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POSET PRODUCT AND BL-ALGEBRAS

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Abstract. We give sufficient conditions for a BL-algebra to be isomorphic to a poset product of BL-chains which are poset product-indecomposable.

Introduction

The variety $\mathcal{BL}$ of BL-algebras is the algebraic counterpart of BL, the logic introduced by Hájek in [12] which includes a fragment common to the most important fuzzy logics (Łukasiewicz, Gödel and product logics). An essential result for studying $\mathcal{BL}$ is the subdirect representation theorem (see [12]), which states that each BL-algebra is a subdirect product of totally ordered BL-algebras (BL-chains). This theorem not only says that each BL-algebra is a subalgebra of the direct product of BL-chains, but also reveals that the main structures in the study of BL-algebras are its totally ordered members. Another important result is the decomposition theorem given by Aglianò and Montagna in [1]. They proved that any BL-chain is an ordinal sum of simpler structures, namely totally ordered Wajsberg hoops. Since the subdirect representation theorem provides an embedding which in general is not surjective and the ordinal sum representation cannot be extended to non totally ordered BL-algebras, none of these tools can be used to obtain a genuine representation theorem for BL-algebras.

Having in mind that every BL-algebra can be embedded into the direct product of BL-chains which in turn admit a further decomposition as ordinal sums, Jipsen and Montagna introduced and studied a construction called poset product with the aim of encompassing both the direct product and ordinal sum constructions (see [13, 14, 15, 16]). Briefly, the poset product is a subset of a direct product which is defined by using a partial order over the index set.

Although it was defined for a larger class of structures, the poset product construction can, however, be used to shed some light into the structure of BL-algebras because it provides many examples of non totally ordered BL-algebras. Indeed, based on the results of [16], it is shown in [6] that

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every BL-algebra is a subalgebra of a poset product of a collection of BL-chains. Moreover, as a consequence of [15], the Di Nola's and Lettieri's representation theorem for finite BL-algebras ([10]) can be rephrased: every finite BL-algebra is isomorphic to the poset product of a finite family of finite MV-chains with respect to a poset which is a forest (*i.e.* the downset of every element is a chain). Unfortunately, the goal of getting a representation for each BL-algebra using the poset product construction cannot be achieved (even when dealing with chains, as explained in [2, 5]). Nevertheless, the poset product construction is an important tool to represent some non-trivial BL-algebras in terms of simpler BL-chains. The present paper is intended to describe a family of BL-algebras that can be represented as a poset product of indecomposable (in the poset product sense) BL-chains.

This article is organized as follows. In the first section we provide preliminaries about BL-algebras and present some key examples that will help to understand the main results of the paper. The second section offers the background on poset product and the definition of indecomposable BL-chain. It also presents a detailed study of the Gödel subalgebra of the poset product of indecomposable BL-chains and the characterization of its (completely) join irreducible elements as far as they are relevant for the third section, which is where the main results are. In Section 3 we introduce the family of sound BL-algebras that depend on its idempotent and prove that these algebras are representable as a poset product indexed by a forest which is isomorphic to the prime spectrum of its Gödel subalgebra. This section is divided into five subsections. In the first one we define those that will be the Gödel subalgebras of representable BL-algebras. In the second we define sound BL-algebras that depends on its idempotent as BL-algebras that satisfy four conditions and we provide examples to show the independence of these conditions. In the third subsection is the morphism which embeds sound BL-algebras that depends of its idempotent into poset products of indecomposable BL-chains. Before proving surjectivity in the last subsection, we show that the index set arising from the embedding theorem is a well partial order. The last section of the paper is devoted to compare our results with some previous one.

To make the paper self-contained we have included all necessary definitions and we have explained the different constructions, providing examples of them. Further details on $\mathcal{BL}$ can be found in [1, 6, 8, 12].

1. Preliminaries

In this first section we offer the necessary background on BL-algebras and fix notation. Then main sources are [1, 6, 9, 12].

1.1. **Basic hoops and BL-algebras.** A *basic hoop* is an algebra $\mathbf{W} = \langle W, \cdot, \to, 1\rangle$ of type $\langle 2, 2, 0\rangle$ such that $\langle W, \cdot, 1\rangle$ is a commutative monoid and for all $a, b, c \in W$:

1. $a \to a = 1$,
2. $a \cdot (a \to b) = b \cdot (b \to a)$,
3. $a \to (b \to c) = (a \cdot b) \to c$,
4. $(((a \to b) \to c) \cdot ((b \to a) \to c)) \to c = 1$.

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Every basic hoop has a residuated lattice structure, where the lattice order is defined by $a \leq b$ if and only if $a \to b = 1$ and the residuation is

$$a \cdot b \leq c \quad \text{if and only if} \quad a \leq b \to c.$$

The lattice operations are defined by

$$a \wedge b = a \cdot (a \to b) \quad \text{and} \quad a \vee b = ((a \to b) \to b) \wedge ((b \to a) \to a),$$

thus 1 is the greatest element. In addition, a basic hoop satisfies the prelinearity identity

$$(a \to b) \vee (b \to a) = 1,$$

which implies that every basic hoop is a subdirect product of totally ordered basic hoops. A *trivial hoop* is a hoop with universe $\{1\}$.

A *BL-algebra* is a bounded basic hoop, that is, it is an algebra $\mathbf{A} = \langle A, \cdot, \to, 0, 1\rangle$ of type $\langle 2, 2, 0, 0\rangle$ such that $\langle A, \cdot, \to, 1\rangle$ is a basic hoop and 0 is the lower bound of the lattice structure. A *BL-chain* is a totally ordered BL-algebra.

We presented the definition of BL-algebras as basic hoops because we will draw upon its hoop structure along the paper. Alternatively, an equivalent definition in terms of residuated lattices can be given (we refer the reader to [6, 11] for details).

1.2. **Filters of BL-algebras.** A *filter* of a BL-algebra (basic hoop) $\mathbf{A}$ is a non-empty subset $F \subseteq A$ satisfying that if $a \in F$ and $a \to b \in F$, then $b \in F$. A filter F of $\mathbf{A}$ is *proper* if $F \neq A$. Filters can also be characterized as non-empty upwards closed subsets of A such that $a \cdot b \in F$ for all $a, b \in F$ (see [12]).

