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By

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Singular integrals with variable kernels in dyadic settings

Hugo Aimar, Raquel Crescimbeni and Luis Nowak *

Abstract

In this paper we explore conditions on variable symbols with respect to Haar systems, defining Calderón-Zygmund type operators with respect to the dyadic metrics associated to the Haar bases.We show that Petermichl's dyadic kernel can be seen as a variable kernel singular integral and we extend it to dyadic systems built on spaces of homogeneous type.

1 Introduction

The seminal work of A.P. Calderón and A. Zygmund during the fifties of the last century, regarding singular integrals and their relation to partial differential equations, can be considered the corner stone of modern Harmonic Analysis, see E. Stein in [16] for historical development of the ideas and their impact in the actual and future research in the area. Let us point out two aspects of their contributions that will help us at introducing the problems that we consider in this paper. These aspects are contained in the two papers [7] and [8]. In [7] the authors consider convolution type singular integral operators and in [8] they introduce non-convolution type kernels, also called variable kernels.

In the Calderón-Zygmund singular integral theory in metric and quasi-metric spaces (see [9], [13],[14], [1] and [10]), the distinction between convolution and non-convolution kernels does not a priori make sense because convolution is not generally defined in this setting. Nevertheless, there is still another way to consider a convolution operator. The idea goes back to the works of Mikhlin, Giraud and Tricomi (see [11], [12] and the references therein) which, aside from the depth of the analytic tools, it becomes relevant at generating convolution type filters in machine learning when the analysis is considered on non euclidean data.This way is provided by the spectral analysis of the operators, when it is available. Let us briefly sketch the basic idea in a general framework. Assume that $\{\varphi_k\}$ is an orthonormal basis for the space $L^2(X,\mu)$, where X is a measure space and μ is a Borel measure. In analogy with the Fourier case we consider convolution type operators, bounded in $L^2(X,\mu)$, as a multiplier operators of the form

$$T_\eta f(x) \ = \ \sum_k \eta_k < f, \varphi_k > \varphi_k(x),$$

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with $\eta = \{\eta_k\}$ a bounded scalar sequence. Here $< f, g >$ denotes the usual scalar product in $L^2(X, \mu)$. On the other hand, if instead of a sequence $\{\eta_k\}$ we consider in the definition of T a sequence of bounded functions of x, $\{\eta_k(x)\}$, i.e.

$$Tf(x) \; = \; \sum_k \eta_k(x) < f, \varphi_k > \varphi_k(x),$$

we say that T is an operator with variable kernel given at least formally by

$$K(x, y) \; = \; \sum_k \eta_k(x)\varphi_k(x)\varphi_k(y).$$

In the analysis of unconditionality wavelet bases in functional Banach spaces, as $L^p(\mathbb{R}^n)$, the operator defined by $T_\eta f(x) = \int_{\mathbb{R}^n} K_\eta(x, y) f(y) dy$ with a kernel given by $K_\eta(x, y) = \sum_{h \in \mathcal{H}} \eta(h)h(x)h(y)$ where $\mathcal{H}$ is the classical Haar system in $\mathbb{R}^n$ and η is some bounded sequence defined on $\mathcal{H}$, is a singular integral operator when we give to $\mathbb{R}^n$ a suitable metric structure (see [4]). Since K_η is not translation invariant, the operator T_η is not a convolution type operator in the classical euclidean sense. Nevertheless, the spectral form of $K_\eta(x, y)$ given by its symbol $\eta : \mathcal{H} \longrightarrow \mathbb{R}$, with respect to the Haar basis $\mathcal{H}$ which is independent of the points x and y, is a good reason to consider K_η as a standard convolution type kernel.

On the other hand, a kernel whose spectral Haar analysis takes the form

$$K_\eta(x, y) = \sum_{h \in \mathcal{H}} \eta(h, x)h(x)h(y)$$

for some $\eta : \mathcal{H} \times \mathbb{R}^n \longrightarrow \mathbb{R}$, can be considered a variable kernel. A special case of variable kernel K_η is considered by S. Petermichl in [15] as we shall see in Section 2.

In this work we aim to explore conditions on the variable symbol $\eta(h, x)$ in order to get kernels defining Calderón-Zygmund type operators with respect to a suitable dyadic metric. The construction of dyadic cubes due to M. Christ (see [6]) in spaces of homogeneous type becomes a basic tool in order to consider the problem in these general settings.

The paper is organized as follows. In Section 2 we consider the variable kernel structure of Petermichl's operator in $\mathbb{R}$. In Section 3 we introduce the basic properties of spaces of homogeneous type and we define the dyadic family $\mathcal{D}$, the Haar system $\mathcal{H}$ and the dyadic metric δ in this general setting. Section 4 is devoted to introduce and prove the main result of this work providing sufficient conditions in the multiplier sequence in order to obtain obtain a Calderón-Zygmund operator. Finally, in Section 5 we build Petermichl type operators on spaces of homogeneous type.

Throughout this work, we denote by C a constant that may change from one occurrence to other.

2 On the Calderón-Zygmund structure of Petermichl's kernel

In [15], S. Petermichl introduce a dyadic kernel given in terms of the Haar functions by

$$P(x, y) \; = \; \sum_{I \in \mathcal{D}} h_I(y)[h_{I^-}(x) - h_{I^+}(x)]$$

IMAL PREPRINT # 2020-0050

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for $x, y \in \mathbb{R}^+$, $\mathcal{D}$ the dyadic intervals in $\mathbb{R}^+$, h_I the Haar wavelets with support in the dyadic interval I and h_{I^-}, h_{I^+} the Haar wavelets in the left and right halves of the dyadic interval I. The corresponding operator is given by

$$\mathcal{P}f(x) \;=\; \sum_{I\in\mathcal{D}} < f, h_I > (h_{I^-}(x) - h_{I^+}(x))\,.$$

This operator is used in [15] to provide an outstanding formula for the Hilbert transform.

In [5] the authors proved that the kernel $P(x, y)$ has a standard Calderón-Zygmund structure when we consider the theory of singular integrals extended to metric measure spaces or, more precisely, to spaces of homogeneous type (see definition in Section 3). In other words, they show that

$$P(x, y) = \frac{\Omega(x, y)}{\delta(x, y)}$$

with $\delta(x, y) = |I(x, y)|$ where $I(x, y)$ is the smallest dyadic interval in $\mathbb{R}^+$ containing x and y. They also prove that Ω is bounded and smooth with respect to the ultrametric δ. Before moving to the abstract setting in order to extend P and $\mathcal{P}$, in this section we prove two elementary properties of the Petermichl's kernel that we shall explore later in the general frame work. Set $\mathcal{H}$ and $\mathcal{D}$ to denote the Haar system and dyadic family respectively in $\mathbb{R}^+$. For $h \in \mathcal{H}$ we denote with $I(h)$ the interval support of h, and we consider as I_h^{--} the left quarter of $I(h)$, I_h^{-+} as the second quarter, I_h^{+-} as the third quarter and I_h^{++} as the last quarter of $I(h)$.

