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JORGE J. BETANCOR, ESTEFANÍA DALMASSO, AND PABLO QUIJANO

Abstract. In this paper we introduce spaces of BLO-type related to Laguerre polynomial expansions. We consider the probability measure on $(0,\infty)$ defined by $d\gamma_\alpha(x) = \frac{2}{\Gamma(\alpha+1)} e^{-x^2} x^{2\alpha+1} dx$ with $\alpha > -\frac{1}{2}$. For every $a > 0$, the space $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$ consists of all those measurable functions defined on $(0,\infty)$ having bounded lower oscillation with respect to γ_α over an admissible family $\mathcal{B}_a$ of intervals in $(0,\infty)$. The space $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$ is a subspace of the space $\mathrm{BMO}_a((0,\infty),\gamma_\alpha)$ of bounded mean oscillation functions with respect to γ_α and $\mathcal{B}_a$. The natural a-local centered maximal function defined by γ_α is bounded from $\mathrm{BMO}_a((0,\infty),\gamma_\alpha)$ into $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$. We prove that the maximal operator, the ρ-variation and the oscillation operators associated with local truncations of the Riesz transforms in the Laguerre setting are bounded from $L^\infty((0,\infty),\gamma_\alpha)$ into $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$. Also, we obtain a similar result for the maximal operator of local truncations for spectral Laplace transform type multipliers.

1. Introduction

We consider, for every $\alpha > -\frac{1}{2}$, the probability measure defined on $(0,\infty)$ by $d\gamma_\alpha(x) = \frac{2}{\Gamma(\alpha+1)} e^{-x^2} x^{2\alpha+1} dx$. This measure has not the doubling property with respect to the usual metric defined by the absolute value $|\cdot|$ on $(0,\infty)$. Then, the triple $((0,\infty),|\cdot|,\gamma_\alpha)$ is not homogeneous in the sense of Coifman and Weiss ([16]). Harmonic analysis in the spaces of homogeneous type can be developed following the model of Euclidean spaces $(\mathbb{R}^n,\|\cdot\|,\lambda)$ where $\|\cdot\|$ denotes a norm and λ is the Lebesgue measure on $\mathbb{R}^n$. When the measure is not doubling the situation is very different and it is necessary to introduce new ideas (see, for instance, [13], [20], [21], [24], [25], [40], [49], [50] and [51]).

Tolsa ([49]) defined BMO-type spaces, that he named RBMO-spaces, on $(\mathbb{R}^n,\mu)$ when μ is a Radon measure on $\mathbb{R}^n$, which is not necessarily doubling, satisfying that $\mu(B(x,r)) \leq Cr^k$, $x \in \mathbb{R}^k$ and $r > 0$, for some $k \in \{1,\dots,n\}$ and $C > 0$. He also proved that $\mathrm{RBMO}(\mathbb{R}^n,\mu)$ has many of the properties of the classical space $\mathrm{BMO}(\mathbb{R}^n)$ of John and Nirenberg. In particular, the integral operators defined by standard Calderón-Zygmund kernels are bounded from $L^\infty(\mathbb{R}^n,\mu)$ into $\mathrm{RBMO}(\mathbb{R}^n,\mu)$.

It is clear that, for every $0 < r \leq x$, $\gamma_\alpha((x-r,x+r)) \leq Cr$. Then, following Tolsa's ideas we can define the space $\mathrm{RBMO}((0,\infty),\gamma_\alpha)$ by replacing $\mathbb{R}^n$ by $(0,\infty)$. However, $\mathrm{RBMO}((0,\infty),\gamma_\alpha)$ is not suitable to study harmonic analysis operators

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associated with Laguerre polynomial expansions because these operators are not defined by standard Calderón-Zygmund kernels ([19], [45] and [46]). Motivated by the results in [37] in the Gaussian setting, the authors and R. Scotto ([5]) defined a local BMO-type space related to the measure γ_α as follows.