The intersection of any family of filters of a BL-algebra $\mathbf{A}$ is still a filter of $\mathbf{A}$. For every subset $W \subseteq A$, the intersection of all filters $F \supseteq W$ is said to be the filter *generated* by W and is denoted $\langle W\rangle$. In particular, for each element w of a BL-algebra $\mathbf{A}$, the filter $\langle w\rangle = \langle\{w\}\rangle$ is called the *principal filter generated by* w.

A filter F of $\mathbf{A}$ is called *prime* provided that it is proper and for all a, $b \in A$, if $a \vee b \in F$, then either $a \in F$ or $b \in F$. Though no topology will be considered, we will refer to the set of prime filters of $\mathbf{A}$ as *prime spectrum* and we will denote it by $\mathrm{Spec}(\mathbf{A})$.

If F is a filter of $\mathbf{A}$, then the binary relation $\equiv_F$ on A defined by

$$a \equiv_F b \quad \text{if and only if} \quad a \to b \in F \text{ and } b \to a \in F$$

is a congruence relation. Given $a \in A$, let a/F be the equivalence class of a with respect to $\equiv_F$. The quotient set A/F endowed with corresponding operations becomes a BL-algebra $\mathbf{A}/F$ called *the quotient algebra* of $\mathbf{A}$ by the filter F. There is a bijective correspondence between the set of filters of $\mathbf{A}$ including F and the set of filters of $\mathbf{A}/F$. In [12] it is shown that the quotient of a BL-algebra modulo a filter is a BL-chain if and only if the filter is prime.

Lemma 1.1. *If $\mathcal{F}$ is a totally ordered family of (prime) filters of a BL-algebra $\mathbf{A}$, then $\bigcap \mathcal{F}$ and $\bigcup \mathcal{F}$ are (prime) filters of $\mathbf{A}$.*

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Proof. The intersection of any family of filters is a filter. The fact that the family $\mathcal{F}$ is totally ordered by inclusion ensures that primality is preserved in the intersection. For the join, let $\mathcal{F} = \{F_i\}_{i\in I}$, so that $F = \bigcup_{i\in I} F_i$. Clearly, F is a non-empty upset which is closed under $\cdot$ because $\mathcal{F}$ is totally ordered. Thus F is a filter of $\mathbf{A}$. Let $F_i \in \mathcal{F}$. Since F_i and F are filters of $\mathbf{A}$ such that $F_i \subseteq F$, F/F_i is a filter of $\mathbf{A}/F_i$. By application of the Second Isomorphism Theorem ([4, Th. 6.15]), it follows that

$$\mathbf{A}/F \cong \frac{\mathbf{A}/F_i}{F/F_i}.$$

Being $\mathbf{A}/F$ the homomorphic image of the BL-chain $\mathbf{A}/F_i$, $\mathbf{A}/F$ is also a BL-chain. Then $F \in \mathrm{Spec}(\mathbf{A})$. □

Lemma 1.2. *Let $\mathbf{A}$ be a BL-algebra and F a filter of $\mathbf{A}$. If $a \in A \setminus F$, then there is a prime filter G such that $F \subseteq G$ and $a \notin G$.*

As a consequence, each proper filter of a BL-algebra is the intersection of all prime filters containing it. In particular,

Corollary 1.3. *For each BL-algebra $\mathbf{A}$, $\bigcap_{F\in\mathrm{Spec}(\mathbf{A})} F = \{1\}$.*

The previous results have a fundamental role in the proof of the subdirect representation theorem for BL-algebras, on which the importance of the study of BL-chains relies.

1.3. **The ordinal sum construction.** The ordinal sum construction has proved to be a very effective tool to describe BL-chains. Indeed, they are characterized as ordinal sums of irreducible hoops in [1].

Definition 1.4. Let $\langle I, \leq\rangle$ be a totally ordered set. For each $i \in I$, let $\mathbf{W}_i = \langle W_i, \cdot_i, \to_i, 1\rangle$ be a totally ordered hoop such that for every $i \neq j$, $W_i \cap W_j = \{1\}$. Then the *ordinal sum* of this family is the hoop $\bigoplus_{i\in I} \mathbf{W}_i = \langle \bigcup_{i\in I} W_i, \cdot, \to, 1\rangle$ where the operations $\cdot$ and $\to$ are given by

$$a \cdot b = \begin{cases} a \cdot_i b & \text{if } a, b \in W_i; \\ a & \text{if } a \in W_i \setminus \{1\}, b \in W_j \text{ and } i < j; \\ b & \text{if } b \in W_i \setminus \{1\}, a \in W_j \text{ and } i < j. \end{cases}$$

and

$$a \to b = \begin{cases} 1 & \text{if } a \in W_i \setminus \{1\}, b \in W_j \text{ and } i < j; \\ a \to_i b & \text{if } a, b \in W_i; \\ b & \text{if } b \in W_i, a \in W_j \text{ and } i < j. \end{cases}$$

Note that the ordinal sum $\bigoplus_{i\in I} \mathbf{W}_i$ of a family $\{\mathbf{W}_i : i \in I\}$ of totally ordered hoops is a BL-chain whenever I has a least element i_0 and $\mathbf{W}_{i_0}$ is lower bounded. This way, the resulting constant bottom in the ordinal sum is the bottom of the first summand.

A *Wajsberg hoop* is a basic hoop which verifies the equation $(a \to b) \to b = (b \to a) \to a$. For totally ordered basic hoops, this condition captures the idea of sum irreducibility. Since each non-trivial BL-chain admits (up to isomorphism) a unique decomposition into an ordinal sum of non-trivial totally ordered Wajsberg hoops, a complete description of BL-chains in terms of ordinal sums can be given.

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Abuse of notation: ordinal sums with bounded summands will be referred as sums of BL-chains instead of hoops reducts of BL-chains.