Proposition 2.1.

(a) The operator $\mathcal{P}$ can be written as a variable kernel singular integral operator, in fact

$$\mathcal{P}f(x) \;=\; \frac{1}{\sqrt{2}} \sum_{h\in\mathcal{H}} \eta(x, h) < f, h > h(x)$$

with $\eta(x, h) = 1$ if $x \in I_h^{--} \cup I_h^{+-}$ and $m(x, h) = -1$ if $x \in I_h^{-+} \cup I_h^{++}$.

(b) If $\mathcal{P}^*$ denotes the adjoint of $\mathcal{P}$, then $\mathcal{P}\mathcal{P}^* = \mathcal{P}^*\mathcal{P} = 2\mathcal{I}$, twice the identity in $L^2(\mathbb{R}^+)$.

Proof. Let us start by proving *(a)*. If we denote with h^- and h^+ the Haar wavelets in the left and right halves of the support of h, respectively, we have that the supports of $h(y)h(x)$ and $h(y)[h^-(x) - h^+(x)]$ coincide as subsets of $(\mathbb{R}^+)^2$. Then in the support of $h(x)h(y)$ we have that

$$\begin{aligned} h(y)[h^-(x) - h^+(x)] &= h(y)\frac{[h^-(x) - h^+(x)]}{h(y)h(x)}h(y)h(x) \\ &= \frac{1}{\sqrt{2}}\eta(x, h)h(y)h(x), \end{aligned}$$

as desired.

In order to prove *(b)* observe that

$$\mathcal{P}^*f(y) \;=\; \sum_{I\in\mathcal{D}} (\langle f, h_{I^-}\rangle - \langle f, h_{I^+}\rangle)\, h_I(y).$$

IMAL PREPRINT # 2020-0050

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On the other hand, from the orthonormality of the system $\mathcal{H}$, for each $I \in \mathcal{D}$ we have that

$$\left\langle \sum_{J\in\mathcal{D}} < f, h_J > (h_{J^-} - h_{J^+}) \, , \, h_{I^-} \right\rangle = < f, h_I >< h_{I^-}, h_{I^-} >$$

and

$$\left\langle \sum_{J\in\mathcal{D}} < f, h_J > (h_{J^-} - h_{J^+}) \, , \, h_{I^+} \right\rangle = < f, h_I >< h_{I^+}, h_{I^+} > .$$

Therefore

$$\begin{aligned} \mathcal{P}^*(\mathcal{P}f)(y) &= \sum_{I\in\mathcal{D}} (< \mathcal{P}f, h_{I^-} > - < \mathcal{P}f, h_{I^+} >)\, h_I(y) \\ &= \sum_{I\in\mathcal{D}} \langle f, h_I \rangle \langle h_{I^-}, h_{I^-} \rangle \, h_I(y) + \sum_{I\in\mathcal{D}} \langle f, h_I \rangle \langle h_{I^+}, h_{I^+} \rangle \, h_I(y) \\ &= 2f, \end{aligned}$$

as desired. □

3 Dyadic families and Haar systems in spaces of homogeneous type

Let us first briefly recall the basic properties of the general theory of spaces of homogeneous type. Assume that X is a set, a nonnegative symmetric function d on $X \times X$ is called a quasi-distance if there exists a constant K such that

$$d(x, y) \le K[d(x, z) + d(z, y)],$$

for every $x, y, z \in X$, and $d(x, y) = 0$ if and only if $x = y$.

We shall say that (X, d, μ) is a space of homogeneous type if d is a quasi-distance on X, μ is a positive Borel measure defined on a σ-algebra of subsets of X which contains the balls, and there exists a constant A such that

$$0 \; < \; \mu(B(x, 2r)) \; \le \; A \; \mu(B(x, r)) \; < \; \infty \tag{3.1}$$

holds for every $x \in X$ and every $r > 0$. This property is usually named as the doubling condition.

The construction of dyadic type families of subsets in metric or quasi-metric spaces with some inner and outer metric control of the sizes of the dyadic sets is given in [6]. These families satisfy all the relevant properties of the usual dyadic cubes in $\mathbb{R}^n$ and are the basic tool to build wavelets on a metric space of homogeneous type (see [1] or [2]). Actually Christ's construction in [6] shows the existence of dyadic families in spaces of homogeneous type. Nevertheless, in order to define Haar wavelets all we need is a dyadic family satisfying the following properties that we state as a definition and we borrow from [2].

Definition 3.1. Let (X, d, μ) be a metric space of homogeneous type. We say that $\mathcal{D} = \bigcup_{j\in\mathbb{Z}} \mathcal{D}^j$ is a dyadic family on X with parameter $\lambda \in (0, 1)$ if each $\mathcal{D}^j$ is a family of Borel subsets Q of X, such that

IMAL PREPRINT # 2020-0050

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Publication date: December 23, 2020

(d.1) *for every* $j \in \mathbb{Z}$ *the cubes in* $\mathcal{D}^j$ *are pairwise disjoint;*

(d.2) *for every* $j \in \mathbb{Z}$ *the family* $\mathcal{D}^j$ *covers* X *in the sense that* $X = \bigcup_{Q\in\mathcal{D}^j} Q$;

(d.3) *if* $Q \in \mathcal{D}^j$ *and* $i < j$, *then there exists a unique* $\tilde{Q} \in \mathcal{D}^i$ *such that* $Q \subseteq \tilde{Q}$;

(d.4) *if* $Q \in \mathcal{D}^j$ *and* $\tilde{Q} \in \mathcal{D}^i$ *with* $i \leq j$, *then either* $Q \subseteq \tilde{Q}$ *or* $Q \cap \tilde{Q} = \emptyset$;

(d.5) *there exist two constants* a_1 *and* a_2 *such that for each* $Q \in \mathcal{D}^j$ *there exists a point* $x \in Q$ *that satisfies* $B(x, a_1\lambda^j) \subseteq Q \subseteq B(x, a_2\lambda^j)$.

The following properties can be deduced from *(d.1)* to *(d.5)*, see [3].

Lemma 3.2. *Let* $\mathcal{D}$ *be a dyadic family, then*

(d.6) *there exists a positive integer* M *depending on* a_i, $i = 1, 2$ *in* $(d.5)$ *and on the doubling constant* A *in* (3.1) *such that for every* $j \in \mathbb{Z}$ *and all* $Q \in \mathcal{D}^j$ *the inequalities* $1 \leq \#(\mathcal{L}(Q)) \leq M$ *hold, where* $\mathcal{L}(Q) = \{Q' \in \mathcal{D}^{j+1} : Q' \subseteq Q\}$ *and* $\#(B)$ *denote the cardinal of* B;

(d.7) *there exists a positive constant* C *such that* $\mu(Q) \leq C\mu(Q')$ *for all* $Q \in \tilde{\mathcal{D}}$ *and every* $Q' \in \mathcal{L}(Q)$.