We consider the function $m(x) = \min\{1, 1/x\}$, $x \in (0,\infty)$. Given $a > 0$, we say that an interval $(x-r, x+r)$, with $0 < r \le x$, is *a-admissible*, or is in the class $\mathcal{B}_a$, when $r \le am(x)$. The measure γ_α has the doubling property on $\mathcal{B}_a$, that is, there exists $C > 0$ such that, for every $0 < r \le x$ being $r \le am(x)$, we have that

$$\gamma_\alpha(I(x, 2r)) \le C\gamma_\alpha((x-r, x+r)),$$

where $I(x,r) := (x-r, x+r) \cap (0,\infty)$ for $x, r > 0$.

A function $f \in L^1((0,\infty), \gamma_\alpha)$ is said to be in $\mathrm{BMO}_a((0,\infty),\gamma_\alpha)$ when

$$\|f\|_{*,\alpha,a} := \sup_{I\in\mathcal{B}_a} \frac{1}{\gamma_\alpha(I)} \int_I |f(y) - f_I| d\gamma_\alpha(y) < \infty,$$

where $f_I = \fint_I f(y)d\gamma_\alpha(y)$, for every $I \in \mathcal{B}_a$. For every $f \in \mathrm{BMO}_a((0,\infty),\gamma_\alpha)$, we define

$$\|f\|_{\mathrm{BMO}_a((0,\infty),\gamma_\alpha)} := \|f\|_{L^1((0,\infty),\gamma_\alpha)} + \|f\|_{*,\alpha,a}.$$

The space $\mathrm{BMO}_a((0,\infty),\gamma_\alpha)$ actually does not depend on $a > 0$. Then, in the sequel we will write $\mathrm{BMO}((0,\infty),\gamma_\alpha)$ and $\|\cdot\|_{*,\alpha}$ instead of $\mathrm{BMO}_a((0,\infty),\gamma_\alpha)$ and $\|\cdot\|_{*,\alpha,a}$, respectively. This space can be identified with the dual space of the Hardy space $H^1((0,\infty),\gamma_\alpha)$ studied in [5] (see [5, Theorem 1.1]).

The space $\mathrm{BLO}(\mathbb{R}^n)$ of functions of bounded lower oscillation on $\mathbb{R}^n$ was introduced by Coifman and Rochberg ([15]). Later, Bennett ([1]) obtained a characterization of the functions in $\mathrm{BLO}(\mathbb{R}^n)$ by using the natural Hardy-Littlewood maximal operators, and Leckband ([29]) proved that certain maximal operators associated with singular integrals are bounded from $L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ into $\mathrm{BLO}(\mathbb{R}^n)$ for certain $1 \le p < \infty$.

Based on Tolsa's ideas, Jiang ([26]) introduced BLO-type spaces in $(\mathbb{R}^n, \mu)$ where μ is a positive non-doubling Radon measure with polynomial growth. BLO-spaces in the Gaussian setting were defined by Liu and Yang ([32]). In [25], Littlewood-Paley functions in non-doubling settings on RBLO spaces were studied. Other results concerning RBLO spaces can be encountered in [30] and [31]. As in happens with RBMO-spaces, RBLO-spaces for γ_α do not work in a correct way in connection with harmonic analysis operators associated to Laguerre polynomial expansions.

In this paper we introduce BLO-spaces associated with the measure γ_α on $(0,\infty)$ by using admissible intervals.

Let $a > 0$. We say that a function $f \in L^1((0,\infty),\gamma_\alpha)$ is in $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$ when

$$\sup_{I\in\mathcal{B}_a} \frac{1}{\gamma_\alpha(I)} \int_I \left(f(y) - \operatorname*{ess\,inf}_{z\in I} f(z)\right) d\gamma_\alpha(y) < \infty.$$

For every $f \in \mathrm{BLO}_a((0,\infty),\gamma_\alpha)$ we define

$$\|f\|_{\mathrm{BLO}_a((0,\infty),\gamma_\alpha)} := \|f\|_{L^1((0,\infty),\gamma_\alpha)} + \sup_{I\in\mathcal{B}_a} \frac{1}{\gamma_\alpha(I)} \int_I \left(f(y) - \operatorname*{ess\,inf}_{z\in I} f(z)\right) d\gamma_\alpha(y).$$

It is not hard to see that

$$L^\infty((0,\infty),\gamma_\alpha) \subset \mathrm{BLO}_a((0,\infty),\gamma_\alpha) \subset \mathrm{BMO}_a((0,\infty),\gamma_\alpha).$$

The main properties of the space $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$ will be established in Section 2.

