^{-1}(x) \left(1 + \frac{|x - y|}{\rho(x)}\right)^{N_0}.$$

Then, we may replace $\rho(y)$ by $\rho(x)$ in the above estimate of the kernel changing the exponent $-N$ by a different one. However, since N is any positive number we call it again N. In this way we get

$$K_{\alpha,\sigma}(x, y) \leq C_N \frac{|x - y|^\alpha \rho^{-\sigma}(x)}{\omega(B(x, |x - y)|)} \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N}. \tag{9}$$

Assume first that $\alpha > \sigma$. In this case, multiplying and dividing by $|x - y|^\sigma$ and using the decay for $N = \sigma$ we are led to

$$\mathcal{L}^{-\alpha/2}V^{\sigma/2}f(x) \leq CJ_{\alpha-\sigma}f(x). \tag{10}$$

Then, the statement in the first item follows from Theorem 2.

For the remaining cases, going back to 9, we may write

$$\mathcal{L}^{-\alpha/2}V^{\sigma/2}f(x) \leq C_N \int \frac{|x - y|^\alpha \rho^{-\sigma}(x)}{\omega(B(x, |x - y|))} \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N} |f(y)|\, \omega(y)dy.$$

Now, splitting the above integral in annulus centered at x and of size $2^k\rho(x)$ with $-\infty < k < \infty$, we are lead to

$$C_N \sum_{-\infty}^{\infty} 2^{k\sigma} \left(1 + 2^k\right)^{-N} \frac{\left(2^k\rho(x)\right)^{\alpha-\sigma}}{\omega(B(x, 2^k\rho(x)))} \int_{B(x,2^k\rho(x))} |f|\, \omega.$$

Then, for $\alpha > \sigma$ and $p = \lambda/(\alpha - \sigma)$, for each k we apply Hölder inequality to obtain

$$\frac{\left(2^k\rho(x)\right)^{\alpha-\beta}}{\omega(B(x, 2^k\rho(x)))} \int_{B(x,2^k\rho(x))} |f|\, \omega \leq \|f\|_{M^\lambda_{\lambda/(\alpha-\sigma)}(\omega)}.$$

Inserting this estimate in the above inequality we get item b) since the series is convergent.

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Next, when $\alpha = \sigma$ we bound the averages by the maximal function, leading to

$$\mathcal{L}^{-\alpha/2}V^{\alpha/2}f(x) \leq C\mathcal{M}_\omega f(x). \tag{11}$$

Therefore the claims concerning to $\mathcal{L}^{-\alpha/2}V^{\alpha/2}$ follow from the properties of $\mathcal{M}_\omega$.

Finally all the statements about the adjoints operators are immediate once we notice that their kernels are also bounded by the right hand side of (9). □

Remark 6. From (9) it follows easily that

$$K_{\alpha,\sigma}(x,y) \leq C_N \frac{|x-y|^{\alpha-\sigma}}{\omega(B(x,|x-y|))}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}, \tag{12}$$

so, based on Lemma 2, we would expect the operators $\mathcal{L}^{-\alpha/2}V^{\sigma/2}$ for $\alpha > \sigma$ to behave as $\mathcal{L}^{-(\alpha-\sigma)/2}$. However item b) reveals that mixed operators are slightly better.

4. Regularity results for S^* and $\mathcal{H}_\alpha$

In this section we prove boundedness on regularity spaces for the operators associated to the degenerate elliptic differential operator $\mathcal{L}_0$. As expected, these results will be used later in analyzing regularity of the corresponding operators in the Schrödinger setting.

Regarding the regularity spaces, it is natural to consider BMO and Lipschitz spaces associated to the space of homogeneous type $(\mathbb{R}^d, |.|, \omega dx)$. We recall that for any $0 \leq \beta < 1$ the space $BMO^\beta(\omega)$ is defined as the set of those functions in $L^1_{loc}(\omega)$ such that

$$\frac{1}{\omega(B)}\int_B |f - f_B|\,\omega \leq Cr^\beta,$$

for any ball $B = B(x,r)$ and where f_B stands for the average of f with respect to ωdx. We recall also that, in order to check that a function belongs to these spaces, it is enough to find any constant c_B, instead of f_B, such that the above inequality holds. Another remark to have in mind is that the above quantity is not a norm but a seminorm and we have to identify functions differing by a constant to make it a normed space. Nevertheless we will use the notation $\|f\|_{BMO^\beta}$ to indicate the least constant C above. For shortness we will not write explicitly the weight in our notation. We think that it will be clear enough from the context.

As it is well known, for the case $\beta = 0$, $BMO(\omega)$, as defined above, coincides with the classical BMO for $\omega \in A_\infty$ (see [MW]). Furthermore, for $\beta > 0$ and ω a doubling weight, they can be identified with their pointwise versions, i.e. those functions f satisfying

$$|f(x) - f(y)| \leq C|x-y|^\beta,$$

and here the weight ω plays no role. So, again, they coincide with integral Lipschitz spaces with respect to the Lebesgue measure, but now just for a doubling weight. Often, we will use the ω-integral versions, specially when we want to work in a unified way, treating all spaces simultaneously. We emphasize again that we will be using f_B to mean the average over B with respect to the measure ωdx and we shall explicitly indicate if a different meaning is given.

We start by considering the maximal semigroup operator S^*. We recall that it is defined as

$$S^*f(x) = \sup_{t>0} \left| \int_{\mathbb{R}^d} h_t(x,y)f(y)\omega(y)dy \right|.$$

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In this case we will give a result a little bit weaker than that for the fractional integral but it will be enough to our future purposes. More precisely, we will analyze the behaviour of S^* over the subspaces $BMO_0^\beta(\omega) = BMO^\beta(\omega) \cap L^\infty$ equipped with the sum of both, the norm and seminorm. The main advantage of this restriction is that, according to Theorem 1, $S^* f$ is a bounded function and hence locally integrable and so there is not need of modifying the definition of the operator. Notice that when $\beta = 0$ the new space is just L^∞. Let us also point out that since, the semigroup comes from the differential operator $\mathcal{L}_0$, we have $S_t 1 \equiv 1$ for any $t > 0$. We recall that the kernel h_t satisfies the smoothness condition with exponent η given in Lemma 1 part b). With this notation we state the following result.

Theorem 5. *Let ω be a weight in $A_2 \cap RD_\nu$ with $\nu > 2$. Then, the maximal operator S^* is bounded on $BMO_0^\beta(\omega)$ for any $0 \le \beta < \eta$.*

Proof. We follow the same steps as in the classical case. First, the result when $\beta = 0$ is just the boundedness on L^∞ contained in Theorem 1. Second, for $\beta > 0$ we may use the pointwise description. So, let $f \in BMO_0^\beta(\omega)$. As we said, $S^* f$ is finite a.e.. Let us take x and z two such points. Then,

$$|S^* f(x) - S^* f(z)| \le \sup_{t>0} \left| \int_{\mathbb{R}^d} [h_t(x,y) - h_t(z,y)] f(y) \omega(y) dy \right|.$$

Next, since $S_t 1(x) = S_t 1(z)$ we have that when f is constant the above integral is zero and therefore we may change, inside the integral, $f(y)$ by the difference $f(y) - f(x)$. Now, we split the integration into two pieces $B = B(x, 4|x-z|)$ and its complement. For the first integral we bound the difference by the sum, giving rise to two terms, that is,

$$\sup_{t>0} \int_B h_t(x,y)|f(y) - f(x)|\omega(y) dy + \sup_{t>0} \int_B h_t(z,y)|f(y) - f(x)|\omega(y) dy.$$

We do the first one and the other follows similarly once we observe that $B \subset B(z, 5|x-z|)$. Clearly, using that $f \in BMO^\beta(\omega)$, the first term is bounded by

$$C \|f\|_{BMO^\beta} |x-z|^\beta \sup_{t>0} \int_B h_t(x,y)\omega(y) dy \le C \|f\|_{BMO^\beta} |x-z|^\beta.$$

As for the integral over B^c, since $y \in B^c$ implies $|x - z| \le |x - y|/4$, we may use Lemma 1 part b). More precisely,

$$, |h_t(x,y) - h_t(z,y)| \le C \left(\frac{|x-z|}{\sqrt{t}} \right)^\eta \frac{e^{-\frac{|x-y|^2}{ct}}}{\omega(B(x,\sqrt{t}))}. \tag{13}$$

Now, our aim is to majorize the right hand side independently of t. To do so we use the estimate $s^\epsilon e^{-s^2} \le C e^{-as^2}$ for some constants a and C with $\epsilon = \eta$ to get the bound

$$C \left(\frac{|x-z|}{|x-y|} \right)^\eta \frac{e^{-\frac{|x-y|^2}{c't}}}{\omega(B(x,\sqrt{t}))}.$$

But, by the reverse doubling and doubling conditions we obtain for $|x - y| \le \sqrt{t}$

$$\omega(B(x,\sqrt{t})) \ge c_1 \left(\frac{\sqrt{t}}{|x-y|} \right)^\nu \omega(B(x, |x-y|)),$$

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and for $|x-y| > \sqrt{t}$

$$\omega(B(x,\sqrt{t})) \geq c_1 \left(\frac{\sqrt{t}}{|x-y|}\right)^{\gamma} \omega(B(x,|x-y|)),$$

where γ is the doubling exponent. In any case, using once again the inequality $s^{\epsilon}e^{-s^2} \leq Ce^{-as^2}$, we arrive at

$$|h_t(x,y) - h_t(z,y)| \leq C \left(\frac{|x-z|}{|x-y|}\right)^{\eta} \frac{e^{-\frac{|x-y|^2}{\tilde{c}t}}}{\omega(B(x,|x-y|))}. \tag{14}$$

After majorizing the exponential by one, we plug this estimate in the integral over B^c. Using again $f \in BMO^{\beta}(\omega)$, we see that it is bounded by

$$C\|f\|_{BMO^{\beta}}|x-z|^{\eta}\int_{B^c} \frac{|x-y|^{\beta-\eta}}{\omega(B(x,|x-y|))}\,\omega(y)dy.$$

Splitting the integral in annulus $2^{k+1}B \setminus 2^kB$ with $k \geq 2$, we get

$$C\|f\|_{BMO^{\beta}}|x-z|^{\eta}\sum_k (2^k|x-z|)^{\beta-\eta}.$$

Since $\beta < \eta$ the sum is convergent we obtain the desired estimate.