It is easy to give examples of dyadic systems $\mathcal{D}$ such that a dyadic cube Q belong to different levels $j \in \mathbb{Z}$. Since we are interested in the identification of those scales and places of partition which shall give rise to the Haar functions, we consider the subfamily $\tilde{\mathcal{D}}$ of $\mathcal{D}$ given by

$$\tilde{\mathcal{D}} = \bigcup_{j\in\mathbb{Z}} \tilde{\mathcal{D}}^j,$$

with

$$\tilde{\mathcal{D}}^j = \{Q \in \mathcal{D}^j : \#(\{Q' \in \mathcal{D}^{j+1} : Q' \subseteq Q\}) > 1\}.$$

Properties *(d.1)* to *(d.6)* allow us to obtain the following aditional properties for $\tilde{\mathcal{D}}$.

(d.8) The families $\tilde{\mathcal{D}}^j$, $j \in \mathbb{Z}$ are pairwise disjoints.

(d.9) The function $\mathcal{J} : \tilde{\mathcal{D}} \longrightarrow \mathbb{Z}$ given by $Q \mapsto \mathcal{J}(Q)$ if $Q \in \tilde{\mathcal{D}}^{\mathcal{J}(Q)}$ is well defined.

Let $\mathcal{D}$ be a dyadic family. We define, for each dyadic cube Q in $\mathcal{D}$, the quadrant of X that contain the cube Q, $\mathbf{C}(Q)$, by

$$\mathbf{C}(Q) = \bigcup_{\{Q'\in\mathcal{D}: Q\subseteq Q'\}} Q'.$$

Following the lines in [2] for the case of Christ's dyadic cube, from *(d.6)* and since all the dyadic cubes Q in $\mathcal{D}$ are spaces of homogeneous type with uniform doubling constant, we get that if (X, d, μ) is a space of homogeneous type and if $\mathcal{D}$ is a dyadic family, then there exists a positive integer N (that depend of the geometric constants of (X, d, μ)) and disjoint dyadic cubes $Q_\alpha, \alpha = 1, ..., N$ such that

$$X = \bigcup_{\alpha=1,...,N} \mathbf{C}_\alpha,$$

where $\mathbf{C}_\alpha = \mathbf{C}(Q_\alpha)$. That is, there exists a finite number of quadrants these are a partition of X and each one of them is a space of homogeneous type (see [2])

IMAL PREPRINT # 2020-0050
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Publication date: December 23, 2020

In the classic euclidean context $\mathbb{R}^n$, the dyadic analysis leads to consider each quadrant separately. Then, without loss of generality, we will assume from now on that X itself is a quadrant for $\mathcal{D}$.

Along this work, given a dyadic family $\mathcal{D}$ we denote by $\delta(x,y)$ the dyadic metric associated to $\mathcal{D}$ for $x,y \in X$. That is δ is the function defined in $X \times X$ given by

$$\delta(x,y) = \begin{cases} min\{\mu(Q) : x,y \in Q, Q \in \tilde{\mathcal{D}}\} & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y. \end{cases} \tag{3.2}$$

Now we state and prove the main result of this section. The proof follow the technique used in [13] where the authors prove that each quasi-metric space (X,d) is metrizable and that d is equivalent to ρ^β, where ρ is a distance on X and $\beta \geq 1$. Moreover, they show that all spaces of homogeneous type (X,d,μ) can be normalized in the sense that there exists a metric ρ on X and two constants C_1 y C_2 such that

$$C_1 r \leq \mu(B_\rho(x,r)) \leq C_2 r, \tag{3.3}$$

where $B_\rho(x,r) = \{y \in X \ : \ \rho(x,y) < r\}$. In general, if ρ satisfies (3.3), we say that (X,ρ,μ) is a normal space of homogeneous type or $1-$Ahlfors.

Lemma 3.3. *Let (X,d,μ) be a space of homogeneous type and let $\mathcal{D}$ be a dyadic family. Then (X,δ,μ) is a normal space of homogeneous type. Moreover, the characteristic functions of dyadic cubes are Lipschitz functions in (X,δ).*

Proof. For each $z \in X$ we write $Q_j(z)$ to denote the unique dyadic cube $Q \in \tilde{\mathcal{D}}_j$ such that $z \in Q$. Without loss of generality we can assume that X in not bounded. Thus, if $x \in X$, $r > 0$, and j_0 is an integer in $\mathbb{Z}$ such that

$$\mu(Q_{j_0}(x)) \ \leq r \ < \ \mu(Q_{j_0-1}(x)), \tag{3.4}$$

then

$$B_\delta(x,r) = Q_{j_0}(x). \tag{3.5}$$

In fact if $y \in Q_{j_0}(x)$ then $x,y \in Q_{j_0}(x)$ and therefore $\delta(x,y) \leq \mu(Q_{j_0}(x)) \ \leq r$ this implies that $Q_{j_0}(x) \subseteq B_\delta(x,r)$. On the other hand, let $y \in B_\delta(x,r)$, if $y \notin Q_{j_0}(x)$ then $Q_{j_0}(x) \cap Q_{j_0}(y) = \emptyset$. Let $n \in \mathbb{N}$ be the first positive integer such that $Q_{j_0}(y) \subseteq Q_{j_0-n}(x)$, then we get that $\delta(x,y) = \mu(Q_{j_0-n}(x)) \geq \mu(Q_{j_0-1}(x)) > r$, this is a contradiction. Hence $y \in Q_{j_0}(x)$ and then $B_\delta(x,r) \subseteq Q_{j_0}(x)$. In orden to prove that (X,δ,μ) is a normal space of homogeneous type, observe that it is not difficult to see that (X,δ) is a metric space (see [2]) moreover, δ is an ultra-metric on X. Let $x \in X$ be and $r > 0$, consider the number j_0 given in (3.4). Since $B_\delta(x,r) = Q_{j_0}(x)$, we get that $\mu(B_\delta(x,r)) = \mu(Q_{j_0}(x)) \leq r$. On the other hand, since $Q_{j_0}(x) \subseteq \mathcal{L}(Q_{j_0-1}(x))$, by the doubling property of the measure (3.1) there exists a positive constant C such that $\mu(Q_{j_0-1}(x)) \leq C\mu(Q_{j_0}(x))$, then from (3.4) and (3.5) we get that

$$r < \mu(Q_{j_0-1}(x)) \leq C\mu(Q_{j_0}(x)) = C\mu(B_\delta(x,r)).$$

Hence, $\frac{r}{C} < \mu(B_\delta(x,r))$. Finally, for the last statement, let $x,y \in X$ and $Q \in \tilde{\mathcal{D}}$. If $x,y \in Q$ or if $y \notin Q$, $x \notin Q$, then $\chi_Q(x) - \chi_Q(y) = 0$. If Q contain only the point x or the point y and $Q(x,y)$ is the smallest dyadic cube such that $x,y \in Q(x,y)$, then $\delta(x,y) = \mu(Q(x,y)) \geq \mu(Q)$. Hence $|\chi_Q(x) - \chi_Q(y)| = 1 \leq \frac{1}{\mu(Q)}\delta(x,y)$. □

IMAL PREPRINT # 2020-0050

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From now on we shall denote by $Q(x, y)$ the smallest dyadic cube such that $x, y \in Q(x, y)$. From each dyadic system $\mathcal{D}$ as above we can associate a Haar type systems that we borrow from ([3]).