1.4. **Key examples of Wajsberg hoops and BL-chains.** In any BL-algebra a unary operation of negation $\neg$ can be defined as $\neg a = a \to 0$.

An *MV-algebra* (see [7]) is a BL-algebra in which the identity $\neg\neg a = a$ holds. The subvariety of MV-algebras is the algebraic semantics for the infinite-valued logic of Łukasiewicz (see [12]). The standard MV-chain $\mathbf{[0,1]_{MV}}$ is the MV-algebra whose universe is the real unit interval $[0,1]$ and the operations $\cdot$ and $\to$ are $a \cdot b = \max(0, a+b-1)$ and $a \to b = \min(1, 1-a+b)$. For $n \geq 2$, $\mathbf{Ł}_n$ is the subalgebra of $\mathbf{[0,1]_{MV}}$ with universe $\left\{\frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-1}{n-1}\right\}$. $\mathbf{Ł}_2$ is simply the two-element Boolean algebra.

A *product algebra* is a BL-algebra that fulfils the identities $a \wedge \neg a = 0$ and $(\neg\neg c \cdot ((a \cdot c) \to (b \cdot c))) \to (a \to b) = 1$. Product algebras correspond to product fuzzy logic (see [12]). The standard product chain is the algebra $\mathbf{[0,1]_{\Pi}} = \langle [0,1], \cdot, \to, 0, 1\rangle$ where $\cdot$ is the usual product over the real interval $[0,1]$ and $\to$ is given by

$$a \to b = \begin{cases} b/a & \text{if } a > b; \\ 1 & \text{if } a \leq b. \end{cases}$$

Totally ordered Wajsberg hoops can be either lower bounded or not. Bounded totally ordered Wajsberg hoops are bottom-free reducts of MV-chains, while unbounded are *cancellative* Wajsberg hoops; that is, they satisfy the equality $a \to (a \cdot b) = b$ (see [3]). A key example of cancellative Wajsberg hoop arises by considering the hoop reduct of the algebra $[0,1]_{\Pi} \setminus \{0\}$; *i.e.* considering the operations on $(0,1]$ as restrictions of the corresponding operations on $\mathbf{[0,1]_{\Pi}}$. This cancellative hoop is denoted by $\mathbf{(0,1]_{\Pi}}$. It is important to note that

$$\mathbf{[0,1]_{\Pi}} \cong \mathbf{Ł}_2 \oplus \mathbf{(0,1]_{\Pi}}.$$

1.5. **Idempotent elements in BL-algebras.** An element i in a BL-algebra $\mathbf{A}$ is called *idempotent* if $i \cdot i = i$. $\mathrm{Id}(\mathbf{A})$ will stand for the set of idempotent elements of $\mathbf{A}$. If $\mathbf{A}$ is a BL-algebra, then $i \in \mathrm{Id}(\mathbf{A})$ if and only if $i \cdot a = i \wedge a$ for all $a \in A$. Hence, if $i \in \mathrm{Id}(\mathbf{A})$, for each a, b in A, $i \leq a \to b$ if and only if $i \wedge a \leq b$. Consequently, the set $\mathrm{Id}(\mathbf{A})$ forms a subalgebra of $\mathbf{A}$ which is a Gödel algebra (see [12]).

Given a subset W of a partially ordered set A, $[W)$ will denote the upset of W; that is, $[W) = \{a \in A : a \geq w \text{ for some } w \in W\}$. Similarly for the downsets $(W]$.

Theorem 1.5. *Let* $\mathbf{A}$ *be a BL-algebra and* $\mathbf{B}$ *a subalgebra of* $\mathbf{Id}(\mathbf{A})$*. If* F *is a filter of* $\mathbf{B}$*, then* $[F)$ *is a filter of* $\mathbf{A}$*. In addition,* $a \equiv_{[F)} b$ *if and only if there exists* $i \in F$ *such that* $a \wedge i = b \wedge i$*.*

Let $\mathbf{A}$ be a BL-algebra (BL-chain) and $i, j \in A$ idempotent elements such that $i < j$. Then the set $[i,j]_{\mathbf{A}} = \{a \in A : i \leq a \leq j\}$ with the lattice and product operations inherited from $\mathbf{A}$ and the implication $\to_{ij}$ given by $a \to_{ij} b = (j \wedge (a \to b)) \vee i$ has a BL-algebra (BL-chain) structure.

Corollary 1.6. *If* $i \in \mathrm{Id}(\mathbf{A})$*, then the mapping* $a/[i) \mapsto a \wedge i$ *defines an isomorphism from the quotient algebra* $\mathbf{A}/[i)$ *onto the segment algebra* $\mathbf{[0,\mathit{i}]_A}$*.*

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Recall from [9] that an element a which is not the bottom element of a lattice $\mathbf{L}$ is called $\vee$-*irreducible* if for all $b, c \in L$, $a = b \vee c$ implies $a = b$ or $a = c$ (alternatively, $b \vee c < a$ whenever $b < a$ and $c < a$). If an element in a lattice is not the supremum of all elements strictly below it, or equivalently if it has a unique subcover, then it is said to be *completely* $\vee$-*irreducible* (see [11]). If $a \in L$ is a completely $\vee$-irreducible element, then $b \prec a$ will mean that b is the predecessor of a in $\mathbf{L}$.

Let $\mathbf{A}$ be a BL-algebra and $i \in \mathrm{Id}(\mathbf{A})$. It is obvious that if $[i) \in \mathrm{Spec}(\mathbf{A})$, then i is a $\vee$-irreducible element. The converse is a straightforward consequence of the prelinearity property of $\mathcal{BL}$. Further, if $i \in \mathrm{Id}(\mathbf{A})$ is $\vee$-irreducible, then $(i] = \{a \in A : a \leq i\}$ is a totally ordered downset of $\vee$-irreducible elements. From now on,

$$\mathcal{J}_{\mathbf{A}} \text{ will denote the set } \{i \in A : i \in \mathrm{Id}(\mathbf{A}) \text{ is } \vee\text{-irreducible in } \mathrm{Id}(\mathbf{A})\}$$

and $\mathcal{S}_{\mathbf{A}}$ will denote its subset of completely $\vee$-irreducible elements.