Our objective is to prove that maximal, variation and oscillation operators defined by singular integrals in the Laguerre settings are bounded from $L^\infty((0,\infty),\gamma_\alpha)$ to $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$.

IMAL PREPRINT # 2023-0063
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We now define the operators we are going to consider. Let $\alpha > -\frac{1}{2}$. The Laguerre polynomial L_k^α of order α and degree $k \in \mathbb{N}$ (see [28]) is

$$L_k^\alpha(x) = \sqrt{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)k!}}e^x x^{-\alpha}\frac{d^k}{dx^k}(e^{-x}x^{\alpha+k}), \quad x \in (0,\infty).$$

The Laguerre differential operator $\widetilde{\Delta_\alpha}$ is given by

$$\widetilde{\Delta_\alpha} := \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{2\alpha+1}{2x} - x\right)\frac{d}{dx} + \alpha + 1, \quad f \in C^2(0,\infty).$$

We define, for every $k \in \mathbb{N}$, $\mathcal{L}_k^\alpha(x) := L_k^\alpha(x^2)$, $x \in (0,\infty)$. Then, the sequence $\{\mathcal{L}_k^\alpha\}_{k\in\mathbb{N}}$ is an orthonormal basis on $L^2((0,\infty),\gamma_\alpha)$. For every $k \in \mathbb{N}$, $\mathcal{L}_k^\alpha$ is an eigenfunction for $\widetilde{\Delta_\alpha}$ associated with the eigenvalue $\lambda_k^\alpha = 2k + \alpha + 1$.

For every $f \in L^1((0,\infty),\gamma_\alpha)$, we define

$$c_k^\alpha(f) := \int_0^\infty f(y)\mathcal{L}_k^\alpha(x)d\gamma_\alpha(x), \quad k \in \mathbb{N}.$$

We consider the operator Δ_α given by

$$\Delta_\alpha f = \sum_{k=0}^\infty \lambda_k^\alpha c_k^\alpha(f)\mathcal{L}_k^\alpha, \quad f \in D(\Delta_\alpha),$$

being

$$D(\Delta_\alpha) = \left\{f \in L^2((0,\infty),\gamma_\alpha) : \sum_{k=0}^\infty (\lambda_k^\alpha |c_k^\alpha(f)|)^2 < \infty\right\}.$$

The space $C_c^\infty(0,\infty)$ of all the smooth functions with compact support in $(0,\infty)$ is contained in $D(\Delta_\alpha)$ and $\Delta_\alpha f = \widetilde{\Delta_\alpha} f$, for any $f \in C_c^\infty(0,\infty)$. The operator Δ_α is self-adjoint and positive in $L^2((0,\infty),\gamma_\alpha)$. Furthermore, the operator $-\Delta_\alpha$ generates a C_0-semigroup of operators $\{W_t^\alpha\}_{t>0}$, where, for every $t > 0$,

$$W_t^\alpha(f) = \sum_{k=0}^\infty e^{-\lambda_k^\alpha t}c_k^\alpha(f)\mathcal{L}_k^\alpha, \quad f \in L^2((0,\infty),\gamma_\alpha).$$

According to [28, (4.17.6)] we have that, for every $x, y, t \in (0,\infty)$,

$$\sum_{k=0}^\infty e^{-kt}\mathcal{L}_k^\alpha(x)\mathcal{L}_k^\alpha(y) = \frac{\Gamma(\alpha+1)}{1-e^{-t}}(e^{-t/2}xy)^{-\alpha}I_\alpha\left(\frac{2e^{-t/2}xy}{1-e^{-t}}\right)\exp\left(-\frac{e^{-t}(x^2+y^2)}{1-e^{-t}}\right), \quad (1.1)$$

being I_α the modified Bessel function of the first kind and order α.

By using (1.1) we can write, for every $f \in L^2((0,\infty),\gamma_\alpha)$ and $t > 0$,

$$W_t^\alpha(f)(x) = \int_0^\infty W_t^\alpha(x,y)f(y)d\gamma_\alpha(y), \quad x \in (0,\infty), \quad (1.2)$$

where

$$W_t^\alpha(x,y) = \frac{\Gamma(\alpha+1)e^{-t(\alpha+1)}}{1-e^{-2t}}(e^{-t}xy)^{-\alpha}I_\alpha\left(\frac{2e^{-t}xy}{1-e^{-2t}}\right)\exp\left(-\frac{e^{-2t}(x^2+y^2)}{1-e^{-2t}}\right),$$

for $x, y, t \in (0,\infty)$.