Definition 3.4. Let $\mathcal{D}$ be a dyadic family on (X, d, μ). A system $\mathcal{H}$ of simple Borel measurable real functions h on X is said to be a Haar system associated to $\mathcal{D}$ if it is an orthonormal basis of $L^2(X, \mu)$ such that

(h.1) *For each* $h \in \mathcal{H}$ *there exists a unique* $j \in \mathbb{Z}$ *and a cube* $Q(h) \in \tilde{\mathcal{D}}^j$ *such that* $\{x \in X : h(x) \neq 0\} \subseteq Q(h)$*, and this property does not hold for any cube in* $\mathcal{D}^{j+1}$.

(h.2) *For every* $Q \in \tilde{\mathcal{D}}$ *there exist exactly* $M_Q = \#(\mathcal{L}(Q)) - 1 \geq 1$ *functions* $h \in \mathcal{H}$ *such that (h.1) holds. We denote with* $\mathcal{H}(Q)$ *the set of all these functions* h.

(h.3) *For each* $h \in \mathcal{H}$ *we have that* $\int_X h d\mu = 0$.

(h.4) *For each* $Q \in \tilde{\mathcal{D}}$ *let* V_Q *denote the vector space of all functions on* Q *which are constant on each* $Q' \in \mathcal{L}(Q)$*. Then the system* $\{\frac{\chi_Q}{(\mu(Q))^{1/2}}\} \bigcup \mathcal{H}(Q)$ *is an orthonormal basis for* V_Q.

(h.5) *There exists a positive constant* C *such that the inequality* $|h(x)| \leq C|h(y)|$ *holds for almost every* x *and* y *in* $Q(h)$ *and every* $h \in \mathcal{H}$.

Observe also that from *(d.7)*, *(h.4)* and *(h.5)* we get that there exists two positive constants C_1 and C_2 such that

$$\frac{C_1}{\mu(Q(h))^{1/2}} \leq |h(x)| \leq \frac{C_2}{\mu(Q(h))^{1/2}}, \tag{3.6}$$

for all $h \in \mathcal{H}$ and $x \in Q(h)$.

4 On convolution and non-convolution type singular integral operators in metric measure spaces.

Let (X, d, μ) a space of homogeneous type, $\mathcal{D}$ and $\mathcal{H}$ the dyadic family of cubes and the Haar system associated given in Definitions 3.1 and 3.4 respectively. For simplicity we denote by $L^2 = L^2(X, \mu)$ of square integrable real functions defined on X. Since $\mathcal{H}$ is an orthonormal basis for L^2, we have the resolution of the identity given by

$$f = \sum_{h \in \mathcal{H}} \langle f, h \rangle h.$$

The operators

$$T_\eta f(x) = \sum_{h \in \mathcal{H}} \eta(h) \langle f, h \rangle h(x), \tag{4.1}$$

with η a bounded function defined on $\mathcal{H}$, or more generally

$$T_\eta f(x) = \sum_{h \in \mathcal{H}} \eta(x, h) \langle f, h \rangle h(x), \tag{4.2}$$

with η a bounded function defined on $X \times \mathcal{H}$, are bounded in L^2.

IMAL PREPRINT # 2020-0050

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With the heuristics described in the introduction we may think that the operator as in (4.1) is of convolution type while that in (4.2) is of non-convolution type singular. In this section we give a sufficient condition on $\eta(x,h)$ in such a way that T_η defined by (4.2) becomes a Calderón-Zygmund type operator in (X,d,μ).

A bounded linear operator $T : L^2 \longrightarrow L^2$ is said to be of Calderón-Zygmund type in (X,δ,μ) if there exists $K \in L^1_{loc}(X\times X\setminus\Delta)$, with Δ the diagonal of $X\times X$, such that

(1) there exists a positive constant C such that $|K(x,y)| \leq \frac{C}{\delta(x,y)}$ for $x,y\in X$ with $x\neq y$,

(2) there exists two positive constants C and γ such that

(2.a) $|K(x',y)-K(x,y)| \leq C\frac{\delta(x',x)^\gamma}{\delta(x,y)^{1+\gamma}}$, if $2\delta(x',x)\leq\delta(x,y)$;

(2.b) $|K(x,y')-K(x,y)| \leq C\frac{\delta(y,y')^\gamma}{\delta(x,y)^{1+\gamma}}$, if $2\delta(y',y)\leq\delta(x,y)$;

(3) for $\varphi,\psi\in\mathcal{S}(\mathcal{H})$, the linear span of $\mathcal{H}$, with $supp\varphi\cap supp\psi=\emptyset$, we have

$$< T(\varphi),\psi > = \int\int_{X\times X} K(x,y)\varphi(x)\psi(y)d(\mu\times\mu)(x,y).$$

The main result of this section is contained in the following statement.

Theorem 4.1. *Let (X,d,μ) a space of homogeneous type, $\mathcal{D}$ a dyadic family, $\mathcal{H}$ a Haar system and δ defined in (3.2). Let $\eta : X\times\mathcal{H}\longrightarrow\mathbb{R}$ be a function such that is a measurable function in $x\in X$ for each $h\in\mathcal{H}$ and there exists a constant $B>0$ such that*

(a) $|\eta(x,h)|\leq B$, for $x\in X$ and $h\in\mathcal{H}$

(b) $|\eta(x',h)-\eta(x,h)|\leq B\frac{\delta(x,x')}{\mu(Q(h))}$, for $h\in\mathcal{H}$ and $x,x'\in X$.

Then the operator

$$T_\eta f(x) \;=\; \sum_{h\in\mathcal{H}} \eta(x,h)\,\langle f,h\rangle\, h(x)$$

is of Calderón-Zygmund type in the space of homogeneous type (X,δ,μ). Hence T_η is bounded on $L^p(X)$ $(1<p<\infty)$ and of weak type $(1,1)$.