For the next remark, recall that a poset $\langle P, \leq \rangle$ is called *forest* if for every $p \in P$ the downset of p is totally ordered.

Remark 1.7. The poset of $\vee$-irreducible elements $\mathcal{J}_{\mathbf{A}}$ of a BL-algebra $\mathbf{A}$ is a forest. Thus $\mathcal{S}_{\mathbf{A}}$ is a forest as well.

2. The poset product construction

We recall the definition of poset product of BL-algebras and some of its properties. This construction was introduced in a more general framework for the study of a class of algebras that properly contains the class $\mathcal{BL}$ (the interested reader should see [13, 15, 16]).

Definition 2.1. Let $P = \langle P, \leq \rangle$ be a poset and let $\{\mathbf{A}_p : p \in P\}$ be a collection of BL-algebras. Up to isomorphism we can (and we will) assume that all $\mathbf{A}_p$ share the same neutral element 1 and the same minimum element 0. The *poset product* $\bigotimes_{p \in P} \mathbf{A}_p$ is the residuated lattice $\mathbf{A} = \langle A, \cdot, \rightarrow, \vee, \wedge, \bot, \top \rangle$ defined as follows:

1. The domain of $\mathbf{A}$ is the set of all maps x belonging to $\prod_{p \in P} A_p$ such that for all $p \in P$, if $x_p \neq 1$, then $x_q = 0$ for all $q > p$.
2. $\top$ is the map whose value in each coordinate is 1, and $\bot$ is the one whose value in each coordinate is 0.
3. The monoid operation and the lattice operations are defined pointwise.
4. The residual is

$$(x \rightarrow y)_p = \begin{cases} x_p \rightarrow_p y_p & \text{if } x_q \leq_q y_q \text{ for all } q < p; \\ 0 & \text{otherwise.} \end{cases}$$

where the subscript $_p$ denotes realization of operations and of order in $\mathbf{A}_p$ (we will often omit subscripts when there is no danger of confusion).

It is worth noting that an element $x \in \prod_{p \in P} A_p$ is in the poset product if and only if $\{p \in P : 0 < h_p < 1\}$ is an antichain and $\{p \in P : x_p = 1\}$ is a downset (hence $\{p \in P : x_p = 0\}$ is an upset).

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2.1. **Representability.** While the poset product construction preserves many properties of residuated lattices (details in [15, 16]), the prelinearity property of $\mathcal{BL}$ is not preserved without some additional requirements.

Theorem 2.2 (see [6]). *Suppose that P is a forest and that for all $p \in P$, $\mathbf{A}_p$ is a BL-chain. Then $\bigotimes_{p\in P} \mathbf{A}_p$ is a BL-algebra.*

So it is natural to wonder when a BL-algebra is isomorphic to a poset product of simpler structures. This question is addressed in [6], where it is shown that every BL-algebra can be embedded into a poset product of a family of MV-chains and product chains. The case of BL-chains is analyzed in [5] by comparing the ordinal sum and the poset product constructions.

An algebra $\mathbf{A}$ is said to be *poset product indecomposable* (*indecomposable*, for short) if $\mathbf{A}$ is non-trivial and if $\mathbf{A}$ is a poset product of two algebras $\mathbf{A}_1$ and $\mathbf{A}_2$, then either $\mathbf{A}_1$ or $\mathbf{A}_2$ is trivial.

Lemma 2.3 (see [5]). *A non-trivial BL-chain $\mathbf{A}$ is indecomposable if and only if $\mathbf{Id}(\mathbf{A}) \cong \mathbf{Ł}_2$.*

Observe that a BL-chain is indecomposable if and only if it is isomorphic to $\mathbf{W} \oplus (\bigoplus_{i\in I} \mathbf{W}_i)$, where $\mathbf{W}$ is an MV-chain and for each $i \in I$, $\mathbf{W}_i$ is an unbounded totally ordered Wajsberg hoop. Therefore MV-chains and product chains are indecomposable. Because of Theorem 2.2, in the present paper we only care for poset products whose factors are indecomposable chains.

Definition 2.4. A BL-algebra is *representable (by a poset product)* if it is isomorphic to the poset product of a family of indecomposable BL-chains indexed by a forest.

2.2. **Idempotent elements in a poset product.** We fix a forest P and a family $\{\mathbf{A}_i : i \in P\}$ of indecomposable BL-chains. Let $\mathbf{A}$ be the poset product

$$\mathbf{A} = \bigotimes_{i\in P} \mathbf{A}_i.$$

For each downset $Q \subseteq P$, its characteristic function χ_Q is an element of $\mathbf{A}$. For each $p \in P$, let $\downarrow p = \{i \in P : i \le p\}$. If $x \in A$, then

1. $O_x = \{i \in P : x_i = 1\}$ is a downset of P.
2. $x \in \mathrm{Id}(\mathbf{A})$ if and only if $x = \chi_{O_x}$.
3. If $x, y \in \mathrm{Id}(\mathbf{A})$, then

$$x \le y \iff O_x \subseteq O_y \iff y_i = 1 \ \forall i \in O_x.$$

The first claim comes from the definition of poset product. The second claim follows from Lemma 2.3 and the third one is evident.

Lemma 2.5. *Let $x \in \mathrm{Id}(\mathbf{A})$.*

1. *$x \in \mathcal{J}_{\mathbf{A}}$ if and only if $O_x = \{i \in P : x_i = 1\}$ is a non-empty downset which is totally ordered.*
2. *$x \in \mathcal{S}_{\mathbf{A}}$ if and only if $x = \chi_{\downarrow p}$ for some $p \in P$ if and only if O_x has a maximum.*

Proof.