The integral in (1.2) is absolutely convergent for every $f \in L^p((0,\infty),\gamma_\alpha)$, $1 \le p < \infty$, and for every $t, x \in (0,\infty)$. By defining $W_t^\alpha(f)$ by (1.2), for every $f \in L^p((0,\infty),\gamma_\alpha)$ and $t > 0$, the family $\{W_t^\alpha\}_{t>0}$ is a C_0-semigroup in $L^p((0,\infty),\gamma_\alpha)$, for every $1 \le p < \infty$. Thus $\{W_t^\alpha\}_{t>0}$ is a symmetric diffusion semigroup in the sense of Stein ([48]).

IMAL PREPRINT # 2023-0063 ISSN 2451-7100
Publication date: February 6, 2023

The study of harmonic analysis in Laguerre settings was begun by Muckenhoupt ([39]) who proved that the maximal operator W_*^α defined by

$$W_*^\alpha(f) = \sup_{t>0} |W_t^\alpha(f)|$$

is bounded from $L^1((0,\infty),\gamma_\alpha)$ into $L^{1,\infty}((0,\infty,\gamma_\alpha)$. This property was generalized by Dinger ([18]) to higher dimensions.

We define the Riesz transform R^α associated with the Laguerre operator Δ_α by

$$R^\alpha(f) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k^\alpha}} c_k^\alpha(f) \frac{d}{dx}\mathcal{L}_k^\alpha, \quad f \in L^2((0,\infty),\gamma_\alpha).$$

Thus R^α defines a bounded operator on $L^2((0,\infty),\gamma_\alpha)$ (see [42]). Furthermore, R^α can be extended from $L^2((0,\infty),\gamma_\alpha) \cap L^p((0,\infty),\gamma_\alpha)$ as a bounded operator on $L^p((0,\infty),\gamma_\alpha)$, for every $1 < p < \infty$, and from $L^1((0,\infty),\gamma_\alpha)$ into $L^{1,\infty}((0,\infty),\gamma_\alpha)$ ([47]). The authors and R. Scotto ([6]) extended the above results by considering variable exponents $L^{p(\cdot)}$-spaces. Also, in [5], endpoint estimates for Riesz transform R^α were established proving that R^α defines a bounded operator from $H^1((0,\infty),\gamma_\alpha)$ into $L^1((0,\infty),\gamma_\alpha)$ and from $L^\infty((0,\infty),\gamma_\alpha)$ into $\mathrm{BMO}((0,\infty),\gamma_\alpha)$.

We can see that R^α is a principal value integral operator. By proceeding as in the proof of [8, Theorem 1.1] we can see that, for every $f \in L^p((0,\infty),\gamma_\alpha)$, $1 \le p < \infty$,

$$R^\alpha(f)(x) = \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon,\ y\in(0,\infty)} R^\alpha(x,y) f(y) d\gamma_\alpha(y), \quad \text{a.e. } x \in (0,\infty),$$

where

$$R^\alpha(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \partial_x W_t^\alpha(x,y) \frac{dt}{\sqrt{t}}, \quad x,y \in (0,\infty),\ x \ne y.$$

For every $\epsilon > 0$, we define the ϵ-truncation of the Riesz transform R^α by

$$R_\epsilon^\alpha(f)(x) = \int_{|x-y|>\varepsilon,\ y\in(0,\infty)} R^\alpha(x,y) f(y) d\gamma_\alpha(y), \quad x \in (0,\infty).$$

The maximal Riesz transform R_*^α is defined by

$$R_*^\alpha(f) = \sup_{\epsilon>0} |R_\epsilon^\alpha(f)|.$$

From the results given by E. Sasso in [47] we can deduced that the maximal operator R_*^α is bounded on $L^p((0,\infty),\gamma_\alpha)$, for every $1 < p < \infty$, and from $L^1((0,\infty),\gamma_\alpha)$ into $L^{1,\infty}((0,\infty),\gamma_\alpha)$.