Proof. The L^2 boundedness of T_η follows from (a) with $\|T_\eta f\|_2\leq\|\eta\|_\infty\|f\|_2$. By testing T_η with simple function in $\mathcal{S}(\mathcal{H})$, we see that

$$K(x,y) \;=\; \sum_{h\in\mathcal{H}} \eta(x,h)h(y)h(x)$$

satisfies property (3) in the above definition of Calderón-Zygmund kernel in the general setting. Let us prove (1) of the definition of Calderón-Zygmund type operator. Let $x\neq y$ in X and $Q(x,y)$ in $\mathcal{D}$ such that $\mu(Q(x,y))=\delta(x,y)$. On the other hand for any cube strictly smaller than $Q(x,y)$ we must have $h(y)=0$ or $h(x)=0$. Hence from $(h.1)$, (3.6) and $(d.6)$ we get

$$|K(x,y)| \;\leq\; C\|\eta\|_\infty \sum_{Q\supseteq Q(x,y)}\;\sum_{\{h\in\mathcal{H}:Q(h)=Q\}}\frac{1}{\mu(Q)}$$

IMAL PREPRINT # 2020-0050

ISSN 2451-7100
Publication date: December 23, 2020

$$\leq \quad C\|\eta\|_\infty M \sum_{Q\supseteq Q(x,y)} \frac{1}{\mu(Q)},$$

where M is as in $(d.6)$ in Lemma 3.2. Notice that we are considering only the cubes in $\tilde{\mathcal{D}}$. Then if Q_m is the $m-$th ancestor of $Q(x,y)$ in $\tilde{\mathcal{D}}$, the measure of this sequence grows geometrically, i.e.

$$\mu(Q_m) \geq (1+\varepsilon)^m \mu(Q(x,y)) \tag{4.3}$$

with a geometric constant $\varepsilon > 0$. Hence

$$|K(x,y)| \;\leq\; \frac{C}{\delta(x,y)}$$

as desired. Let us now prove the smoothness properties of K. Notice first that, from (4.3) we get that

$$\begin{aligned}
\sum_{\substack{Q\in\tilde{\mathcal{D}}\\ Q\supseteq Q(x,y)}} \frac{1}{(\mu(Q))^2} \quad &= \sum_{m\in\mathbb{N}} \frac{1}{(\mu(Q_{m-1}))^2}\\
&\leq \sum_{m\in\mathbb{N}} \left(\frac{1}{(1+\varepsilon)^2}\right)^{m-1} \frac{1}{(\mu(Q_0))^2}\\
&= \frac{1}{(\mu(Q_0))^2} \sum_{m\in\mathbb{N}} \left(\frac{1}{(1+\varepsilon)^2}\right)^{m-1}\\
&= \frac{C}{(\mu(Q_0))^2},
\end{aligned} \tag{4.4}$$

where $Q_0 = Q(x,y)$ in $\tilde{\mathcal{D}}$. On the other hand, notice that for $h \in \mathcal{H}$ if $Q = Q(h) \in \tilde{\mathcal{D}}$, then

$$h(x) \;=\; \sum_{Q'\in\mathcal{L}(Q)} \beta_{Q'}\chi_{Q'}(x),$$

where $\beta_{Q'} \in \mathbb{R}$. Thus, since the characteristic functions on dyadic cube are Lipschitz functions on (X,δ), from dyadic doubling property, $(d.6)$ and (3.6) there exists a positive constant C such that if $x, x' \in X$ we get that

$$\begin{aligned}
|h(x) - h(x')| \quad &\leq \sum_{Q'\in\mathcal{L}(Q(h))} |\beta_{Q'}| \left|\chi_{Q'}(x) - \chi_{Q'}(x')\right|\\
&\leq \sum_{Q'\in\mathcal{L}(Q(h))} \|h\|_\infty \left|\chi_{Q'}(x) - \chi_{Q'}(x')\right|\\
&\leq \frac{C}{\sqrt{\mu(Q(h))}} \sum_{Q'\in\mathcal{L}(Q(h))} \left|\chi_{Q'}(x) - \chi_{Q'}(x')\right|\\
&\leq \frac{C}{\sqrt{\mu(Q(h))}} \sum_{Q'\in\mathcal{L}(Q(h))} \frac{\delta(x,x')}{\mu(Q')}
\end{aligned}$$

IMAL PREPRINT # 2020-0050

ISSN 2451-7100
Publication date: December 23, 2020

$$
\begin{aligned}
&\leq \frac{C^2}{\sqrt{\mu(Q(h))}} \sum_{Q' \in \mathcal{L}(Q(h))} \frac{\delta(x,x')}{\mu(Q(h))} \\
&\leq \frac{C^2\delta(x,x')}{(\mu(Q(h)))^{\frac{3}{2}}} \# \mathcal{L}(Q(h)) \\
&\leq MC^2 \frac{\delta(x,x')}{(\mu(Q(h)))^{\frac{3}{2}}} \\
&\leq C\frac{\delta(x,x')}{\mu(Q(h))^{\frac{3}{2}}}.
\end{aligned} \tag{4.5}
$$

Observe now that if $x, y, x' \in X$ satisfy $2\delta(x', x) \leq \delta(x, y)$ then $x' \in Q(x, y)$ and moreover

$$Q(x, y) = Q(x', y).$$

In fact, if $x' \notin Q(x, y)$ then $\delta(x, x') > \delta(x, y)$, which is a contradiction. On the other hand, since $Q(x, y) \in \tilde{\mathcal{D}}$, there exists two different dyadic cubes Q' and $\hat{Q}$ in $\mathcal{L}(Q(x, y))$ such that $y \in Q'$ and $x \in \hat{Q}$. So, if $x' \in X$ satisfies $2\delta(x', x) \leq \delta(x, y)$ and we suppose that $x' \notin \hat{Q}$, then

$$\delta(x, x') \; = \; \mu(Q(x, y)) \; = \; \delta(x, y),$$

which is again a contradiction. Then if $2\delta(x', x) \leq \delta(x, y)$ we have $Q(x, y) = Q(x', y)$, this implies that $\delta(x, y) = \delta(x', y)$. Hence in such case, from the conditions (a) and (b) on η, (4.5), (4.4) and (3.6) we get that

$$
\begin{aligned}
\left|\big(\eta(x',h)h(x') - \eta(x,h)h(x)\big)\, h(y)\right| &= \big(\left|\eta(x',h) - \eta(x,h)\right| |h(x')| + |\eta(x,h)| \left|h(x') - h(x)\right|\big)\, |h(y)| \\
&\leq \left(\frac{CB\delta(x,x')}{(\mu(Q(h)))^{3/2}} + B\frac{MC^2\delta(x,x')}{(\mu(Q(h)))^{3/2}} \right) |h(y)| \\
&\leq \left(\frac{CB\delta(x,x')}{(\mu(Q(h)))^{2}} + B\frac{MC^2\delta(x,x')}{(\mu(Q(h)))^{2}} \right) \\
&= C\frac{\delta(x,x')}{(\mu(Q(h)))^2}.
\end{aligned}
$$

Then from the above estimate and (4.4) we get that

$$
\begin{aligned}
|K(x',y) - K(x,y)| &= \left| \sum_{h\in\mathcal{H}} \big(\eta(x',h)h(x') - \eta(x,h)h(x)\big)\, h(y) \right| \\
&= \left| \sum_{\substack{Q\in\tilde{\mathcal{D}} \\ Q\supseteq Q(x,y)}} \sum_{\substack{h\in\mathcal{H} \\ Q(h)=Q}} \big(\eta(x',h)h(x') - \eta(x,h)h(x)\big)\, h(y) \right| \\
&= C \sum_{\substack{Q\in\tilde{\mathcal{D}} \\ Q\supseteq Q(x,y)}} \sum_{\substack{h\in\mathcal{H} \\ Q(h)=Q}} \frac{\delta(x,x')}{(\mu(Q(h)))^2}
\end{aligned}
$$

IMAL PREPRINT # 2020-0050

ISSN 2451-7100
Publication date: December 23, 2020

$$\leq C \sum_{\substack{Q \in \tilde{\mathcal{D}} \\ Q \supseteq Q(x,y)}} \frac{\delta(x,x')}{(\mu(Q))^2}$$

$$= C \frac{\delta(x,x')}{(\mu(Q(x,y)))^2}$$

$$= C \frac{\delta(x,x')}{(\delta(x,y))^2},$$

this complete the proof of $(2.a)$. In a similar way we can prove $(2.b)$.