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1. Let $x \in \mathcal{J}_{\mathbf{A}}$, which implies that $O_x \neq \emptyset$. Assume that j, $k \in O_x$ are two incomparable elements and define y, $z \in \mathrm{Id}(\mathbf{A})$ as follows:

$$y_i = \begin{cases} x_i & \text{if } i \text{ and } j \text{ are comparable;} \\ 0 & \text{otherwise.} \end{cases} \qquad z_i = \begin{cases} 0 & \text{if } i \geq j; \\ x_i & \text{if } i \not\geq j. \end{cases}$$

   Then $y < x$, $z < x$ and $x = y \vee z$, so x is not $\vee$-irreducible in $\mathbf{Id}(\mathbf{A})$, which is a contradiction. Thus O_x must be totally ordered. Conversely, assume that O_x is non-empty and totally ordered (so that $x > 0$) and let y, $z \in \mathrm{Id}(\mathbf{A})$ be such that $x = y \vee z$. Hence $O_x = O_y \cup O_z$ is a totally ordered set which is the join of two downsets. Then either $O_y = O_x$ or $O_z = O_x$. Thus either $x = y$ or $x = z$, so $x \in \mathcal{J}_{\mathbf{A}}$.
2. Let $x \in \mathcal{S}_{\mathbf{A}}$ and let y be the predecessor of x in $\mathbf{Id}(\mathbf{A})$. Since $y = \chi_{O_y} < \chi_{O_x} = x$, $O_y \subsetneq O_x$. Let $p \in O_x \setminus O_y$. Clearly, $y < \chi_{\downarrow p} \leq x$ and it may not occur that $\chi_{\downarrow p} < x$ because y is the unique lower cover of x in $\mathbf{Id}(\mathbf{A})$. Hence $x = \chi_{\downarrow p}$. For the converse, let $x = \chi_{\downarrow p}$. To see that it is completely $\vee$-irreducible in $\mathbf{Id}(\mathbf{A})$, consider $y \in \mathrm{Id}(\mathbf{A})$ defined as

$$y_i = \begin{cases} 1 & \text{if } i < p; \\ 0 & \text{otherwise.} \end{cases}$$

   and see that y is the unique subcover of x. It is obvious that $x \in \mathcal{S}_{\mathbf{A}}$ if and only if O_x has a maximum. □

The following lemma says that each element $x \in \mathrm{Id}(\mathbf{A})$ can be written as the supremum of all idempotent completely $\vee$-irreducible elements below it.

Lemma 2.6. *For all $x \in \mathrm{Id}(\mathbf{A})$, $x = \bigvee_{i \in O_x} \chi_{\downarrow i}$.*

Proof. Given $x \in \mathrm{Id}(\mathbf{A})$, clearly $\chi_{\downarrow i} \leq x$ for all $i \in O_x$. On the other hand, if $\chi_{\downarrow i} \leq y$ for all $i \in O_x$, then $O_x \subseteq O_y$. Therefore $x \leq y$ and $x = \bigvee_{i \in O_x} \chi_{\downarrow i}$. □

Note. Lemma 2.6 holds trivially when $x = 0$.

3. Representation by poset product

In this section we study under which conditions an algebra is isomorphic to a poset product of indecomposable BL-chains. The way to achieve the result is long as it requires a deep investigation of the set of prime filters of poset products. We first define and analyze the Gödel subalgebras of those BL-algebras that are going to be representable. Then we establish sufficient conditions for a BL-algebra to be embeddable into a poset product and provide examples of algebras that do not satisfy such conditions. Finally, we prove an embedding theorem and, after analyzing the prime spectrum of these BL-algebras, we prove that the embedding is also surjective.

3.1. **Principal Gödel algebras.** A Gödel algebra $\mathbf{A}$ is said to be *principal* if every prime filter $F \in \mathrm{Spec}(\mathbf{A})$ is principal. Of course, finite Gödel algebras are principal. The following examples will play a major role in Section 3.2.

Example 3.1. Let $\mathbf{A} = \bigoplus_{\mathbb{N}} \mathbf{Ł}_2$. From the fact that $\langle A, \leq \rangle$ is a well-ordered set, it is easily seen that if F is a (prime) filter of $\mathbf{A}$, then $F = \langle i \rangle$ for some $i \in A$. Hence $\mathbf{A}$ is a principal Gödel chain. Similarly, $\mathbf{B} = (\bigoplus_{\mathbb{N}} \mathbf{Ł}_2) \oplus \mathbf{Ł}_2 \cong$

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$\bigoplus_{\mathbb{N}\cup\{t\}} \text{Ł}_2$ is a principal Gödel chain, where t denotes the greatest element in the order of the index set $\langle \mathbb{N}\cup\{t\}, \leq\rangle$.

Example 3.2. The Gödel chain $\mathbf{A} = \bigoplus_P \text{Ł}_2$, where $P = \langle \{b\}\cup\mathbb{Z}^-, \leq\rangle$ is a totally ordered set endowed with the usual order in $\mathbb{Z}^-$ and $b < z$ for all $z \in \mathbb{Z}^-$, is not a principal Gödel chain because the filter $\{a \in A : a > 0\}$ is not principal.

As we will see in the following results, the subset of $\vee$-irreducible elements of a principal Gödel algebra satisfies some interesting properties.

Lemma 3.3. *Let* $\mathbf{A}$ *be a principal Gödel algebra. Then each non-empty totally ordered subset of* $\mathcal{J}_{\mathbf{A}}$ *has minimum and supremum.*

Proof. Let $\emptyset \neq \mathcal{C} \subseteq \mathcal{J}_{\mathbf{A}}$ be a totally ordered set, so that $\mathcal{F} = \{\langle i\rangle\}_{i\in\mathcal{C}}$ is a totally ordered family of $\mathrm{Spec}(\mathbf{A})$. Therefore

$$F = \bigcup \mathcal{F} = \{a \in A : a \geq i \text{ for some } i \in \mathcal{C}\} \in \mathrm{Spec}(\mathbf{A})$$

by Lemma 1.1. Moreover, since $\mathbf{A}$ is principal, $F = \langle m\rangle = [m)$ for some $m \in \mathcal{J}_{\mathbf{A}}$ and m is thus the greatest lower bound of $\mathcal{C}$. If it were $m \notin \mathcal{C}$, then it would exists $i \in \mathcal{C}$ causing the absurd $m \leq i < m$. Therefore $m = \min\mathcal{C}$. Analogously, there is an $s \in \mathcal{J}_{\mathbf{A}}$ such that the prime filter

$$G = \bigcap \mathcal{F} = \{a \in A : a \geq i \text{ for all } i \in \mathcal{C}\}$$

is generated by s, meaning that $s = \sup\mathcal{C}$. □

Example 3.4. Let $\mathbf{A}$ and $\mathbf{B}$ be the principal Gödel chains as defined in Example 3.1.