We are going to consider the following local maximal Riesz transform operators. For every $a > 0$, we define the maximal operator $R_{*,a}^\alpha$ by

$$R_{*,a}^\alpha(f)(x) = \sup_{0<\epsilon\le am(x)} |R_\epsilon^\alpha(f)(x)|, \quad x \in (0,\infty).$$

Let $\rho > 0$. If $\{c_t\}_{t>0}$ is a subset of complex numbers, we define the ρ-variation $\mathcal{V}_\rho(\{c_t\}_{t>0})$ of $\{c_t\}_{t>0}$ by

$$\mathcal{V}_\rho(\{c_t\}_{t>0}) = \sup_{0<t_n<t_{n-1}<\cdots<t_1,\ n\in\mathbb{N}} \left(\sum_{j=1}^{n-1} |c_{t_j} - c_{t_{j+1}}|^\rho \right)^{1/\rho}.$$

If $\{T_t\}_{t>0}$ is a family of bounded operators in $L^p((0,\infty),\gamma_\alpha)$, with $1 \le p < \infty$, we define the ρ-variation operator $\mathcal{V}_\rho(\{T_t\}_{t>0})$ of $\{T_t\}_{t>0}$ by

$$\mathcal{V}_\rho(\{T_t\}_{t>0})(f)(x) = \mathcal{V}_\rho(\{T_t(f)(x)\}_{t>0}).$$

Since Bourgain ([10]) studied variational inequalities involving martingales (see also [27]), ρ-variation operators has been extensively studied in ergodic theory

IMAL PREPRINT # 2023-0063

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and harmonic analysis. Campbell, Jones, Reinhold and Wierdl ([11]) proved L^p-boundedness properties for ρ-variation operators associated to the family of truncations for the Hilbert transform. In [12] those results were extended by considering Riesz transforms in higher dimensions. In order to obtain L^p-boundedness for ρ-variation operators it is usual to ask for the condition $\rho > 2$ (see [44]). For the exponent $\rho = 2$, oscillation operators are commonly considered.

Let $\{t_j\}_{j\in\mathbb{Z}}$ be an increasing sequence of positive real numbers satisfying that $\lim_{j\to-\infty} t_j = 0$ and $\lim_{j\to+\infty} t_j = +\infty$. If $\{c_t\}_{t>0}$ is a set of complex numbers, we define the oscillation with respect to $\{t_j\}_{j\in\mathbb{Z}}$ by

$$\mathcal{O}(\{c_t\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}}) = \left(\sum_{j=-\infty}^{+\infty} \sup_{t_j\le\epsilon_j<\epsilon_{j+1}<t_{j+1}} |c_{\epsilon_j} - c_{\epsilon_{j+1}}|^2\right)^{1/2}.$$

If $\{T_t\}_{t>0}$ is a family of bounded operators in $L^p((0,\infty),\gamma_\alpha)$, with $1 \le p < \infty$, we define the oscillation operator $\mathcal{O}(\{T_t\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})$ as follows

$$\mathcal{O}(\{T_t\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})(f)(x) = \mathcal{O}(\{T_t(f)(x)\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}}).$$

L^p-boundedness properties of the oscillation operators defined by the family of truncations of Hilbert transform and Euclidean Riesz transforms were established in [11] and [12], respectively.

After [11] and [12], the study of ρ-variation and oscillation operators defined by singular integrals has been an active working area (see, for instance, [3], [14], [17], [22], [33], [34], [35], [36] and [38]). Variation and oscillation operators give information about convergence properties for the family $\{T_t\}_{t>0}$.

Being $\{T_t\}_{t>0}$ and $\{t_j\}_{j\in\mathbb{Z}}$ as above, we are going to consider the local ρ-variation and oscillation operators defined as follows. Let $a > 0$. The a-local ρ-variation operator $\mathcal{V}_{\rho,a}(\{T_t\}_{t>0})$ is given by

$$\mathcal{V}_{\rho,a}(\{T_t\}_{t>0})(f)(x) = \sup_{0<t_n<t_{n-1}<\cdots<t_1\le am(x),\ n\in\mathbb{N}} \left(\sum_{j=1}^{n-1} |T_{t_j}(f)(x) - T_{t_{j+1}}(f)(x)|^\rho\right)^{1/\rho}.$$

The a-local oscillation operator $\mathcal{O}_a(\{T_t\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})$ is defined by

$$\mathcal{O}_a(\{T_t\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})(f)(x) = \left(\sum_{j\in\mathbb{Z},\ t_j\le am(x)} \sup_{t_j\le\epsilon_j<\epsilon_{j+1}<t_{j+1}} |T_{\epsilon_j}(f)(x) - T_{\epsilon_{j+1}}(f)(x)|^2\right)^{1/2}.$$

Our first result is the following.