Finally, using again $\|S^*f\|_{\infty} \leq C\|f\|_{\infty}$, we complete the proof of the theorem. □

Remark 7. Looking at the proof, notice that in fact we have obtained the stronger inequality $\sup_{t>0}|S_tf(x) - S_tf(z)| \leq C\|f\|_{BMO^{\beta}}|x-z|^{\beta}$. That is, when estimating the oscillation the L^{∞}-norm of f does not play any role.

Now we turn our attention to the fractional operator $\mathcal{L}_0^{-\alpha/2}$. In Lemma 2 we already estimated the kernel size for $0 < \alpha < \nu$. However, to deal with these operators acting on functions in $M_p^{\lambda}(\omega)$ with $p \geq \lambda/\alpha$ or in $BMO^{\beta}(\omega)$, we need not only size but also some smoothness of H_{α}, property that will be derived from the smoothness of the semigroup kernel, contained in Lemma 1.

Lemma 3. *Let ω be a weight in $A_2 \cap RD_{\nu}$ with $\nu > 2$. Then we have*

$$|H_{\alpha}(x,y) - H_{\alpha}(z,y)| \leq C \left(\frac{|x-z|}{|x-y|}\right)^{\eta} \frac{|x-y|^{\alpha}}{\omega(B(x,|x-y|))}, \tag{15}$$

provided $|x-z| \leq |x-y|/4$.

Proof. To estimate the left hand side above, using expression 4, it is enough to bound

$$\int_0^{\infty} |h_t(x,y) - h_t(z,y)|\, t^{\alpha/2}\frac{dt}{t}$$

that, according to 1, is bounded by

$$\left(\frac{|x-z|}{|x-y|}\right)^{\eta}\int_0^{\infty}\left(\frac{|x-y|}{\sqrt{t}}\right)^{\eta}\frac{e^{-\frac{|x-y|^2}{\tilde{c}t}}}{\omega(B(x,\sqrt{t}))}\,t^{\alpha/2}\frac{dt}{t}.$$

As in Lemma 2, performing the change of variable $t = |x-y|^2s$ we get

$$|x-y|^{\alpha}\left(\frac{|x-z|}{|x-y|}\right)^{\eta}\int_0^{\infty}\frac{s^{(\alpha-\eta)/2}e^{-1/\tilde{c}s}}{\omega(B(x,\sqrt{s}|x-y|))}\frac{ds}{s}.$$

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Now, splitting the integral into $(0,1)\cup[1,\infty)$ and using that $\omega\in D_\gamma$ and $\omega\in RD_\nu$ respectively, we get the bound

$$\left(\frac{|x-z|}{|x-y|}\right)^\eta \frac{|x-y|^\alpha}{\omega(B(x,|x-y)|))}\left(\int_0^1 s^{(\alpha-\eta-\gamma)/2}e^{-1/\tilde{c}s}\frac{ds}{s}+\int_1^\infty s^{(\alpha-\eta-\nu)/2}\frac{ds}{s}\right),$$

and both integrals are convergent since $\alpha<\nu$, completing the proof of the Lemma. □

Now we are ready to state and prove new results for the operator $\mathcal{L}_0^{-\alpha/2}$. First we are going to modify our operator so it makes sense for all functions in $M_p^\lambda(\omega)$ with $\lambda/\alpha\le p<\lambda/(\alpha-\eta)^+$, which means $\lambda/\alpha\le p<\infty$ when $\alpha\le\eta$ and $\lambda/\alpha\le p<\lambda/(\alpha-\eta)$, otherwise. In fact, our definition works on functions in the larger spaces $M_1^{\alpha-\beta}(\omega)$, for any $0\le\beta<\alpha$ and $\beta<\eta$. Clearly these spaces contain any of the aforementioned ones since, by Hölder's inequality, $M_p^\lambda(\omega)\subset M_1^{\lambda/p}(\omega)$ and we $\beta=\alpha-\lambda/p$ satifies the above conditions for $\lambda/\alpha\le p<\lambda/(\alpha-\eta)^+$. Then, we introduce for $f\in M_1^{\alpha-\beta}(\omega)$ and $B_1=B(0,1)$

$$\tilde{\mathcal{H}}_\alpha f(x)=\int[H_\alpha(x,y)-H_\alpha(0,y)\chi_{B_1^c}]f(y)\omega(y)dy.$$

First we notice that the right hand side gives a locally integrable function. Clearly, it will be enough to show integrability in balls $B(0,R)$ with $R\ge 2$. In fact, from the definition, we have

$$\tilde{\mathcal{H}}_\alpha f(x)=\int_{B_1}H_\alpha(x,y)f(y)\omega(y)dy+\int_{B_1^c}[H_\alpha(x,y)-H_\alpha(0,y)]f(y)\omega(y)dy.$$

Let us call the above terms $I(x)$ and $II(x)$, respectively. Then, by the Fubini-Tonelli theorem and (6), we have

$$\int_{B(0,R)}|I(x)|\omega(x)dx\le\int_{B_1}|f(y)|\left(\int_{B(y,2R)}\frac{|x-y|^\alpha}{\omega(B(x,|x-y|))}\omega(x)dx\right)\omega(y)dy,$$

since $B(0,R)\subset B(y,2R)$. Now, splitting in annulus of thickness $2^{-k}R$, the inner integral is bounded by CR^α and hence, being f locally integrable with respect to the weihgt ω, the above quantity is finite.

Regarding the integrability of II, we observe that

$$|II(x)|\le\int_{B(0,R)}H_\alpha(x,y)|f(y)|\omega(y)dy+\int_{B(0,R)\setminus B_1}H_\alpha(0,y)|f(y)|\omega(y)dy$$
$$+\int_{B(0,R)^c}|H_\alpha(x,y)-H_\alpha(0,y)||f(y)|\omega(y)dy=II_1(x)+II_2(x)+II_3(x).$$

The local integrability of II_1 follows as for I. Regarding II_2, we notice that $H_\alpha(0,y)\le C\frac{R^\alpha}{\omega(B(0,1))}$ for $y\in B(0,R)\setminus B_1$ and then $II(x)$ is a bounded function. Finally, to bound II_3, we use Lemma 3 obtaining

$$II_3(x)\le C|x|^\eta\int_{B(0,R)^c}\frac{|y|^{\alpha-\eta}}{\omega(B(0,|y|))}|f(y)|\omega(y)dy.$$

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To see that the integral is finite we must use that $f \in M_1^{\alpha-\beta}(\omega)$. In fact, dividing the integral in annulus $B(0, 2^{k+1}R) \setminus B(0, 2^k R)$ we have

$$\begin{aligned} II_3(x) &\leq C|x|^\eta \sum_{k\geq 0} \frac{(2^k R)^{\alpha-\eta}}{\omega(B(0,2^k R))} \int_{B(0,2^{k+1}R)} |f(y)|\omega(y)dy \\ &\leq C\|f\|_{M_1^{\alpha-\beta}} |x|^\eta R^{\beta-\eta} \sum_{k\geq 0} 2^{k(\beta-\eta)}. \end{aligned}$$

Since $\beta < \eta$ the series converges and II_3 is also a locally integrable function with respect to the weight ω.

Having proved the good definition of our modified operator $\tilde{\mathcal{H}}_\alpha$ we establish continuity properties.

Theorem 6. *Let ω be weight in $A_2 \cap RD_\nu$ with $\nu > 2$. Then, for $0 < \alpha < \nu$, the operator $\tilde{\mathcal{H}}_\alpha$ maps continuously $M_1^{\alpha-\beta}(\omega)$ into $BMO^\beta(\omega)$ for any given β with $0 \leq \beta < \min\{\eta, \alpha\}$. Furthermore, when f is also of compact support $\tilde{\mathcal{H}}_\alpha f$ coincides with $\mathcal{L}_0^{-\alpha/2} f$ as functions in $BMO^\beta(\omega)$.*

Proof. We know that $\tilde{\mathcal{H}}_\alpha f$ is a locally integrable function and hence finite a.e.. Moreover, following a similar argument, we will show that for any given ball $B = B(x_0, r)$, setting $\tilde{B} = 2B$, we have that

$$a_B = \int_{\mathbb{R}^d} [H_\alpha(x_0, y)\chi_{\tilde{B}^c} - H_\alpha(0, y)\chi_{B_1{}^c}] f(y)\omega(y)dy$$

is a finite constant.