1. Although $1 = \sup\{i \in \mathcal{S}_{\mathbf{A}} : i \leq 1\}$, 1 is not a maximum because $1 \notin \mathcal{S}_{\mathbf{A}}$.
2. $\{i \in \mathcal{J}_{\mathbf{B}} : i < 1\}$ has a maximum, namely 0_t, the minimun element of the topmost summand. It is worth mentioning that $0_t \notin \mathcal{S}_{\mathbf{B}}$.

Corollary 3.5. *Let* $\mathbf{A}$ *be a principal Gödel algebra. If* $i, k \in A$ *are such that* $i < k$ *and* k *is a* $\vee$*-irreducible element, then* $\mathcal{C} = \{j \in A : i < j \leq k\}$ *has a minimum element* $m \in \mathcal{S}_{\mathbf{A}}$.

Proof. From Remark 1.7 we know that $\mathcal{C}$ is a non-empty totally ordered subset of $\mathcal{J}_{\mathbf{A}}$ (because $k \in \mathcal{C}$ is a $\vee$-irreducible element). Then $\mathcal{C}$ has a minimum by Lemma 3.3, say m, which actually is a completely $\vee$-irreducible element because $i \prec m$. □

Remark 3.6.

1. Corollary 3.5 says that for every $a \in \mathcal{J}_{\mathbf{A}}$ there is an $i \in \mathcal{S}_{\mathbf{A}}$ such that $i \leq a$. As a consequence,
$$a = \bigvee\{i \in \mathcal{S}_{\mathbf{A}} : i \leq a\} \text{ for each } a \in \mathcal{J}_{\mathbf{A}}.$$
This claim is trivial when $a \in \mathcal{S}_{\mathbf{A}}$. In case $a \notin \mathcal{S}_{\mathbf{A}}$, a is the supremum of all elements *strictly* below it.
2. If $i = 0$ in Corollary 3.5, then for each $k \in \mathcal{J}_{\mathbf{A}}$ there exists $m_k \in \mathcal{J}_{\mathbf{A}}$ such that $0 \prec m_k$. So m_k is minimal in $\mathcal{J}_{\mathbf{A}}$, and it is an *atom* of $\mathbf{A}$.

Theorem 3.7. *If* $\mathbf{A}$ *is a principal Gödel algebra, then*

$$\bigcap_{i\in\mathcal{S}_{\mathbf{A}}} \langle i\rangle = \{1\}.$$

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Proof. From Corollary 1.3 we know that the intersection of the prime filters of $\mathbf{A}$ is the trivial filter $\{1\}$. Under the hypothesis that $\mathbf{A}$ is principal, we also know that every prime filter is of the form $\langle i\rangle$ for some $i \in \mathcal{J}_{\mathbf{A}}$. Then

$$\{1\} = \bigcap_{i\in\mathcal{J}_{\mathbf{A}}} \langle i\rangle \subseteq \bigcap_{i\in\mathcal{S}_{\mathbf{A}}} \langle i\rangle.$$

Assume now that $a \in \bigcap_{i\in\mathcal{S}_{\mathbf{A}}}\langle i\rangle$, so that $a \geq i$ for each i in $\mathcal{S}_{\mathbf{A}}$. By Remark 3.6, since every $i \in \mathcal{J}_{\mathbf{A}}$ is the supremum of all elements in $\mathcal{S}_{\mathbf{A}}$ below it, $a \geq i$ for every $i \in \mathcal{J}_{\mathbf{A}}$. Hence

$$\bigcap_{i\in\mathcal{S}_{\mathbf{A}}} \langle i\rangle \subseteq \bigcap_{i\in\mathcal{J}_{\mathbf{A}}} \langle i\rangle,$$

as desired. □

To close this section we prove a result that will play a crucial role when dealing with representable Gödel algebras which are principal.

Lemma 3.8. *If $\mathbf{A} = \bigotimes_P \text{Ł}_2$ is a principal Gödel algebra, then P does not have infinite antichains.*

Proof. By the way of contradiction, assume the lemma is false. Let

$$F = \{x \in A : |\{i \in \mathcal{C} : x_i = 0\}| < \infty\},$$

where $\mathcal{C} \subseteq P$ is an infinite antichain with the property of being maximal in the sense that for all $i \in P$, $\mathcal{C} \cup \{i\}$ is not an antichain. It is fairly easy to see that F is a filter. Consider an infinite set $\mathcal{C}' \subsetneq \mathcal{C}$ such that $\mathcal{C} \setminus \mathcal{C}'$ is also infinite and define $x, y \in A$ satisfying that

$$x_i = \begin{cases} 0 & \text{if } i \in \mathcal{C}'; \\ 1 & \text{if } i \in \mathcal{C} \setminus \mathcal{C}'. \end{cases} \quad \text{and} \quad y_i = \begin{cases} 1 & \text{if } i \in \mathcal{C}'; \\ 0 & \text{if } i \in \mathcal{C} \setminus \mathcal{C}'. \end{cases}$$

Although $x \vee y \in F$, neither x nor y belong to F. Hence F is a proper filter of $\mathbf{A}$ which is not prime. Let $z \in A \setminus F$ be such that

$$z_i = \begin{cases} 0 & \text{if } i \in \mathcal{C}; \\ 1 & \text{if } i < j \text{ for some } j \in \mathcal{C}. \end{cases}$$

By Lemma 1.2, there is a $G = \langle m\rangle \in \mathrm{Spec}(\mathbf{A})$ such that $F \subseteq G$ and $z \notin G$. Furthermore, G properly contains F because $m \notin F$, so that $|\{i \in \mathcal{C} : m_i = 0\}| = \infty$. However, $|\{i \in \mathcal{C} : m_i = 1\}| \geq 1$, since otherwise it would be $m \leq z$ (implying the absurd $z \in G$) due to the maximality of $\mathcal{C}$. Pick an $i_0 \in \mathcal{C}$ such that $m_{i_0} = 1$. Then the element $w \in A$ with

$$w_i = \begin{cases} 0 & \text{if } i = i_0; \\ 1 & \text{if } i \in \mathcal{C} \setminus \{i_0\}. \end{cases}$$

is in F but not in G; contrary to the fact that $F \subseteq G$. Thus P has no infinite antichains.