Theorem 1.1. *Let $\alpha > -\frac{1}{2}$, $a > 0$ and $\rho > 2$. Suppose that $\{t_j\}_{j\in\mathbb{Z}}$ is an increasing sequence of positive real numbers such that $t_{j+1} \le \theta t_j$, $j \in \mathbb{Z}$, for some $\theta > 1$, $\lim_{j\to-\infty} t_j = 0$ and $\lim_{j\to+\infty} t_j = +\infty$. The operators $R^\alpha_{*,a}$, $\mathcal{V}_{\rho,a}(\{R^\alpha_\epsilon\}_{\epsilon>0})$, and $\mathcal{O}_a(\{R^\alpha_\epsilon\}_{\epsilon>0}, \{t_j\}_{j\in\mathbb{Z}})$ are bounded from $L^\infty((0,\infty),\gamma_\alpha)$ into $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$.*

We shall now introduce multiplier operators in the Laguerre setting. A measurable complex function M defined on $[0,\infty)$ is said to be of Laplace transform type when

$$M(x) = x\int_0^\infty \phi(t)e^{-xt}dt, \quad x > 0,$$

where $\phi \in L^\infty(0,\infty)$.

IMAL PREPRINT # 2023-0063
ISSN 2451-7100
Publication date: February 6, 2023

Suppose that M is of Laplace transform type. We denote by T_M^α the spectral multiplier for the Laguerre operator Δ_α defined by $M - M(0)$. For every $f \in L^2((0,\infty),\gamma_\alpha)$, $T_M^\alpha(f)$ is given by

$$T_M^\alpha(f) = \sum_{k=1}^{\infty} M(k) c_k^\alpha(f) \mathcal{L}_k^\alpha.$$

Since M is bounded on $(0,\infty)$, T_M^α is bounded on $L^2((0,\infty),\gamma_\alpha)$. Since $\{W_t^\alpha\}_{t>0}$ is a symmetric diffusion semigroup, T_M^α is bounded on $L^p((0,\infty),\gamma_\alpha)$, for every $1 < p < \infty$ ([48, Corollary 3, p. 121]). The authors and R. Scotto ([6, Theorem 1.1 (d)]) extended the last result establishing variable $L^{p(\cdot)}$-boundedness properties for T_M^α. On the other hand, Sasso ([45]) proved that T_M^α defines a bounded operator from $L^1((0,\infty),\gamma_\alpha)$ into $L^{1,\infty}((0,\infty),\gamma_\alpha)$. In [5], the authors with R. Scotto established the endpoint estimate for T_M^α from $L^\infty((0,\infty),\gamma_\alpha)$ into $\mathrm{BMO}((0,\infty),\gamma_\alpha)$.

From [8, Theorem 1.1] we deduce that there exists a function $\Lambda \in L^\infty(0,\infty)$ such that, for every $f \in L^p((0,\infty),\gamma_\alpha)$, $1 \le p < \infty$,

$$T_M^\alpha(f)(x) = \lim_{\varepsilon\to 0^+} \left(\Lambda(\varepsilon) f(x) + \int_{|x-y|>\varepsilon,\ y\in(0,\infty)} K_\phi^\alpha(x,y) f(y)\, d\gamma_\alpha(y) \right),$$

for a.e. $x \in (0,\infty)$, where

$$K_\phi^\alpha(x,y) = -\int_0^\infty \phi(t) \partial_t W_t^\alpha(x,y)\, dt, \quad x,y \in (0,\infty),\ x \neq y.$$

A special case of T_M^α is the imaginary power $\Delta_\alpha^{i\alpha}$ that appears when $M_\eta(x) = x^{i\eta}$ for $x \in (0,\infty)$ and $\eta \in \mathbb{R} \setminus \{0\}$. For these values of η,

$$M_\eta(x) = x \int_0^\infty \phi_\eta(t) e^{-xt}\, dt, \quad x \in (0,\infty),$$

where $\phi_\eta(t) = \frac{t^{-i\eta}}{\Gamma(1+i\eta)}$, $t > 0$. Note that $|\phi_\eta'(t)| \le C/t$, $t \in (0,\infty)$.