In fact, take a ball $B^* = B(x_0, R)$ with R large enough so that it contains $2B_1 \cup \tilde{B}$, for example we may choose $R = 2(|x_0| + r + 1)$. Then

$$\begin{aligned} |a_B| \leq &\int_{B^*\setminus\tilde{B}} H_\alpha(x_0, y)|f(y)|\omega(y)dy + \int_{B^*\setminus B_1} H_\alpha(0, y)|f(y)|\omega(y)dy \\ &+ \int_{B^{*c}} |H_\alpha(x_0, y) - H_\alpha(0, y)||f(y)|\omega(y)dy = A_1 + A_2 + A_3. \end{aligned}$$

For the first two terms we have that the kernel is bounded since, in the first integral $2r \leq |x_0 - y| < R$, and in the second $2 \leq |y| \leq |x_0| + R$. Therefore the finiteness of both terms follows using the local integrability of f. Regarding the last term, we have $|x_0 - y| \geq R > 2|x_0|$ and so Lemma 3 can be applied. Proceeding as for II_3 above we obtain for $f \in M_1^{\alpha-\beta}(\omega)$

$$A_3 \leq C\|f\|_{M_1^{\alpha-\beta}(\omega)} |x_0|^\eta R^{\beta-\eta} < \infty.$$

In this way we have proved that for any ball B

(16)

$$\begin{aligned} \tilde{\mathcal{H}}_\alpha f(x) &= \int_{\tilde{B}} H_\alpha(x, y) f(y)\omega(y)dy + \int_{\tilde{B}^c} [H_\alpha(x, y) - H_\alpha(x_0, y)] f(y)\omega(y)dy + a_B \\ &= J_1(x) + J_2(x) + a_B. \end{aligned}$$

Now we show that $\tilde{\mathcal{H}}_\alpha f$ belongs to $BMO^\beta(\omega)$. We fix a ball $B = B(x_0, r)$ and we use the above expression for that specific ball. First we integrate $J_1(x)$ over B and we argue as for $I(x)$ above, that is, changing the order of integration and evaluating the inner integral. In this way we arrive at

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$$\int_B |J_1(x)|\omega(x)dx \le Cr^\alpha \int_{\tilde{B}} |f(y)|\omega(y)dy \le C\|f\|_{M_1^{\alpha-\beta}(\omega)} r^\beta \omega(B).$$

Next, to calculate the oscillation over B of the remaining terms we may subtract a constant, for example a_B. Therefore we only have to estimate the integral of $|J_2(x)|$. We apply again Lemma 3. In this manner, since $|x-y| \simeq |x_0-y|$ and $\omega(B(x_0,|x_0-y|)) \simeq \omega(B(x,|x-y|))$, we get

$$|J_2(x)| \le C|x_0-x|^\eta \int_{\tilde{B}^c} \frac{|x_0-y|^{\alpha-\eta}}{\omega(B(x_0,|x_0-y|))} |f(y)|\omega(y)dy.$$

Splitting the integral in annulus $2^{k+1}B \setminus 2^k B$ with $k>1$ we obtain

$$|J_2(x)| \le C\|f\|_{M_1^{\alpha-\beta}(\omega)} r^\beta.$$

Averaging with respect to ωdx we obtain the desired estimate. Finally, notice that if f has compact support we may take in (16) a ball B large enough so that the second term is zero. This means that

$$\tilde{\mathcal{H}}_\alpha f(x) = \int_{\mathbb{R}^d} H_\alpha(x,y) f(y)\omega(y)dy + a_B,$$

and hence $\tilde{\mathcal{H}}_\alpha f$ equals to $\mathcal{L}_0^{-\alpha/2} f$ as functions in $BMO^\beta(\omega)$. □

Remark 8. In the non-degenerate case of $\omega \equiv 1$, it is possible to prove that for any $s>1$, weak-$L^s \subset M_1^{d/s}$ (see for example Lemma 4.1 [HSV] with $w \equiv 1$). Therefore our result above recovers the boundedness of the modified classical fractional integral of order α from weak-$L^{d/\alpha}$ into BMO or, more generally, from weak-L^p into BMO^β for $p \ge d/\alpha$ and $\beta = \alpha - p/d < 1$ (see Theorems 1.1 and 1.2 in [GV] and Theorem 2.5 in[HSV]). Moreover, we get an improved version since, as it was proved in [GHI], the inclusion of weak-L^s into $M_1^{d/s}$ is strict.

Our next step concerns with the behaviour of $\mathcal{L}_0^{-\alpha/2}$ on $BMO^\beta(\omega)$ spaces. As in the previous case, we have to give a different definition for the operator to make sense on this kind of functions and having in mind that whenever they have compact support both definitions must coincide upon a constant since that is the meaning of equality in $BMO^\beta(\omega)$ spaces. We have estimated size and smoothness of the kernel H_α. Now we reveal a further property. Since for the semigroup $\{S_t\}_{t>0}$ we know that $\int h_t(x,y)\omega(y)dy = 1$ for any x (see for example [St]), we have

$$\int_0^\infty (\int [h_t(x,y) - h_t(x_0,y)]\,\omega(y)dy)\, t^{\alpha/2}\,\frac{dt}{t} = 0.$$

Now, if we take absolute value inside and reverse the order of integration, it is easy to check that the iterated integral is finite. In fact, if we divide the integration on $\mathbb{R}^d$ into $B(x,2|x-x_0|)$ and its complement, the first piece is bounded by

$$\int_{B(x,2|x-x_0|)} H_\alpha(x,y)\omega(y)dy + \int_{B(x_0,3|x-x_0|)} H_\alpha(x_0,y)\omega(y)dy,$$

and both are finite in view of the size of H_α. To estimate the integration over the complement we use smoothness of h_t and it is exactly the calculation we made in

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Lemma 3, so we obtain the bound

$$C|x-x_0|^\eta \int_{B^c(x,2|x-x_0|)} \frac{|x-y|^{\alpha-\eta}}{\omega(B(x,|x-y|))}\omega(y)dy,$$

and the integral is finite if we assume $\alpha < \eta$.

Therefore the order of integration can be reversed obtaining

$$\int [H_\alpha(x,y) - H_\alpha(x_0,y)]\omega(y)dy = 0, \tag{17}$$

for $0 < \alpha < \eta$.

Now, given a function $f \in BMO^\beta(\omega)$, $0 \le \beta < \eta$, we fix some x_0 and define the following operator

$$\mathring{\mathcal{H}}_\alpha f(x) = \int [H_\alpha(x,y) - H_\alpha(x_0,y)]f(y)\omega(y)dy.$$

With this definition we state and prove the following result.

Theorem 7. *Let ω be a weight in $A_2 \cap RD_\nu$ with $\nu > 2$. Then, for $0 < \alpha < \eta$, the operator $\mathring{\mathcal{H}}_\alpha$ maps continuously $BMO^\beta(\omega)$ into $BMO^{\beta+\alpha}(\omega)$ for any given $\beta > 0$ such that $0 \le \beta + \alpha < \eta$. Furthermore, when f is also of compact support, $\mathring{\mathcal{H}}_\alpha f$ coincides with $\mathcal{L}_0^{-\alpha/2} f$ as functions in $BMO^{\beta+\alpha}(\omega)$.*

Proof. We begin observing that 17 allows us to substitute $f(y)$ by $f(y)-c$ inside the integral, therefore the definition is independent of the member of the equivalence class.

Next, we check that for $f \in BMO^\beta(\omega)$ with $0 < \alpha + \beta < \eta$, it defines a locally integrable function, in fact it is locally bounded. First notice that, given a ball B and $j \in \mathbb{Z}$, adding and subtracting intermediate averages, we get

$$\frac{1}{\omega(2^jB)} \int_{2^jB} |f|\omega \le \|f\|_{BMO^\beta} c(j,\beta)\, r^\beta + |f|_B, \tag{18}$$

where for $\beta > 0$ is either $c(j,\beta) = 2^{j\beta}$ when $j > 0$ and $c(j,\beta) = c$ for $j < 0$ or $c(j,\beta) = j$ when $\beta = 0$. Now we show that the integral in the definition converges absolutely for any pair x, x_0. Take $B = B(x,r)$ with $r = 2|x-x_0|$ and divide the integral in B and B^c. As usual, over B we majorize by the sum while in B^c we use the smoothness of the kernel. In this way,

$$\int_B H_\alpha(x,y)|f(y)|\omega(y)dy \le Cr^\alpha \sum_{j \le 0} 2^{j\alpha} \frac{1}{\omega(2^jB)} \int_{2^jB} |f|\omega\,.$$

Then, using (18) we get the bound

$$C|x-x_0|^\alpha \left(\|f\|_{BMO^\beta}|x-x_0|^\beta + |f|_B\right).$$

Noting that $|f|_{B(x,2|x-x_0|)}$ is a continuous function of x, that is also true for the above function and so our original integral is a locally bounded function. A similar argument holds for the other integral on B.

For the integral on B^c, in view of the smoothness of H_α, we have

$$\int_{B^c} |H_\alpha(x,y) - H_\alpha(x_0,y)||f(y)|\omega(y)dy \le C|x-x_0|^\eta \int_{B^c} \frac{|x-y|^{\alpha-\eta}}{\omega(B(x,|x-y|))}|f(y)|\omega(y)dy.$$

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Splitting the integrals in annulus $2^{j+1}B \setminus 2^jB$, $j > 0$, we get the bound

$$C|x-x_0|^\eta \sum_{j>0} \left(2^j|x-x_0|\right)^{\alpha-\eta} \frac{1}{\omega(2^jB)} \int_{2^jB} |f|\omega,$$

and applying (18) the last expression is bounded by

$$C|x-x_0|^\alpha (c|f|_{B(x,2|x-x_0|)} + |x-x_0|^\beta \|f\|_{BMO^\beta} \sum_{j>0} c(j,\beta) 2^{j(\alpha-\eta)}),$$

and the sum is convergent recalling that $\alpha+\beta<\eta$. Again, since this bound is a continuous function of x, our initial quantity is locally bounded.

Therefore we have proved that $\mathring{\mathcal{H}}_\alpha f$ is well defined and, in fact, finite for any x. Moreover, it is independent of the choice of x_0. If we take another point, say x_1 in the definition of $\mathring{\mathcal{H}}_\alpha$, since we showed that the integral is absolutely convergent, the difference between the two possible definitions is

$$\int [H_\alpha(x_1,y) - H_\alpha(x_0,y)] f(y)\omega(y)dy,$$

giving a finite constant.