□

5 Petermichl's type operators in spaces of homogeneous type

In this section we introduce Petermichl type operators $\mathcal{P}$ on spaces of homogeneous type. We prove, using Theorem 4.1, that this operator is a Calderón-Zygmund type operator on a suitable space of homogeneous type and we show that $\mathcal{P}^*$ is almost the identity operator in a sense that shall be made precise.

Let (X, d, μ) be a space of homogeneous type, $\mathcal{D}$ a dyadic family, $\mathcal{H}$ a Haar system associated to $\mathcal{D}$ and $(\alpha_h)_{h\in\mathcal{H}}$ a bounded sequence in $\mathbb{R}$. For $f \in L^2(X, \mu)$ we consider the operator $\mathcal{P}$ defined as

$$\mathcal{P}f(x) = \sum_{Q \in \tilde{\mathcal{D}}} \sum_{\substack{h \in \mathcal{H} \\ Q(h)=Q}} < f, h > \left(\sum_{\substack{\tilde{h} \in \mathcal{H}(R) \\ R \in \mathcal{L}(Q)}} \alpha_{\tilde{h}} \tilde{h}(x) \right)$$

where we recall that $\mathcal{H}(R)$ is given in $(h.2)$.

Proposition 5.1. *Let (X, d, μ) be a space of homogeneous type, $\mathcal{D}$ the dyadic family, $\mathcal{H}$ the Haar system associated to $\mathcal{D}$ and $(\alpha_h)_{h\in\mathcal{H}}$ a bounded sequence in $\mathbb{R}$. Then the operator $\mathcal{P}$ satisfies the following properties*

(1)

$$\mathcal{P}f(x) = \int_{y\in X} N(x,y) f(y) d\mu(y),$$

where $N(x,y) = \sum_{Q \in \tilde{\mathcal{D}}} \sum_{\substack{h \in \mathcal{H} \\ Q(h)=Q}} h(y) \left(\sum_{\substack{\tilde{h} \in \mathcal{H}(R) \\ R \in \mathcal{L}(Q)}} \alpha_{\tilde{h}} \tilde{h}(x) \right)$ *and f is a simple function in $\mathcal{S}(\mathcal{H})$.*

(2)

$$\mathcal{P}^* f(x) = \int_{y\in X} N^*(x,y) f(y) d\mu(y),$$

where $N^*(z,w) = N(w,z)$ and f is a simple function in $\mathcal{S}(\mathcal{H})$.

IMAL PREPRINT # 2020-0050

ISSN 2451-7100
Publication date: December 23, 2020

(3)

$$\mathcal{P}^*(\mathcal{P}f)(x) = \sum_{h\in\mathcal{H}} C(Q) < f, h > h(x),$$

with $1 \leq C(Q) \leq M^2$*, with* M *as in* $(d.6)$ *in Lemma 3.2.*

Proof. In order to prove (1), we observe that for f in $\mathcal{S}(\mathcal{H})$ the sum in the definition of $\mathcal{P}f(x)$ is finite and therefore we have that

$$\begin{aligned} \mathcal{P}f(x) &= \int_{y\in X} \left(\sum_{Q\in\tilde{\mathcal{D}}} \sum_{\substack{h\in\mathcal{H} \\ Q(h)=Q}} h(y) \left(\sum_{\substack{\tilde{h}\in\mathcal{H}(R) \\ R\in\mathcal{L}(Q)}} \alpha_{\tilde{h}} \tilde{h}(x) \right) \right) f(y)d\mu(y) \\ &= \int_{y\in X} N(x,y)f(y)d\mu(y), \end{aligned}$$

where

$$N(x,y) = \sum_{Q\in\tilde{\mathcal{D}}} \sum_{\substack{h\in\mathcal{H} \\ Q(h)=Q}} h(y) \left(\sum_{\substack{\tilde{h}\in\mathcal{H}(R) \\ R\in\mathcal{L}(Q)}} \alpha_{\tilde{h}} \tilde{h}(x) \right).$$

On the other hand,

$$\mathcal{P}^*f(z) = \int_{w\in X} N^*(z,w)f(w)d\mu(w),$$

for $N^*(z,w) = N(w,z)$.
Finally we compute the action of $\mathcal{P}^*$ on $\mathcal{P}$. By Fubini's theorem we get that

$$\begin{aligned} \mathcal{P}^*(\mathcal{P}f)(x) &= \int_{y\in X} N^*(x,y)\mathcal{P}f(y)d\mu(y) \\ &= \int_{y\in X} N^*(x,y) \int_{z\in X} N(y,z)f(z)d\mu(z)d\mu(y) \\ &= \int_{y\in X} N(y,x) \int_{z\in X} N(y,z)f(z)d\mu(z)d\mu(y) \\ &= \int_{z\in X} \left(\int_{y\in X} N(y,x)N(y,z)d\mu(y) \right) f(z)d\mu(z) \\ &= \int_{z\in X} U(x,z)f(z)d\mu(z), \end{aligned}$$

where

$$\begin{aligned} U(x,z) &= \int_{y\in X} N(y,x)N(y,z)d\mu(y) \\ &= \sum_{Q\in\tilde{\mathcal{D}}} \sum_{\substack{h\in\mathcal{H} \\ Q(h)=Q}} \sum_{Q'\in\tilde{\mathcal{D}}} \sum_{\substack{h'\in\mathcal{H} \\ Q(h')=Q}} h(x)h'(z) \int_{y\in X} \sum_{\substack{\tilde{h}\in\mathcal{H}(R) \\ R\in\mathcal{L}(Q)}} \alpha_{\tilde{h}} \tilde{h}(y) \sum_{\substack{\hat{h}\in\mathcal{H}(R') \\ R'\in\mathcal{L}(Q')}} \alpha_{\hat{h}} \hat{h}(y)d\mu(y). \end{aligned}$$