We define, for every $\epsilon > 0$, the truncations

$$Q_{\phi,\epsilon}^\alpha(f)(x) = \int_{|x-y|>\epsilon,\ y\in(0,\infty)} K_\phi^\alpha(x,y) f(y)\, d\gamma_\alpha(y), \quad x \in (0,\infty),$$

and consider, for every $a > 0$, the a-local maximal operator $Q_{\phi,*,a}^\alpha$, which is given by

$$Q_{\phi,*,a}^\alpha(f)(x) = \sup_{0<\epsilon\le am(x)} |Q_{\phi,\epsilon}^\alpha(f)(x)|.$$

Theorem 1.2. *Let $\alpha > -\frac{1}{2}$ and $a > 0$. The maximal operator $Q_{\phi,*,a}^\alpha$ is bounded from $L^\infty((0,\infty),\gamma_\alpha)$ into $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$ provided that $|\phi'(t)| \le C/t$ for some $C > 0$, and each $t \in (0,\infty)$.*

The paper is organized as follows. In Section 2 we state the main properties for the spaces $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$. In the subsequent sections we prove Theorems 1.1 and 1.2.

Throughout this paper C and c will always denote positive constants than may change in each occurrence.

2. The spaces $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$

In this section we state the main properties of the spaces $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$. This properties will be useful in the following sections and they can be proved as the corresponding properties for the Gaussian BLO_a space given in [32, Theorem 3.1, Proposition 3.1 and Theorem 3.2] (see also [1] for the Euclidean case and [23] for the non-doubling measure case).

IMAL PREPRINT # 2023-0063

ISSN 2451-7100
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Let $a>0$. The local natural maximal operator $\mathcal{M}_a^\alpha$ associated with the measure γ_α on $(0,\infty)$ is defined by

$$\mathcal{M}_a^\alpha(f)(x)=\sup_{I\in\mathcal{B}_a(x)}\frac{1}{\gamma_\alpha(I)}\int_I f(y)d\gamma_\alpha(y),\ x\in(0,\infty),$$

for every measurable function f on $(0,\infty)$ such that $\int_0^\delta|f(y)|d\gamma_\alpha(y)<\infty$, $\delta>0$.

Proposition 2.1. *Let $a>0$. There exists $C>0$ such that for every $I\in\mathcal{B}_a$ and every measurable function f on $(0,\infty)$ such that $\|f\|_{*,\alpha}<\infty$,*

$$\frac{1}{\gamma_\alpha(I)}\int_I\mathcal{M}_a^\alpha(f)(y)d\gamma_\alpha(y)\leq C\|f\|_{*,\alpha}+\operatorname*{ess\,inf}_{x\in I}\mathcal{M}_a^\alpha(f)(x).$$

Furthermore, the natural maximal operator $\mathcal{M}_a^\alpha$ defines a bounded operator from $\mathrm{BMO}((0,\infty),\gamma_\alpha)$ into $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$.

The space $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$ can be characterized by using the local natural maximal operator.

Proposition 2.2. *Let $a>0$. A measurable function f belongs to $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$ if and only if $f\in L^1((0,\infty),\gamma_\alpha)$ and $\mathcal{M}_a^\alpha(f)-f\in L^\infty((0,\infty),\gamma_\alpha)$. In addition, we have that*

$$\|\mathcal{M}_a^\alpha(f)-f\|_{L^\infty((0,\infty),\gamma_\alpha)}=\sup_{I\in\mathcal{B}_a}\left(\frac{1}{\gamma_\alpha(I)}\int_I f(y)d\gamma_\alpha(y)-\operatorname*{ess\,inf}_{x\in I}f(x)\right).$$

By combining Proposition 2.1 and Proposition 2.2 we can establish the following characterization of $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$ involving the space $\mathrm{BMO}((0,\infty),\gamma_\alpha)$ and the local natural maximal operator.