Next we prove the continuity result. We may use the pointwise description in $BMO^{\beta+\alpha}(\omega)$ since $\alpha+\beta>0$. Let x and z be two points and set $B = B(x, 2|x-z|)$. As we observed, from (17) we can replaced f by $f - f_B$ in the definition of the operator. So we have to bound

$$\int |H_\alpha(x,y) - H_\alpha(z,y)||f(y) - f_B|\omega(y)dy = I_1 + I_2,$$

where I_1 and I_2 are the integrals over B and B^c respectively. This time we are going to use the estimate

$$\frac{1}{\omega(2^jB)} \int_{2^jB} |f - f_B|\,\omega \le c(j,\beta)\|f\|_{BMO^\beta}|x-z|^\beta, \tag{19}$$

where $c(j,\beta)$ has the same meaning as in (18). This inequality follows from just observing $|f - f_B| \le |f - f_{2^jB}| + \sum_{i=2}^{j} |f_{2^iB} - f_{2^{i-1}B}|$. Thus, for I_1 we proceed as before, majorizing the difference of the kernels by its sum. For each piece we decompose in rings arriving at

$$|x-z|^\alpha \sum_{j\le 0} 2^{j\alpha} \frac{1}{\omega(2^jB)} \int_{2^jB} |f-f_B|\omega \le C|x-z|^{\alpha+\beta}\|f\|_{BMO^\beta} \sum_{j\le 0} c(j,\beta)2^{j\alpha},$$

as we wished, since the series is convergent. The other integral over B is the same once we notice $B \subset B(z, 3|x-z|)$. Regarding I_2 we may apply the smoothness and proceed as above, using this time (19). In this way we get

$$I_2 \le C|x-z|^{\alpha+\beta}\|f\|_{BMO^\beta} \sum_{j>0} 2^{j(\alpha+\beta-\eta)}$$

and the series is convergent since $\alpha+\beta<\eta$. □

Remark 9. Notice that in the proof of Theorem 6, we just use the size and smoothness of the kernel and the doubling property of the weight. Therefore, a more general result could be obtained for an integral operator with kernel satisfying (6) and (15). In particular both conditions hold for K_α, the kernel of $\mathcal{L}^{-\alpha/2}$ as we shall see in the next section. Moreover, with a little of extra work, a similar result could be obtained in the context of a space of homogeneous type. Regarding Theorem 7

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we use an additional property, namely, the mean value zero for the difference of the kernel at distinct points, as stated in (17). Consequently, an analogous extension could be derived for a kernel having the three named properties. However, let us to remark that the third condition is not satisfied by the kernel of $\mathcal{L}^{-\alpha/2}$. Finally we point out that the reason to ask more assumptions on the weight ω in Theorems 6 and 7 is just to guarantee that $\mathcal{L}_0^{-\alpha/2}$ satisfies all the needed estimates.

5. Regularity results for operators related to $\mathcal{L}$

In this section we are going to deal with the degenerate Schrödinger case, analyzing the behaviour of the maximal operator of the semigroup, fractional integration as well as the mixed operators $\mathcal{L}^{-\alpha/2}V^{\sigma/2}$.

Following [MSTZ], we take the point of view of proving a general theorem concerning continuity on regularity spaces, and then we will apply it to the aforementioned operators. In doing so, the results of the previous section will be quite helpful.

First, let us introduce the suitable regularity spaces in this context. Associate to ω and V, we recalled in the introduction the definition of the critical radius function ρ whenever $V \in RH_q$ with $q > \gamma/2$ (see (2)). As in the non-degenerate case, given $0 \leq \beta < 1$, we define the space $BMO_\rho^\beta(\omega)$ as the locally integrable functions with respect to $\omega\, dx$ such that

$$\frac{1}{\omega(B)} \int_B |f - f_B|\, \omega \leq c r^\beta \tag{20}$$

for any ball $B = B(x, r)$ and moreover

$$\frac{1}{\omega(B(x,r))} \int_{B(x,r)} |f|\, \omega \leq C r^\beta \quad \text{if } r \geq \rho(x). \tag{21}$$

In this definition f_B denotes the average with respect to the measure ω. Even it is the same notation that for Lebesgue averages, most of the time we will use it with that meaning. Otherwise we shall point it out explicitly. The sum of the infima of the constants c and C actually gives a norm. Also, for a doubling weight, it can be proved that it is enough to ask the second condition only for critical balls, i.e., when $r = \rho(x)$, while the first suffices just for subcritical balls. Besides, as in the classical case, note that for $\beta = 0$, if we replace $\omega\, dx$ by the Lebesgue measure everywhere in the definition, we get the same space as long as $\omega \in A_\infty$. In fact, for the oscillation is just the result given in [MW] and the average can be obtained adding and subtracting the corresponding average. Since we are asking the weight to be in A_2 in the degenerate Schrodinger context, that characterization applies.

Also, when $\beta > 0$ and ω is doubling, our functions can be described by the following pointwise inequalities

$$|f(x) - f(z)| \leq C|x - y|^\beta \quad \text{if} \quad |x - y| < \rho(x)$$

and

$$|f(x)| \leq C\rho^\beta(x).$$

As a consequence, for $\beta > 0$ and ω doubling, the integral space $BMO_\rho^\beta(\omega)$ defined above also coincides with the integral version corresponding to $\omega \equiv 1$. Nevertheless, we shall keep the weight in the notation of the spaces since we will very often work with that characterization, even though we will omit it in the subscript of the norm.

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Another feature of these spaces is that for any $\beta \geq 0$, the following consequence of John-Nirenberg inequality is also valid in this context, namely

$$\left(\frac{1}{\omega(B)}\int_B |f-f_B|^p\omega\right)^{1/p} \leq C\|f\|_{BMO^\beta_\rho}\, r^\beta, \tag{22}$$

for any ball $B = B(x, r)$ and

$$\left(\frac{1}{\omega(B(x,r)}\int_{B(x,r)} |f|^p\omega\right)^{1/p} \leq C\|f\|_{BMO^\beta_\rho} r^\beta \tag{23}$$

for balls such that $r \geq \rho(x)$.

In fact (22) comes from the results about John-Nirenberg type inequalities on spaces of homogeneous type as given in [FPW]. As for (23) it is enough to prove it when $r = \rho(x)$ and that follows writing $f = (f - f_B) + f_B$ and applying Minkowski's inequality and (22).

Let us point out that, in order to define the above spaces and prove the stated properties, we only need ω to be a doubling weight, excepting for the identification between $BMO_\rho(\omega)$ and the corresponding space with $\omega \equiv 1$, that it will not be used in what follows. Also notice that the potential does not appear explicitly in the definition but hidden into the critical radius function ρ. As we mentioned, under the assumptions made on V, the associate ρ given by (2) satisfies the inequalities (3). Moreover, the latter property on ρ is all we need to work with these spaces.

Since our first goal is to obtain regularity results in a general framework, from now on we assume that we are given a doubling weight ω and a critical radius function ρ, i.e., a function $\rho : \mathbb{R}^d \mapsto \mathbb{R}^+$ satisfying (3). So, at this point, there will be not any mention neither to the potential nor to other assumptions on ω. In fact, they will be needed at the moment we want to apply our general theorem to concrete examples coming from the Schrödinger context.

In the next lemma we put together some technical properties of functions in these spaces that will be needed in the sequel.

Lemma 4. *Let ω be a doubling weight and $f \in BMO^\beta_\rho(\omega)$. Then we have*

(a) For any critical ball $B = B(x_0, \rho(x_0))$ and $k \geq 0$

$$\frac{1}{\omega(2^{-k}B)}\int_{2^{-k}B} |f|\,\omega \leq \|f\|_{BMO^\beta_\rho}\; c(k,\beta)\; \rho(x_0)^\beta, \tag{24}$$

with $c(k, \beta) = k$ when $\beta = 0$ and $c(k, \beta) = c$ when $0 < \beta < 1$.

(b) For any subcritical ball $B = B(x_0, r)$ with $r \leq \rho(x_0)$ and $k \geq 0$

$$\frac{1}{\omega(2^k B)}\int_{2^k B} |f-f_B|\,\omega \leq \|f\|_{BMO^\beta_\rho}\; a(k,\beta)\; r^\beta \tag{25}$$

with $a(k, \beta) = k$ when $\beta = 0$ and $a(k, \beta) = 2^{k\beta}$ when $0 < \beta < 1$.

Proof. For $\beta = 0$ let us observe that

$$|f| \leq |f - f_{2^{-k}B}| + |f_{2^{-k}B} - f_{2^{-k+1}B}| + ... + |f_{2^{-1}B} - f_B| + |f_B|.$$

Since for any $0 \leq j < k - 1$ we have

$$|f_{2^{-j-1}B} - f_{2^{-j}B}| \leq \frac{C}{\omega(2^{-j}B)}\int_{2^{-j}B} |f - f_{2^{-j}B}|\omega \leq C\|f\|_{BMO_\rho},$$

averaging over $2^{-k}B$ and using (21) we obtain the desired estimate.

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For the case $\beta > 0$, since ω is doubling, we may use the pointwise estimate $|f(x)| \leq \|f\|_{BMO^\beta_\rho}\, \rho(x)^\beta$ and, having in mind that $x \in 2^{-k}B(x_0, \rho(x_0))$ implies $\rho(x)) \simeq \rho(x_0))$, the conclusion is straightforward.

As for item b) we just recall that $BMO^\beta_\rho(\omega) \subset BMO^\beta(\omega)$ and then it follows from (19). □

Following the approach in [MSTZ], given a doubling weight and a critical radius function, we want to enclose the operators we are handling in a general class, so we can obtain regularity results for all of them simultaneously. Let us remind that one of the conditions they imposed in the non-degenerate case is the L^p-boundedness. However, as it was shown in section 2, there were cases where continuity on $L^p(\omega)$ might not hold so, somehow, we must change this hypothesis. Examples of such operators were $\mathcal{L}^{-\alpha/2}$ and $\mathcal{L}^{-\alpha/2}V^{\sigma/2}$ with $\alpha > \sigma$.