In case $\mathcal{C}$ were not maximal, the proof works as long as the filter F is a subset of $\{x \in A : x_i = 1$ for each $j \in \mathcal{C}$ such that i and j are incomparable$\}$. □

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3.2. **Sound BL-algebras that depend on its idempotents.** We will establish some conditions under which a BL-algebra can be embedded into a poset product of indecomposable BL-chains.

We will say that a BL-algebra $\mathbf{A}$ is *sound* if

(a) $\mathbf{Id}(\mathbf{A})$ is principal.
(b) For every $F \in \mathrm{Spec}(\mathbf{Id}(\mathbf{A}))$, $[F) \in \mathrm{Spec}(\mathbf{A})$.

In Section 3.1 we studied some properties of the set $\mathcal{J}_{\mathbf{A}} \subseteq \mathrm{Id}(\mathbf{A})$ of a sound BL-algebra $\mathbf{A}$. Note that it is always the case that $\{i \in \mathrm{Id}(\mathbf{A}) : i \text{ is } \vee\text{-irreducible in } \mathbf{A}\} \subseteq \mathcal{J}_{\mathbf{A}}$ and the opposite inclusion is obvious when $\mathbf{A}$ is a chain. Then condition (b) ensures that both sets coincide if $\mathbf{A}$ is sound; in other words, the attribute of being $\vee$-irreducible is preserved from $\mathbf{Id}(\mathbf{A})$ to $\mathbf{A}$. Finite BL-algebras and principal Gödel algebras are easy examples of sound BL-algebras. The next ones are more interesting.

Example 3.9. Let $\mathbf{A} = (\bigoplus_{\mathbb{N}} \mathbf{Ł}_2) \oplus \mathbf{(0,1]_\Pi}$ and $\mathbf{B} = (\bigoplus_{\mathbb{N}} \mathbf{Ł}_2) \oplus \mathbf{(0,1]_\Pi} \oplus \mathbf{B}_t$, where $\mathbf{B}_t$ is an indecomposable BL-chain. By Example 3.1, $\mathbf{Id}(\mathbf{A}) \cong \bigoplus_{\mathbb{N}} \mathbf{Ł}_2$ and $\mathbf{Id}(\mathbf{B}) \cong (\bigoplus_{\mathbb{N}} \mathbf{Ł}_2) \oplus \mathbf{Ł}_2$ are principal Gödel chains and clearly each $\vee$-irreducible element of $\mathbf{Id}(\mathbf{A})$ (or $\mathbf{Id}(\mathbf{B})$) generates a prime filter in $\mathbf{A}$ (respectively, in $\mathbf{B}$). Therefore $\mathbf{A}$ and $\mathbf{B}$ are sound BL-chains.

The independence of conditions (a) and (b) are now illustrated.

Example 3.10. Let $\mathbf{A} = \bigotimes_{i \in \mathbb{Z}^-} \mathbf{A}_i$ be the poset product of a family of indecomposable BL-chains. It is easy to check that $\mathbf{Id}(\mathbf{A}) \cong \bigotimes_{\mathbb{Z}^-} \mathbf{Ł}_2$ is isomorphic to $\bigoplus_{\{b\} \cup \mathbb{Z}^-} \mathbf{Ł}_2$. Although $[F) \in \mathrm{Spec}(\mathbf{A})$ for every $F \in \mathrm{Spec}(\mathbf{Id}(\mathbf{A}))$, $\mathbf{A}$ is not a sound BL-chain because $\mathbf{Id}(\mathbf{A})$ is a non-principal Gödel chain by Example 3.2.

Example 3.11. Let $\mathbf{A} = \mathbf{Ł}_2 \oplus (\mathbf{(0,1]_\Pi} \times \mathbf{(0,1]_\Pi})$ be the ordinal sum of $\mathbf{Ł}_2$ and the unbounded Wajsberg hoop $\mathbf{(0,1]_\Pi} \times \mathbf{(0,1]_\Pi}$. Trivially, $\mathbf{Id}(\mathbf{A}) \cong \mathbf{Ł}_2$ is a principal Gödel algebra and $\{1\}$ is its unique prime filter. But $[1) \notin \mathrm{Spec}(\mathbf{A})$. Thus $\mathbf{A}$ is not a sound BL-algebra.

For an $i \in \mathcal{J}_{\mathbf{A}}$, let $F_i = \langle i \rangle$ be the corresponding prime filter of $\mathbf{Id}(\mathbf{A})$ and $[F_i)$ the prime filter of $\mathbf{A}$ induced by F_i. Observe that the order in $\mathcal{J}_{\mathbf{A}}$ is such that

$$i \leq j \iff \langle j \rangle \subseteq \langle i \rangle \iff [F_j) \subseteq [F_i).$$

Lemma 3.12. *Let $\mathbf{A}$ be a sound BL-algebra. For each $i \in \mathcal{S}_{\mathbf{A}}$,*

$$\mathbf{A}/[F_i) \cong \mathbf{B}_i \oplus \mathbf{A}_i,$$

where $\mathbf{B}_i$ is a (possibly trivial) BL-chain and $\mathbf{A}_i$ is an indecomposable BL-chain.