IMAL PREPRINT # 2020-0050

ISSN 2451-7100
Publication date: December 23, 2020

Now, by the orthogonality of the Haar system

$$\begin{aligned}\int_{y\in X}\sum_{\substack{\tilde{h}\in\mathcal{H}(R)\\ R\in\mathcal{L}(Q)}}\alpha_{\tilde{h}}\tilde{h}(y)\sum_{\substack{\hat{h}\in\mathcal{H}(R')\\ R'\in\mathcal{L}(Q')}}\alpha_{\hat{h}}\hat{h}(y)d\mu(y) &= \sum_{\substack{\tilde{h}\in\mathcal{H}(R)\\ R\in\mathcal{L}(Q)}}\sum_{\substack{\hat{h}\in\mathcal{H}(R')\\ R'\in\mathcal{L}(Q')}}\alpha_{\tilde{h}}\alpha_{\hat{h}}\int_{y\in X}\tilde{h}(y)\hat{h}(y)d\mu(y)\\ &= \sum_{\substack{\tilde{h}\in\mathcal{H}(R)\\ R\in\mathcal{L}(Q)}}\alpha_{\tilde{h}}^2\\ &= (\#(\mathcal{L}(Q)))\,(\#(\mathcal{L}(R)-1))\\ &= C_{(Q)}.\end{aligned}$$

Therefore

$$\begin{aligned}U(x,z) &= \int_{y\in X}N(y,x)N(y,z)d\mu(y)\\ &= \sum_{Q\in\tilde{\mathcal{D}}}\sum_{\substack{h\in\mathcal{H}\\ Q(h)=Q}}C_{(Q)}h(x)h(z).\end{aligned}$$

Thus

$$\mathcal{P}^*(\mathcal{P}f)(x)=\int_{z\in X}\left(\sum_{Q\in\tilde{\mathcal{D}}}\sum_{\substack{h\in\mathcal{H}\\ Q(h)=Q}}C_{(Q)}h(x)h(z)\right)f(z)d\mu(z).$$

with

$$1\ \le C_{(Q)}=\ (\#(\mathcal{L}(Q))\,(\#(\mathcal{L}(R)-1)\ \le\ M^2$$

as desired.

□

As an application of Theorem 4.1 we obtain the boundedness of these operators in Lebesgue spaces.

Theorem 5.2. *Let (X,d,μ) be a space of homogeneous type. Let $\mathcal{D}$, $\mathcal{H}$ and δ be a dyadic family, a Haar systems associated to $\mathcal{D}$ and the dyadic metric induced by $\mathcal{D}$ respectively. Let $(\alpha_h)_{h\in\mathcal{H}}$ be a bounded sequence in $\mathbb{R}$. Then the operator*

$$\mathcal{P}f(x)=\sum_{Q\in\tilde{\mathcal{D}}}\sum_{\substack{h\in\mathcal{H}\\ Q(h)=Q}}<f,h>\left(\sum_{\substack{\tilde{h}\in\mathcal{H}(R)\\ R\in\mathcal{L}(Q)}}\alpha_{\tilde{h}}\tilde{h}(x)\right)$$

is a Calderón-Zygmund type operator on the space (X,δ,μ). Hence $\mathcal{P}$ is bounded in $L^p(X)$ $(1<p<\infty)$ and of weak type $(1,1)$.

IMAL PREPRINT # 2020-0050

ISSN 2451-7100
Publication date: December 23, 2020

Proof. By Theorem 4.1 it is enough to prove that the operator $\mathcal{P}$ can be written as

$$\mathcal{P}f(x) = \sum_{h\in\mathcal{H}} \eta(x,h)\,\langle f,h\rangle\, h(x)$$

for some function $\eta : X\times\mathcal{H} \longrightarrow \mathbb{R}$ satisfying the hypothesis in Theorem 4.1. In fact for $h\in\mathcal{H}$ with $Q = Q(h)\in\mathcal{D}$ we have that

$$h(x) = \sum_{R\in\mathcal{L}(Q(h))} h_R\ \chi_R(x),$$

where $h_R\in\mathbb{R}$. Thus, as h is different from zero on $Q(h)$, we define for $x\in X$,

$$\eta(x,h) = \sum_{R\in\mathcal{L}(Q(h))}\left(\sum_{\tilde{h}\in\mathcal{H}(R)}\frac{\alpha_{\tilde{h}}}{h_R}\tilde{h}(x)\right)\chi_R(x),$$

which is a measurable function for $x\in X$. Then we get that

$$\eta(x,h)h(x) = \sum_{\substack{\tilde{h}\in\mathcal{H}(R)\\ R\in\mathcal{L}(Q)}} \alpha_{\tilde{h}}\tilde{h}(x)$$

and therefore

$$\mathcal{P}f(x) = \sum_{h\in\mathcal{H}} \eta(x,h)\,\langle f,h\rangle\, h(x).$$

Let us first prove that the function η satisfies condition (a) in the Theorem 4.1. Notice that if $h\in\mathcal{H}$ and $x\notin Q(h)$ then $\eta(x,h)=0$. On the other hand if $x\in Q(h)$, from (3.6), doubling property on dyadic cubes $(d.7)$, $(d.6)$ and $(h.2)$ we get

$$\begin{aligned}
|\eta(x,h)| &\leq \sum_{R\in\mathcal{L}(Q(h))}\ \sum_{\tilde{h}\in\mathcal{H}(R)} \frac{|\alpha_{\tilde{h}}|}{|h_R|}|\tilde{h}(x)|\,|\chi_R(x)| \\
&\leq \sum_{R\in\mathcal{L}(Q(h))}\ \sum_{\tilde{h}\in\mathcal{H}(R)} \|(\alpha_{\tilde{h}})\|_\infty \frac{\sqrt{\mu(Q(h))}}{C_1}\frac{C_2}{\sqrt{\mu(Q(\tilde{h}))}} \\
&\leq \|(\alpha_{\tilde{h}})\|_\infty\sqrt{C}\frac{C_2}{C_1}\left(\sum_{R\in\mathcal{L}(Q(h))}\ \sum_{\tilde{h}\in\mathcal{H}(R)} 1\right) \\
&\leq M^2\|(\alpha_{\tilde{h}})\|_\infty\sqrt{C}\frac{C_2}{C_1} = B, \qquad (5.1)
\end{aligned}$$

where M is as in $(d.6)$ in Lemma 3.2.

In order to prove that the function η satisfies (b) in Theorem 4.1, take $h\in\mathcal{H}$ with $Q = Q(h)\in\mathcal{D}$ as above $h(x) = \sum_{R\in\mathcal{L}(Q(h))} h_R\ \chi_R(x)$. We split the proof in five cases.

Case 1. $x, x'\notin Q(h)$. Then $|\eta(x,h)-\eta(x',h)| = 0$.