To be precise, given a doubling weight ω, a critical radius function ρ and a parameter $0 \leq \alpha$, we associate a class of operators that we call α-Schrödinger-Calderón-Zygmund with respect to the measure ωdx. We distinguish two cases:

Case $\alpha > 0$: T is an integral operator with respect to the measure ωdx, given by a kernel $K(x, y)$ that satisfies

(a) For any $N > 0$ there is a constant C_N such that

$$|K(x,y)| \leq C_N \frac{|x-y|^\alpha}{\omega(B(x,|x-y|))}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N}. \tag{26}$$

(b) There exists some $0 < \delta < 1$ such that

$$|K(x,y) - K(z,y)| \leq C \left(\frac{|x-z|}{|x-y|}\right)^\delta \frac{|x-y|^\alpha}{\omega(B(x,|x-y|))}. \tag{27}$$

provided $|x - z| < |x - y|/2$.

Case $\alpha = 0$: T is a linear bounded operator on $L^p(\omega)$, $1 < p < \infty$ having an associated kernel $K(x, y)$ in the sense that, for any $L^p(\omega)$-function with compact support

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y)\omega(y)dy \quad \text{for } x \notin supp(f).$$

Furthermore, K satisfies conditions a) and b) above with $\alpha = 0$.

Remark 10. The condition (27) involves certain regularity on the kernel outside the diagonal and, as it is easy to check, the factor 1/2 could be replaced by any different fraction $0 < \tau < 1$. More precisely, if a kernel satisfies (26) and (27) for $|x - z| < \tau|x - y|$, (27) also holds for $|x - z| < |x - y|/2$.

After giving this definition, we check that T is well defined for functions in $BMO^\beta_\rho(\omega)$.

First we deal with the case $\alpha > 0$. The size condition on the kernel allows to see that the integral

$$\int_{\mathbb{R}^d} K(x,y) f(y)\omega(y)dy$$

is absolutely convergent for almost any x and, moreover, it gives a locally integrable function. In fact, let $B = B(x_0, \rho(x_0))$ be a critical ball and let us split the above integral in $2B$ and its complement. For the first piece, since $x \in B$ implies $B \subset$

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$B(x, 3\rho(x_0)) = B_x$, using 26 and dividing into the annulus $2^{-k}B_x \setminus 2^{-k-1}B_x$ we have

$$\int_{2B} \frac{|x-y|^\alpha}{\omega(B(x,|x-y|))}|f(y)|\omega(y)dy \leq \sum_{k=0}^{\infty} \frac{(2^{-k}\rho(x))^\alpha}{\omega(2^{-k}B_x)} \int_{2^{-k}B_x} |f(y)|\omega(y)dy,$$

and, in view of (24), we get the bound $C\rho(x)^{\alpha+\beta}\|f\|_{BMO_\rho^\beta}$ for any $0 \leq \beta < 1$. Now, since $x \in B$ implies $\rho(x) \simeq \rho(x_0)$, we have that the left hand side above is a function bounded by $C\rho(x_0)^{\alpha+\beta}\|f\|_{BMO_\rho^\beta}$ on B.

Next we consider the integral over $(2B)^c$. Here we use the extra decay of the kernel and that for $x \in B$ and $y \notin 2B$, $|x-y| \simeq |x_0-y|$, $\omega(B(x,|x-y|)) \simeq \omega(B(x_0,|x_0-y|))$ and $\rho(x) \simeq \rho(x_0)$. So, splitting into annulus $2^kB \setminus 2^{k-1}B$, we have

(28)
$$\int_{(2B)^c} \frac{|x-y|^\alpha}{\omega(B(x,|x-y|))}|f(y)|\omega(y)dy \leq C_N\rho(x_0)^\alpha \sum_{k>0} \frac{2^{k(\alpha-N)}}{\omega(2^kB)} \int_{2^kB} |f(y)|\omega(y)dy.$$

Since the averages of f are bounded by $2^{k\beta}\rho(x_0)^\beta\|f\|_{BMO_\rho^\beta}$ we get

(29)
$$\int_{(2B)^c} \frac{|x-y|^\alpha}{\omega(B(x,|x-y|))}|f(y)|\omega(y)dy \leq C_N\rho(x_0)^{\alpha+\beta}\|f\|_{BMO_\rho^\beta} \sum_{k>0} 2^{k(\alpha+\beta-N)},$$

and the series is convergent choosing N large enough.

Therefore we have shown that the integral is absolutely convergent and, as a function of x, is bounded over any critical ball.

Now we take care of the case $\alpha = 0$. In this case, to apply the operator to $f \in BMO_\rho^\beta(\omega)$ has the following meaning: take a critical ball B as above and $x \in B$, then

$$Tf(x) = T(f\chi_{2B})(x) + \int_{(2B)^c} K(x,y)f(y)\omega(y)dy.$$

and both terms make sense. For the first, notice that $f\chi_{2B}$ has compact support and it is in $L^p(\omega)$ because of the John-Nirenberg property for f (see (22) and (23) above), and hence, by the assumption on the boundedness in $L^p(\omega)$, $T(f\chi_{2B})$ is also in $L^p(\omega)$. So, in particular, is locally integrable and finite a.e.. As for the second term, as in the previous case, we may use the kernel for representing the operator and moreover, the estimate made above for the integral over $(2B)^c$ also holds when $\alpha = 0$. We leave to the reader to check that the value of $Tf(x)$ is independent of the chosen ball B.

Finally we notice that in both cases, either $\alpha > 0$ or $\alpha = 0$, we may apply the operator T to $f \equiv 1$, since it belongs to $BMO_\rho(\omega)$, no matter what ρ is.

Now we are ready to present the announced general theorem.

Theorem 8. *Let ω be a doubling weight and ρ a critical radius function. Suppose T is an α-Schrödinger-Calderón-Zygmund operator with respect to ωdx that further satisfies the following $T1$-condition:*

There exists $\epsilon > 0$ and a constant C such that for any ball $B = B(x_0, r)$ with $r < \rho(x_0)$

$$\frac{1}{r^\alpha\omega(B)} \int_B |T1(x) - (T1)_B|\, \omega(x)dx \leq C\left(\frac{r}{\rho(x_0)}\right)^\epsilon.$$

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Then, T maps continuously $BMO_\rho^\beta(\omega)$ into $BMO_\rho^{\beta+\alpha}(\omega)$ for any $0 \le \beta \le \epsilon$ and such that $0 \le \alpha + \beta < \min\{1, \delta\}$.

Proof. We are going to follow similar steps to those for Theorem 1.1 in [MSTZ], so we shall be more precise only when there is a difference. Let $f \in BMO_\rho^\beta(\omega)$. First we check the condition for averages over critical balls. In the case $\alpha > 0$ it is almost done, since we have seen that given a critical ball $B = B(x_0, \rho(x_0))$ the inequality $|Tf|(x) \le C\rho(x_0)^{\alpha+\beta}\|f\|_{BMO_\rho^\beta}$ holds for $x \in B$. Averaging over B, we get the right estimate for $|Tf|_B$.

When $\alpha = 0$, to take care of the average of $|T(f\chi_{2B})|$, we use the $L^p(\omega)$-boundedness of T and (23), the John-Nirenberg property of f, to get

$$
\begin{aligned}
\frac{1}{\omega(B)}\int_B |T(f\chi_{2B})(x)|\omega(x)dx &\le \left(\frac{1}{\omega(B)}\int_B |T(f\chi_{2B})(x)|^p\omega(x)dx\right)^{1/p} \\
&\le C\left(\frac{1}{\omega(B)}\int_{2B}|f(y)|^p\omega(y)dy\right)^{1/p} \qquad (30)\\
&\le C\rho(x_0)^\beta\ \|f\|_{BMO_\rho^\beta},
\end{aligned}
$$

since the weight is doubling. As for $|Tf\chi_{(2B)^c}|$, when checking the good definition, we have already seen that it is bounded by $C\rho(x_0)^{\beta+\alpha}\|f\|_{BMO_\rho^\beta}$ in both cases, either $\alpha > 0$ or $\alpha = 0$, and so we get also the needed estimate.

Now we take care of the oscillation. Let$B = B(x_0, r)$ with $r < \rho(x_0)$. We set

$$f = (f - f_B)\chi_{4B} + (f - f_B)\chi_{(4B)^c} + f_B = f_1 + f_2 + f_3.$$

For $\alpha > 0$ it is clear that it is enough to estimate the oscillation of $T(f_i)$, $i = 1, 2, 3$, noting that we may subtract a different constant in each case. For $\alpha = 0$ that follows arguing as in [MSTZ] (see (3.3) there).

Now, for Tf_1, we choose the constant zero and proceed as above when estimating $T(f\chi_B)$. When $\alpha = 0$ we use that f satisfies (22) while for $\alpha > 0$ we reverse the order of integration and evaluate the inner integral to get

(31)

$$
\begin{aligned}
\int_B\Big(\int_{4B} K(x,y)|f_1(y)|\omega(y)dy\Big)\,\omega(x)dx &\le \int_{4B}\Big(\int_{B(y,3r)} K(x,y)\omega(x)dx\Big)\,|f_1(y)|\omega(y)dy \\
&\le Cr^\alpha\int_{4B}|f(y) - f_B|\omega(y)dy \\
&\le C\|f\|_{BMO_\rho^\beta}\omega(B)\,r^{\alpha+\beta},
\end{aligned}
$$

where for the last inequality we add and subtract f_{4B} and we use the doubling property of ω.