Proof. Let $F_i = \langle i \rangle$ with $i \in \mathcal{S}_{\mathbf{A}}$. The soundness of $\mathbf{A}$ guarantees that $\mathbf{A}/[F_i)$ is a BL-chain. From Corollary 1.6, $\mathbf{A}/[F_i) \cong \mathbf{[0,\mathit{i}]_A}$. Since i is completely $\vee$-irreducible in $\mathbf{Id}(\mathbf{A})$, there exists a unique idempotent element j such that $j \prec i$. Clearly, we have

$$\mathbf{A}/[F_i) \cong \mathbf{[0,\mathit{i}]_A} \cong \mathbf{[0,\mathit{j}]_A} \oplus \mathbf{[\mathit{j},\mathit{i}]_A}$$

and $\mathbf{A}_i = \mathbf{[\mathit{j},\mathit{i}]_A}$ is an indecomposable BL-chain. Set $\mathbf{B}_i = \mathbf{[0,\mathit{j}]_A}$. Observe that if $0 \prec i$, then $\mathbf{B}_i$ is trivial. $\square$

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A sound BL-algebra $\mathbf{A}$ *depends on its idempotents* if it satisfies:

(c) Let $i \in \mathcal{S}_{\mathbf{A}}$ and $a \in A$. If $a \notin [F_i)$ but $a \in [F_k)$ for all $k \in \mathcal{S}_{\mathbf{A}}$ such that $k < i$, then $a \geq j$, being $j \prec i$ in $\mathcal{J}_{\mathbf{A}}$.

(d) $\bigcap_{i \in \mathcal{S}_{\mathbf{A}}} [F_i) = \{1\}$.

The most immediate example of a BL-algebra that is sound and depends on its idempotents is the one of an indecomposable BL-chain. Some other examples are presented next:

Example 3.13. Finite BL-algebras are sound BL-algebras that depend on its idempotents. Since every of $\vee$-irreducible element is completely $\vee$-irreducible, condition (c) is immediate, as well as (d).

Example 3.14. Principal Gödel algebras are sound and due to Theorem 3.7 they satisfy condition (d), too. Let $i \in \mathcal{S}_{\mathbf{A}}$ and $j \prec i$ in $\mathcal{J}_{\mathbf{A}}$. Since $\{k \in \mathcal{S}_{\mathbf{A}} : k \leq j\} = \{k \in \mathcal{S}_{\mathbf{A}} : k < i\}$, if $a \in [F_k)$ for all $k \in \mathcal{S}_{\mathbf{A}}$ such that $k < i$, then $a \geq j = \bigvee\{k \in \mathcal{S}_{\mathbf{A}} : k \leq j\}$. Then condition (c) holds for principal Gödel algebras. Hence principal Gödel algebras depend on its idempotents.

Example 3.15. Consider the sound BL-chains $\mathbf{A}$ and $\mathbf{B}$ from Example 3.9. Recall that $\mathbf{Id}(\mathbf{A}) \cong \bigoplus_{\mathbb{N}} \mathbf{Ł}_2$ and $\mathbf{Id}(\mathbf{B}) \cong (\bigoplus_{\mathbb{N}} \mathbf{Ł}_2) \oplus \mathbf{Ł}_2$.

1. Let $i \in \mathcal{S}_{\mathbf{A}}$ and $j \prec i$ in $\mathbf{Id}(\mathbf{A})$, which is also a completely $\vee$-irreducible element in $\mathbf{Id}(\mathbf{A})$. If $a \in A$ is not in $[F_i) = [i)$ but $a \in [F_k)$ for all $k \in \mathcal{S}_{\mathbf{A}}$ such that $k < i$, then clearly $a = j$. Hence $\mathbf{A}$ satisfies condition (c). On the other hand, $\bigcap_{i \in \mathcal{S}_{\mathbf{A}}} [F_i) = (0, 1]$, so condition (d) fails for $\mathbf{A}$.
2. Since $1 \in \mathcal{S}_{\mathbf{B}}$, $\mathbf{B}$ verifies condition (d). Choosing $a = 1/2$ in B we get that $a \notin \{1\}$ and $k \leq a < 1$ for all $k \in \mathcal{S}_{\mathbf{B}} \setminus \{1\}$, thus $a \in [F_k)$ for all $k \in \mathcal{S}_{\mathbf{B}} \setminus \{1\}$. However, $a \notin B_t$ and thus (c) does not hold for $\mathbf{B}$.
3. $\mathbf{A} \oplus \mathbf{A}$ and $\mathbf{A} \times \mathbf{A}$ are sound BL-algebras in which neither (c) nor (d) holds.

3.3. Embedding into a poset product.

Theorem 3.16. *Let $\mathbf{A}$ be a sound BL-algebra that depends on its idempotents. Then $\mathbf{A}$ can be embedded into $\bigotimes_{i \in \mathcal{S}_{\mathbf{A}}} \mathbf{A}_i$, where each $\mathbf{A}_i$ is an indecomposable BL-chain.*

Proof. For each $i \in \mathcal{S}_{\mathbf{A}}$, by Lemma 3.12, we have

$$\mathbf{A}/[F_i) \cong \mathbf{B}_i \oplus \mathbf{A}_i$$

with $\mathbf{A}_i$ an indecomposable BL-chain. The universe of $\mathbf{A}_i$ is the totally ordered set $A_i = [j/[F_i), i/[F_i)]$, where $j \prec i$ in $\mathcal{J}_{\mathbf{A}}$. For simplicity we denote $0_i = j/[F_i)$ and $1_i = i/[F_i)$.

Consider the function $h \colon \mathbf{A} \to \prod_{i \in \mathcal{S}_{\mathbf{A}}} \mathbf{A}_i$ given by

$$h(a)_i = \begin{cases} a/[F_i) & \text{if } a/[F_i) \in A_i; \\ 0_i & \text{otherwise,} \end{cases} \tag{3.1}$$

where $a \in A$ and $i \in \mathcal{S}_{\mathbf{A}}$. Observe that

$$h(a)_i = 0_i \vee a/[F_i).$$

We will prove that h is an embedding. Before this, we need two technical facts.