IMAL PREPRINT # 2020-0050

ISSN 2451-7100
Publication date: December 23, 2020

Case 2. $x, x' \in Q'$ for some $Q' \in \mathcal{L}(R_0)$ and some $R_0 \in \mathcal{L}(Q(h))$. Then, since in such case $\tilde{h}(x) = \tilde{h}(x')$ for every $\tilde{h} \in \mathcal{H}(R_0)$, we have that

$$\begin{aligned}
|\eta(x,h) - \eta(x',h)| &= \left| \sum_{R \in \mathcal{L}(Q(h))} \left[\left(\sum_{\tilde{h} \in \mathcal{H}(R)} \frac{\alpha_{\tilde{h}}}{h_R} \tilde{h}(x) \right) \chi_R(x) - \left(\sum_{\tilde{h} \in \mathcal{H}(R)} \frac{\alpha_{\tilde{h}}}{h_R} \tilde{h}(x') \right) \chi_R(x') \right] \right| \\
&= \left| \sum_{\tilde{h} \in \mathcal{H}(R_0)} \left(\frac{\alpha_{\tilde{h}}}{h_{R_0}} \tilde{h}(x) - \frac{\alpha_{\tilde{h}}}{h_{R_0}} \tilde{h}(x') \right) \chi_{R_0}(x) \right| = 0
\end{aligned}$$

Case 3. $x \in Q$ and $x' \in Q'$ with $Q, Q' \in \mathcal{L}(R_0)$ and $R_0 \in \mathcal{L}(Q(h))$. Then, from (4.5), (3.6), doubling property on dyadic cubes $(d.7)$, $(d.6)$ and $(h.2)$ we get that

$$\begin{aligned}
|\eta(x,h) - \eta(x',h)| &= \left| \sum_{R \in \mathcal{L}(Q(h))} \left[\left(\sum_{\tilde{h} \in \mathcal{H}(R)} \frac{\alpha_{\tilde{h}}}{h_R} \tilde{h}(x) \right) \chi_R(x) - \left(\sum_{\tilde{h} \in \mathcal{H}(R)} \frac{\alpha_{\tilde{h}}}{h_R} \tilde{h}(x') \right) \chi_R(x') \right] \right| \\
&= \left| \left(\sum_{\tilde{h} \in \mathcal{H}(R_0)} \frac{\alpha_{\tilde{h}}}{h_{R_0}} \tilde{h}(x) - \sum_{\tilde{h} \in \mathcal{H}(R_0)} \frac{\alpha_{\tilde{h}}}{h_{R_0}} \tilde{h}(x') \right) \chi_{R_0}(x) \right| \\
&= \left| \sum_{\tilde{h} \in \mathcal{H}(R_0)} \frac{\alpha_{\tilde{h}}}{h_{R_0}} \left(\tilde{h}(x) - \tilde{h}(x') \right) \right| \\
&\leq \|(\alpha_{\tilde{h}})\|_\infty \frac{1}{|h_{R_0}|} M C^2 \delta(x,x') \sum_{\tilde{h} \in \mathcal{H}(R_0)} \frac{1}{(\mu(Q(\tilde{h})))^{3/2}} \\
&\leq \|(\alpha_{\tilde{h}})\|_\infty \frac{(\mu(Q(h)))^{1/2}}{C_1} M C^2 \delta(x,x') \sum_{\tilde{h} \in \mathcal{H}(R_0)} \frac{(\mu(Q(h)))^{3/2}}{(\mu(Q(h)))^{3/2} (\mu(Q(\tilde{h})))^{3/2}} \\
&\leq \|(\alpha_{\tilde{h}})\|_\infty \frac{M C^{5/2}}{C_1} \frac{\delta(x,x')}{\mu(Q(h))} \left(\sum_{\tilde{h} \in \mathcal{H}(R_0)} 1 \right) \\
&\leq \|(\alpha_{\tilde{h}})\|_\infty \frac{M^2 C^{5/2}}{C_1} \frac{\delta(x,x')}{\mu(Q(h))}.
\end{aligned}$$

Case 4. $x \in Q(h)$ and $x' \notin Q(h)$ then $\eta(x',h) = 0$, also $\delta(x,x') > \mu(Q(h))$. Hence, from (5.1) we obtain that

$$\begin{aligned}
|\eta(x,h) - \eta(x',h)| &= |\eta(x,h)| \\
&\leq B \\
&\leq B \frac{\delta(x,x')}{\mu(Q(h))}.
\end{aligned}$$

Case 5. $x \in R_1$ and $x' \in R_2$ with $R_1, R_2 \in \mathcal{L}(Q(h))$different. Then $\delta(x,x') = \mu(Q(h))$ and hence from (5.1) we get that

$$|\eta(x,h) - \eta(x',h)| \leq |\eta(x,h)| + |\eta(x',h)|$$

IMAL PREPRINT # 2020-0050

ISSN 2451-7100
Publication date: December 23, 2020

$$
\begin{aligned}
&\le 2B \\
&= 2B\frac{\delta(x,x')}{\mu(Q(h))}.
\end{aligned}
$$

as desired. □

References

[1] H. Aimar, *Integrales singulares y aproximaciones de la identidad en espacios de tipo homogéneo*, Doctoral thesis, Universidad Nacional de Buenos Aires, PEMA-INTEC. Available in http://www.imal.santafe-conicet.gov.ar/TesisIMAL/tesisAimarH.pdf, (1983).

[2] H. Aimar, A. Bernardis and B. Iaffei, *Multiresolution approximation and unconditional bases on weighted Lebesgue spaces on spaces of homogeneous type*, J. Approx. Theory, **148** (2007) 12–34.

[3] H. Aimar, A. Bernardis and L. Nowak, *Equivalence of Haar bases associated to different dyadic systems.*Journal of Geometric Analysis **21** (2011) 288–304.

[4] H. Aimar, J. Comesatti, I. Gómez and L. Nowak, *Partial derivates, singular integrals and Sobolev spaces in dyadic settings.* arXiv:2004.10940v1 23 Apr 2020.

[5] H. Aimar and I. Gómez, *On the Calderón-Zygmund structure of Petermichl's kernel*, C.R. Acad. Sci. Paris, Ser.I **356** (2018) 509–516.

[6] M. Christ, *A T(b) theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60/61** (2) (1990), 601–628.

[7] A.P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.

[8] A.P. Calderón and A. Zygmund, *On singular integrals with variable kernels*, Applicable Analysis, **7** (3) (1978), 221–238.

[9] R.Coifman and G. Weiss, *Analyse harmonique non-conmutative sur certains espaces homogenes.* Lecture Notes in Math., **242**. Springer, Berlin (1971).

[10] G. David and S. Semmes, *Singular integrals and rectificable sets in* $\mathbb{R}^n$*. Ausdelà des graphes lipschitziens.* Astèrique. **1**93 (1991)

[11] S. Mikhlin, *Singular integral equations.* Uspehi Mat. Nauk. **3** (1948) pp. 29–112

[12] S. Mikhlin, *Concerning a theorem on the boundedness of a singular integral operator.* Uspehi Mat. Nauk vol. **8** (1953) pp. 213–217.

[13] R. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type.* Adv. in Math. **33** (1979), 271–309.

[14] R. Macías and C. Segovia, *A decomposition into atoms of distributions on spaces of homogeneous type*, Adv. Math. **33** (1979), 271– 309.

IMAL PREPRINT # 2020-0050
ISSN 2451-7100
Publication date: December 23, 2020

[15] S. Petermichl, *Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol.* C.R. Acad. Sci. Paris, **330**, Série I (2000), 455–460.

[16] E.M. Stein, *Calderón and Zygmund Theory of Singular Integrals.* Chapter 1 of Harmonic analysis and partial differential equations. University of Chicago Press, edited by Michael Christ, Carlos E. Kenig and Cora Sadosky (1999), 1–26.

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