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For Tf_2, we subtract its average, use the smoothness of the kernel and split the integrals in annulus to get

$$
\begin{aligned}
|Tf_2(x) - Tf_2(z)| &\leq C|x-z|^{\delta} \int_{(4B)^c} \frac{|x-y|^{\alpha-\delta}}{\omega(B(x,|x-y|))} |f(y) - f_B|\omega(y)dy \\
&\leq Cr^{\alpha} \sum_{k=1}^{\infty} 2^{k(\alpha-\delta)} \frac{1}{\omega(2^k B)} \int_{2^k B} |f(y) - f_B|\omega(y)dy \qquad (32)\\
& Cr^{\alpha+\beta} \|f\|_{BMO_{\rho}^{\beta}} \sum_{k=1}^{\infty} c(k,\beta) 2^{k(\alpha-\delta)},
\end{aligned}
$$

where we have used (25). Since the sum is finite for $\alpha + \beta < \delta$, averaging over B in x and z, we obtain the desired inequality.

Finally for Tf_3, after subtracting its average, we use Lemma 4 with $B = B(x_0, \rho(x_0)))$ and $k = k_0$, the smallest non-negative integer such that $2^{k_0} r \geq \rho(x_0)$, together with the assumption on $T1$ to get

(33)

$$
\begin{aligned}
|f_B| \int_B |T1(x) - (T1)_B|\omega(x)dx &\leq \|f\|_{BMO_{\rho}^{\beta}} c(k_0,\beta)\omega(B) r^{\alpha} \rho(x_0)^{\beta} \left(\frac{r}{\rho(x_0)}\right)^{\epsilon} \\
&\leq C\|f\|_{BMO_{\rho}^{\beta}} \omega(B) r^{\alpha+\beta} c(k_0,\beta) \left(\frac{\rho(x_0)}{r}\right)^{\beta-\epsilon}.
\end{aligned}
$$

In the case $\beta > 0$ we have $c(k_0, \beta) = c$ and the desired estimate follows recalling $\beta \leq \epsilon$. When $\beta = 0$ we have $c(k_0, 0) \simeq 1 + \log \frac{\rho(x_0)}{r}$ and then $c(k_0, 0) \leq (\rho(x_0)/r)^{\epsilon}$, completing the proof of the theorem. □

Remark 11. The above result can also be stated in the vector valued setting. Assume we have a linear operator acting on functions defined on $\mathbb{R}^d$ and taking values in a Banach space $\mathcal{E}$ and that it satisfies all the conditions with absolute value replaced by the $\mathcal{E}$-norm, then the conclusion also holds. We shall need this version for $\alpha = 0$, so we must require boundedness from $L^p(\omega)$ into $L^p_{\mathcal{E}}(\omega)$ with $1 < p < \infty$.

Now we turn to the applications of our general theorem in order to obtain regularity results for the maximal operator of the degenerate Schrödinger semigroup, T^*, non-negative powers $\mathcal{L}^{-\alpha/2}$ and the mixed operators $\mathcal{L}^{-\alpha/2}V^{\sigma/2}$. Therefore we go back to our initial assumptions on ω, that is, ω is a A_2-weight such that $\omega \in RD_{\nu} \cap D_{\gamma}$ with $\nu > 2$ and the function ρ is derived from ω and some potential $V \in RH_q$ with $q > \gamma/2$, according to (2). The additional assumptions on ω will be used to show that our specific operators satisfy all the conditions required in Theorem 8.

5.1. **The maximal operator T^*.** We may look at T^* as the $L^{\infty}(\mathbb{R}^+)$-norm of the vector valued operator $\mathcal{T}f = (T_t f)_{t>0}$. Then, Theorem 1 gives the boundedness of $\mathcal{T}$ from $L^p(\omega)$ into $L^p_{\mathcal{E}}(\omega)$ with $\mathcal{E} = L^{\infty}(\mathbb{R}^+)$. By Remark 11 we must check the three other conditions with absolute value replaced by the $L^{\infty}(\mathbb{R}^+)$-norm. First, for the size, we know from Lemma 1 item c) that

$$
\|k_t(x,y)\|_{\mathcal{E}} \leq \sup_{t>0} \frac{e^{-\frac{|x-y|^2}{ct}}}{\omega(B(x,\sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N}. \qquad (34)
$$

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Now, for $\sqrt{t} \le |x-y|$ we have by the doubling property of ω

$$\omega(B(x,\sqrt{t})) \ge c_1 \left(\frac{\sqrt{t}}{|x-y|}\right)^{\gamma} \omega(B(x,|x-y|)) \tag{35}$$

and also

$$1+\frac{\sqrt{t}}{\rho(x)} \ge \left(1+\frac{|x-y|}{\rho(x)}\right)\frac{\sqrt{t}}{|x-y|}.$$

In the other case, $\sqrt{t} \ge |x-y|$, we just use that the two above functions are decreasing so we may replace $\sqrt{t}$ by $|x-y|$. So, coming back to (34) and using that $s^{\varepsilon}e^{-s^2} \le C_{\varepsilon}$ we get

$$\|k_t(x,y)\|_{\mathcal{E}} \le \frac{C}{\omega(B(x,|x-y|))}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}. \tag{36}$$

Let δ be a number such that $0<\delta<\min\{\eta,\delta_0\}$ with δ_0, η as given in Lemma 1.

For the smoothness we just observe that Lemma 1 gives the same estimates for k_t and h_t with δ instead of η. Therefore, we obtain an estimate like (14), and majorizing the exponential by one we arrive at

$$\sup_{t>0}|k_t(x,y)-k_t(z,y)| \le C\left(\frac{|x-z|}{|x-y|}\right)^{\delta}\frac{1}{\omega(B(x,|x-y|))}, \tag{37}$$

provided $|x-z|<|x-y|/4$. Therefore, the smoothness condition holds having in mind Remark 10.

Now we go for the $T1$-condition. Let $B=B(x_0,r)$ with $r<\rho(x_0)$ and let x,z two points in B. Let B_ρ the ball $B(x_0,2\rho(x_0))$ and ψ_{B_ρ} a smooth function with support in $2B_\rho$ and such that $\psi_{B_\rho}\equiv 1$ on B_ρ, $0\le\psi_{B_\rho}\le 1$. We write

(38)

$$\begin{aligned}\sup_{t>0}|T_t1(x)-T_t1(z)| &\le \sup_{t>0}|(T_t-S_t)\psi_{B_\rho}(x)-(T_t-S_t)\psi_{B_\rho}(z)|\\ &+\sup_{t>0}|S_t\psi_{B_\rho}(x)-S_t\psi_{B_\rho}(z)|\\ &+\sup_{t>0}|T_t(1-\psi_{B_\rho})(x)-T_t(1-\psi_{B_\rho})(z)| = I+II+III.\end{aligned}$$

For I we are going to use Lemma 1 item d) and e) together with Remark 1 as follows.

$$I \le \int_{2B_\rho}|q_t(x,y)-q_t(z,y)|\omega(y)dy = \left(\int_{2B}+\int_{2B_\rho\setminus 2B}\right)|q_t(x,y)-q_t(z,y)|\omega(y)dy.$$

For the integral over $2B$ we bound the difference by the sum and apply item d). In this way we are led to

$$\int_{2B}|q_t(x,y)|\omega(y)dy \le C\left(\frac{\sqrt{t}}{\rho(x_0)}\right)^{\delta_0}\int_{2B}\frac{e^{-\frac{|x-y|^2}{ct}}}{\omega(B(x,\sqrt{t}))}\omega(y)dy.$$

When $\sqrt{t}\le r$ we may replace $\sqrt{t}$ by r and the integral, extended to the whole space, is bounded by a constant, independently of t. In turn, when $\sqrt{t}\ge r$, we use the reverse doubling condition and bound the exponential by one; in this way, by the doubling property, we get

$$\int_{2B}|q_t(x,y)|\omega(y)dy \le \frac{C}{\omega(B)}\left(\frac{\sqrt{t}}{\rho(x_0)}\right)^{\delta_0}\left(\frac{r}{\sqrt{t}}\right)^{\nu}\int_{2B}\omega(y)dy \le C\left(\frac{r}{\rho(x_0)}\right)^{\delta_0},$$

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since $\delta_0 < \nu$ and $r \le \sqrt{t}$.

Now for the integral in $2B_\rho \setminus 2B$ we may apply Lemma 1, item e) and Remark 1, to get the bound

$$C\left(\frac{r}{\rho(x)}\right)^\delta \int_{2B_\rho\setminus 2B} \frac{e^{-\frac{|x-y|^2}{ct}}}{\omega(B(x,\sqrt{t}))}\omega(y)dy \le C\left(\frac{r}{\rho(x)}\right)^\delta.$$

Altogether we obtain

$$I \le C\left(\frac{r}{\rho(x_0)}\right)^\delta,$$

since $0 < \delta < \delta_0$.

Next, to take care of II, we use the regularity result for S^* obtained in the previous section, more precisely, Remark 7. Since ψ_{B_ρ} is in $BMO^\delta(\omega)$ and it is also a bounded function

$$II \le C|x-z|^\delta \|\psi_{B_\rho}\|_{BMO^\delta}.$$

Easy calculations show that the semi-norm above is like $c/\rho^\delta(x_0)$, obtaining for II the same bound as for I.

It remains to look at III. In this case we may apply the smoothness inequality obtained in (37) to get

$$III \le C|x-z|^\delta \int_{B_\rho^c} |x-y|^{-\delta}\frac{1}{\omega(B(x,|x-y|))}\omega(y)dy.$$

Splitting the integral in annulus of thickness $2^k\rho(x_0)$, it is easy to check that the integral behaves like $\rho(x_0)^{-\delta}$, and we obtain the same bound also for III.

After averaging in x and z with respect to ω, we have shown that $\mathcal{T}$ satisfies the $T1$ condition with $\alpha = 0$ and $\epsilon = \delta$, where δ, as we said above, is any number such that $0 < \delta < \delta_1 = \min\{\eta, \delta_0\}$, with η and δ_0 as given in Lemma 1.

Therefore, with this notation, an application of Theorem 8 in its vector valued version (see Remark 11) gives the following result.

Theorem 9. *Let ω be an A_2-weight such that $\omega \in RD_\nu \cap D_\gamma$ with $\nu > 2$ and $V \in RH_q$ with $q > \gamma/2$. Then, the operator T^* is bounded on $BMO^\beta_\rho(\omega)$ for any $0 \le \beta < \delta_1$.*

Remark 12. Notice that from the general theorem, we obtain an estimate on $BMO^\beta_{\mathcal{E},\rho}(\omega)$ for $\mathcal{T}f$. Nevertheless, as it is easy to see, $\|T^*f\|_{BMO^\beta_\rho} \le \|\mathcal{T}f\|_{BMO^\beta_{\mathcal{E},\rho}}$.

5.2. **Fractional integral operator.** We deal now with $\mathcal{L}^{-\alpha/2}$. We proceed as in the previous case, checking that all the requirements of Theorem 8 for the case $\alpha > 0$ are fulfilled. That the size condition holds for the kernel K_α follows from Lemma 2, item b). Regarding the smoothness, it is enough to bound

$$\int_0^\infty |k_t(x,y) - k_t(z,y)|\, t^{\alpha/2}\frac{dt}{t}$$

that, according to item f) of Lemma 1, is bounded by

$$\left(\frac{|x-z|}{|x-y|}\right)^\delta \int_0^\infty \left(\frac{|x-y|}{\sqrt{t}}\right)^\delta \frac{e^{-\frac{|x-y|^2}{\tilde{c}t}}}{\omega(B(x,\sqrt{t}))}\, t^{\alpha/2}\frac{dt}{t},$$

where, as above, δ is such that $0 < \delta < \delta_1$ and $|x-z| < 1/4|x-y|$.

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Notice that the bound for smoothness of k_t is quite similar to that of h_t (see items b) and f) in Lemma 1), so we may proceed as in the proof of Lemma 3 with δ instead of η to obtain

$$|K_\alpha(x,y) - K_\alpha(z,y)| \leq C \left(\frac{|x-z|}{|x-y|}\right)^\delta \frac{|x-y|^\alpha}{\omega(B(x,|x-y|))}. \tag{39}$$

provided $|x-z| < 1/4|x-y|$. It remains to check the $T1$-condition. We further assume that $\alpha < \delta_1$ and pick δ such that $\alpha < \delta < \delta_1$. Let $B = B(x_0, r)$ with $r < \rho(x_0)$ and x and z two points in B. Proceeding as for T^* and, with the same notation there, we write

(40)

$$\begin{aligned} |\mathcal{L}^{-\alpha/2}1(x) - \mathcal{L}^{-\alpha/2}1(z)| &\leq |(\mathcal{L}^{-\alpha/2} - \mathcal{L}_0^{-\alpha/2})\psi_{B_\rho}(x) - (\mathcal{L}^{-\alpha/2} - \mathcal{L}_0^{-\alpha/2})\psi_{B_\rho}(z)| \\ &\quad + |\mathcal{L}_0^{-\alpha/2}\psi_{B_\rho}(x) - \mathcal{L}_0^{-\alpha/2}\psi_{B_\rho}(z)| \\ &\quad + |\mathcal{L}^{-\alpha/2}(1-\psi_{B_\rho})(x) - \mathcal{L}^{-\alpha/2}(1-\psi_{B_\rho})(z)| \\ &= I + II + III. \end{aligned}$$

For I, let us call D_α the kernel of the operator $\mathcal{L}^{-\alpha/2} - \mathcal{L}_0^{-\alpha/2}$. Then, it is enough to estimate

$$\int_{2B_\rho} |D_\alpha(x,y) - D_\alpha(z,y)|\omega(y)dy.$$

We divide again the integral in $2B$ and $2B_\rho \setminus 2B$ and, as above, for the first term we bound the difference by the sum and for the other we would like to apply some smoothness condition. So we need estimates on the size as well as on the smoothness of D_α. First observe

$$\begin{aligned} |D_\alpha(x,y)| &\leq \int_0^\infty |q_t(x,y)| t^{\alpha/2} \frac{dt}{t} \\ &\leq \frac{C}{\rho(x_0)^\delta} \int_0^\infty \frac{e^{-\frac{|x-y|^2}{\tilde{c}t}}}{\omega(B(x,\sqrt{t}))} t^{\alpha/2+\delta/2} \frac{dt}{t}, \end{aligned} \tag{41}$$

where we use item d) of Lemma 1 and that $\rho(x) \simeq \rho(x_0)$. Clearly we have $\omega(B(x,\sqrt{t})) \geq \omega(B(x,|x-y|))$ when $\sqrt{t} \geq |x-y|$. Otherwise we use (35). In this way we can bound the integral above by

$$\frac{|x-y|^{\alpha+\delta}}{\omega(B(x,|x-y|))} \int_0^\infty e^{-\frac{|x-y|^2}{\tilde{c}t}} \left(1 + \frac{|x-y|}{\sqrt{t}}\right)^\gamma \left(\frac{\sqrt{t}}{|x-y|}\right)^{\alpha+\delta} \frac{dt}{t},$$

and changing variables the integral is a finite constant independent of x and y. Therefore, going back to (41) and integrating over $2B$, we get

$$\int_{2B} |D_\alpha(x,y)|\omega(y)dy \leq \frac{C}{\rho(x_0)^\delta} \int_{2B} \frac{|x-y|^{\alpha+\delta}}{\omega(B(x,|x-y|))} \omega(y)dy \leq Cr^\alpha \left(\frac{r}{\rho(x_0)}\right)^\delta,$$

where the last inequality follows by splitting the integral domain into the annulus $2^{-k}B \setminus 2^{-(k+1)}B$ and using the doubling property of ω.

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For the integral over $2B_\rho \setminus 2B$, we use the estimate given in item e) of Lemma 1 and that $\rho(x_0) \simeq \rho(x)$, leading to

$$(42)\quad \begin{aligned} |D_\alpha(x,y) - D_\alpha(z,y)| &\leq \int_0^\infty |q_t(x,y) - q_t(z,y)| t^{\alpha/2} \frac{dt}{t} \\ &\leq \left(\frac{|x-z|}{\rho(x_0)}\right)^\delta \int_0^\infty \frac{e^{-\frac{|x-y|^2}{\bar{c}t}}}{\omega(B(x,\sqrt{t}))} t^{\alpha/2} \frac{dt}{t} \\ &\leq C \left(\frac{r}{\rho(x_0)}\right)^\delta \frac{|x-y|^\alpha}{\omega(B(x,|x-y|))}, \end{aligned}$$

where, to get the last inequality, we use the calculus of the integral made for (41) but this time with $\delta = 0$. Now, integrating over $2B_\rho \setminus 2B$, we obtain

$$(43)\quad \begin{aligned} \int_{2B_\rho \setminus 2B} |D_\alpha(x,y) - D_\alpha(z.y)| \omega(y) dy &\leq C \left(\frac{r}{\rho(x_0)}\right)^\delta \int_{2B_\rho} \frac{|x-y|^\alpha}{\omega(B(x,|x-y|))} \omega(y) dy \\ &\leq C \left(\frac{r}{\rho(x_0)}\right)^\delta \rho(x_0)^\alpha \leq C \left(\frac{r}{\rho(x_0)}\right)^{\delta-\alpha} r^\alpha. \end{aligned}$$

To take care of II we use Theorem 7 for $\beta = \delta - \alpha$, that certainly satisfies $\alpha + \beta < \eta$, and with $f = \psi_{B_\rho}$. Notice that ψ_{B_ρ} is smooth and with compact support so $\mathring{\mathcal{H}}_\alpha \psi_{B_\rho} = \mathcal{L}_0^{-\alpha/2} \psi_{B_\rho}$. Therefore,

$$II \leq C|x-z|^\delta \|\psi_{B_\rho}\|_{BMO^{\delta-\alpha}} \leq C r^\delta \rho(x_0)^{\alpha-\delta} \leq C \left(\frac{r}{\rho(x_0)}\right)^{\delta-\alpha} r^\alpha.$$

Finally we estimate III. As above, here we are in condition to apply the smoothness of the kernel K_α to get

$$III \leq C|x-z|^\delta \int_{B_\rho^c} \frac{|x-y|^{\alpha-\delta}}{\omega(B(x,|x-y|))} \omega(y) dy \leq C r^\delta \rho(x_0)^{\alpha-\delta} \leq C \left(\frac{r}{\rho(x_0)}\right)^{\delta-\alpha} r^\alpha.$$

Combining all the estimates and having in mind that $r < \rho(x_0)$ we have obtained

$$|\mathcal{L}^{-\alpha/2}1(x) - \mathcal{L}^{-\alpha/2}1(z)| \leq C \left(\frac{r}{\rho(x_0)}\right)^{\delta-\alpha} r^\alpha,$$

for any $\alpha < \delta < \delta_1$.

Consequently, averaging in x and z over B, we have proved that $\mathcal{L}^{-\alpha/2}$ satisfies the $T1$-condition with $\epsilon = \delta - \alpha$ for any $\alpha < \delta < \delta_1$. Therefore an application of Theorem 8 gives the following result.

Theorem 10. *Let ω be an A_2-weight such that $\omega \in RD_\nu \cap D_\gamma$ with $\nu > 2$ and $V \in RH_q$ with $q > \gamma/2$. Then, the operator $\mathcal{L}^{-\alpha/2}$ is bounded from $BMO_\rho^\beta(\omega)$ into $BMO_\rho^{\alpha+\beta}(\omega)$ for any $\beta \geq 0$ and such that $0 < \alpha + \beta < \delta_1$.*

5.3. **The operators $\mathcal{L}^{-\alpha/2} V^{\sigma/2}$ for $\alpha \geq \sigma$.** Along this section we are going to assume that V satisfies RH_∞, which implies that $V(y) \leq \rho^{-2}(y)$. First of all, when $\alpha = \sigma$, by (11), the operator is bounded on $L^p(\omega)$ for $1 < p \leq \infty$. When $\alpha \geq \sigma$, in

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view of the inequality 9, multiplying and dividing by $|x-y|^\sigma$ and using the decay we have

$$K_{\alpha,\sigma}(x,y) = K_\alpha(x,y)\,V^{\sigma/2}(y) \le C_N \frac{|x-y|^{\alpha-\sigma}}{\omega(B(x,|x-y|))}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}.$$

So the kernel has order size $\alpha-\sigma$. As for the smoothness we want to use that of K_α, but we need first an improved version of (39), involving decay at infinity. To do that we use a kind of interpolation between size and smoothness estimates. Notice that for any pair of numbers we have $|u-v| \le (|u|+|v|)^\theta\,|u-v|^{1-\theta}$ for any fixed $0<\theta<1$. In our case we set $u = K_\alpha(x,y)$ and $v = K_\alpha(z,y)$. We recall that according to item b) of Lemma 2,

$$0 \le K_\alpha(z,y) \le C_N \frac{|z-y|^\alpha}{\omega(B(z,|z-y|))}\left(1+\frac{|z-y|}{\rho(z)}\right)^{-N},$$

for any positive N and our aim is to check that we may replace z by x on the right hand side provided $|x-z| \le 1/2|x-y|$, so u and v have the same bound. To do so observe that in such case $|x-y| \simeq |z-y|$ and the doubling property of ω gives $\omega(B(x,|x-y)|) \simeq \omega(B(z,|z-y)|)$. Besides, from (3) and using again $|x-z| \le 1/2|x-y|$ we get

$$\frac{1}{\rho(z)} \ge c\,\frac{1}{\rho(x)}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-\frac{N_0}{N_0+1}}.$$

So multiplying by $|z-y|$ and adding the obvious inequality $1 \ge \left(1+\frac{|x-y|}{\rho(x)}\right)^{-\frac{N_0}{N_0+1}}$ we arrive at

$$\left(1+\frac{|z-y|}{\rho(z)}\right)^{-N} \le C\left(1+\frac{|x-y|}{\rho(x)}\right)^{-\tilde N},$$

with $\tilde N = N - NN_0/(N_0+1)$, as we wanted.

Therefore, inserting the estimates for $|u-v|$ and $|u|+|v|$, we obtain

(44)
$$|K_\alpha(x,y) - K_\alpha(z,y)| \le C\left(\frac{|x-z|}{|x-y|}\right)^{\delta(1-\theta)} \frac{|x-y|^\alpha}{\omega(B(x,|x-y|))}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-\theta N}.$$

Since (39) was valid for any $\delta<\delta_1$, choosing θ small enough and then N sufficiently large we get

$$|K_\alpha(x,y) - K_\alpha(z,y)| \le C_N\left(\frac{|x-z|}{|x-y|}\right)^{\delta} \frac{|x-y|^\alpha}{\omega(B(x,|x-y|))}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}, \tag{45}$$

for any $0<\delta<\delta_1$. From here we easily obtain the smoothness for $K_{\alpha,\sigma}$ in view of the inequality

$$|K_{\alpha,\sigma}(x,y) - K_{\alpha,\sigma}(z,y)| \le |K_\alpha(x,y) - K_\alpha(z,y)|V^{\sigma/2}(y),$$

and from

(46)
$$V^{\sigma/2}(y) \le C\rho(y)^{-\sigma} \le C\rho(x)^{-\sigma}\left(1+\frac{|x-y|}{\rho(x)}\right)^{N_0} \le C|x-y|^{-\sigma}\left(1+\frac{|x-y|}{\rho(x)}\right)^{N_0+\sigma}.$$

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Therefore, from the last three inequalities, choosing $N = N_0 + \sigma$, it follows for $0 < \delta < \delta_1$ that

$$|K_{\alpha,\sigma}(x,y) - K_{\alpha,\sigma}(z,y)| \le C \left(\frac{|x-z|}{|x-y|}\right)^{\delta} \frac{|x-y|^{\alpha-\sigma}}{\omega(B(x,|x-y|))}. \tag{47}$$

Finally we check the $T1$-condition. Let $B = B(x_0, r)$ with $r < \rho(x)$ and x and z two points in B. Then we have

(48)

$$\begin{aligned}|\mathcal{L}^{-\alpha/2}V^{\sigma/2}1(x) - \mathcal{L}^{-\alpha/2}V^{\sigma/2}1(z)| &\le |\mathcal{L}^{-\alpha/2}\left(V^{\sigma/2}\chi_{5B}\right)(x) - \mathcal{L}^{-\alpha/2}\left(V^{\sigma/2}\chi_{5B}\right)(z)| \\ &+ \int_{(5B)^c} |K_{\alpha,\sigma}(x,y) - K_{\alpha,\sigma}(z,y)|\omega(y)dy \\ &= I + II.\end{aligned}$$

For the first term, according to Theorem 6 together with Remark 9 , $\mathcal{L}^{-\alpha/2}$ may be extended to a bounded operator from $M_1^{\alpha-\beta}$ into $BMO^{\beta}(\omega)$ for $\beta < \min\{\alpha, \delta_1\}$, since its kernel satisfies the appropriate size and smoothness estimates. (see Lemma 2 and (39)). On the other hand, $V^{\sigma/2}\chi_{5B}$ is a function in $M_1^{\alpha-\beta}$ and with compact support. In fact, if we take $Q = B(x_1, s)$ any ball

$$\frac{s^{\alpha-\beta}}{\omega(Q)}\int_Q V^{\sigma/2}\chi_{5B}\,\omega \le Cs^{\alpha-\beta}\rho(x_0)^{-\sigma}\ \frac{\omega(Q\cap 5B)}{\omega(Q)}.$$

Assume $Q \cap 5B \ne \emptyset$. If $s \le 5r$ we bound the above quantity by $cr^{\alpha-\beta}\rho(x_0)^{-\sigma}$, having in mind that $\rho(y) \simeq \rho(x_0)$ for $y \in 5B$. Otherwise, $|x_1 - x_0| \le 2s$ and also $Q \subset \tilde{B} = B(x_0, 3s) \subset 5Q$ and being ω doubling and ν-reverse doubling we have $\omega(Q) \simeq \omega(\tilde{B}) \ge (s/r)^{\nu}\omega(B) \ge C\,(s/r)^{\alpha-\beta}\,\omega(B)$; this, together with the obvious inequality $\omega(Q\cap 5B) \le \omega(5B)$, gives also the bound $r^{\alpha-\beta}\rho(x_0)^{-\sigma}$ when $5r \le s$.

Altogether we conclude

$$\|V^{\sigma/2}\chi_{5B}\|_{M_1^{\alpha-\beta}} \le Cr^{\alpha-\beta}\rho(x_0)^{-\sigma}.$$

Going back to the estimate for I, it follows

$$I \le Cr^{\alpha-\beta}\rho(x_0)^{-\sigma}r^{\beta} = Cr^{\alpha-\sigma}\left(\frac{r}{\rho(x_0)}\right)^{\sigma}.$$

As for the second term we may use the smoothness of the kernel since in our situation $|x-y| \ge 4r \ge 2|x-z|$, but instead of (47) we will use a somehow stronger variant, namely

$$|K_{\alpha,\sigma}(x,y) - K_{\alpha,\sigma}(z,y)| \le \frac{C}{\rho(x_0)^{\sigma}}\left(\frac{|x-z|}{|x-y|}\right)^{\delta}\frac{|x-y|^{\alpha}}{\omega(B(x,|x-y|))},$$

which holds also from (45) and from (46) just stopping before the last inequality and using that, in our case, $\rho(x) \simeq \rho(x_0)$. Plugging that estimate in II we obtain

$$II \le \frac{C}{\rho(x_0)^{\sigma}}|x-z|^{\delta}\int_{(5B)^c}\frac{|x-y|^{\alpha-\delta}}{\omega(B(x,|x-y|))}\omega(y)dy.$$

Since the integral is bounded by $Cr^{\alpha-\delta}$, we get

$$II \le \frac{C}{\rho(x_0)^{\sigma}}r^{\alpha} \le r^{\alpha-\sigma}\left(\frac{r}{\rho(x_0)}\right)^{\sigma},$$

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which is the same estimate that we have for the first term.

In this way we have shown that $T1$-condition holds with $\epsilon = \sigma$.

Collecting estimates, we have proved that $\mathcal{L}^{-\alpha/2}V^{\sigma/2}$ is an $(\alpha - \sigma)$-Schrödinger-Calderón-Zygmund operator with respect to ωdx and having smoothness of order δ for any $0 < \delta < \delta_1$. Moreover, the $T1$ condition holds with exponent σ. Therefore, an application of Theorem 8 gives the following result.

Theorem 11. *Let ω be an A_2-weight such that $\omega \in RD_\nu$ with $\nu > 2$. Assume that $V \in RH_\infty$ and let ρ be its associate critical radius function. Then, given α and σ, with $\alpha \geq \sigma > 0$, the operator $\mathcal{L}^{-\alpha/2}V^{\sigma/2}$ is bounded from $BMO_\rho^\beta(\omega)$ into $BMO_\rho^{\beta+\alpha-\sigma}(\omega)$ for any $0 \leq \beta \leq \sigma$ and such that $0 < \beta + \alpha - \sigma < \delta_1$.*

Declaration

Data availability: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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