Proposition 2.3. *Let $a>0$. A measurable function f belongs to $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$ if and only if $f=\mathcal{M}_a^\alpha(g)+h$, where $g\in\mathrm{BMO}((0,\infty),\gamma_\alpha)$ and $h\in L^\infty((0,\infty),\gamma_\alpha)$. Furthermore,*

$$\|f\|_{\mathrm{BLO}_a((0,\infty),\gamma_\alpha)}\sim\inf\{\|g\|_{\mathrm{BMO}((0,\infty),\gamma_\alpha)}+\|h\|_{L^\infty((0,\infty),\gamma_\alpha)}\},$$

where the infimum is taken over all the pairs (g,h) for which $f=\mathcal{M}_a^\alpha(g)+h$ with $(g,h)\in\mathrm{BMO}((0,\infty),\gamma_\alpha)\times L^\infty((0,\infty),\gamma_\alpha)$.

3. Proof of Theorem 1.1

3.1. **Local variation operators.** Let $f\in L^\infty((0,\infty),\gamma_\alpha)$. Since the variation operator $\mathcal{V}_\rho(\{R_\epsilon^\alpha\}_{\epsilon>0})$ is bounded on $L^2((0,\infty),\gamma_\alpha)$ (see [9, Theorem 1.3]) it follows that

$$\begin{aligned}\int_0^\infty\mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f)(x)d\gamma_\alpha(x)&\leq\left(\int_0^\infty\left(\mathcal{V}_\rho(\{R_\epsilon^\alpha\}_{\epsilon>0})(f)(x)\right)^2d\gamma_\alpha(x)\right)^{1/2}\\&\leq C\left(\int_0^\infty|f(x)|^2d\gamma_\alpha(x)\right)^{1/2}\\&\leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}.\end{aligned}$$

According to Proposition 2.2 the proof will be finished when we see that

$$\|\mathcal{M}_a^\alpha(\mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f))-\mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f)\|_{L^\infty((0,\infty),\gamma_\alpha)}\leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}.$$

Notice that

$$\begin{aligned}0&\leq\mathcal{M}_a^\alpha(\mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f))(x)-\mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f)(x)\\&=\sup_{I\in\mathcal{B}_a(x)}\frac{1}{\gamma_\alpha(I)}\int_I\mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f)(z)d\gamma_\alpha(z)-\mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f)(x),\end{aligned}$$

for almost every $x\in(0,\infty)$, where $I\in\mathcal{B}_a(x)$ indicates that $I\in\mathcal{B}_a$ and $x\in I$.

IMAL PREPRINT # 2023-0063
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Let $x,\ x_0,\ r_0 \in (0,\infty)$ such that $I = I(x_0, r_0) \in \mathcal{B}_a(x)$. We decompose f as follows

$$f = f\chi_{4I} + f\chi_{(0,\infty)\setminus 4I} = f_1 + f_2.$$

We can write

$$\begin{aligned}
&\frac{1}{\gamma_\alpha(I)}\int_I \mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f)(z)d\gamma_\alpha(z) - \mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f)(x)\\
&\qquad\leq \frac{1}{\gamma_\alpha(I)}\int_I \mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f_1)(z)d\gamma_\alpha(z)\\
&\qquad\quad + \frac{1}{\gamma_\alpha(I)}\int_I \left(\mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f_2)(z) - \mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f_2)(x)\right)d\gamma_\alpha(z)\\
&\qquad\quad + \mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f_2)(x) - \mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f)(x)\\
&\qquad := J_1 + J_2 + J_3.
\end{aligned}$$

By using again that the variation $\mathcal{V}_\rho(\{R_\epsilon^\alpha\}_{\epsilon>0})$ is bounded on $L^2((0,\infty),\gamma_\alpha)$ we get

$$\begin{aligned}
J_1 &\leq \left(\frac{1}{\gamma_\alpha(I)}\int_I \left(\mathcal{V}_\rho(\{R_\epsilon^\alpha\}_{\epsilon>0})(f_1)(z)\right)^2 d\gamma_\alpha(z)\right)^{1/2}\\
&\leq C\left(\frac{1}{\gamma_\alpha(I)}\int_I |f(z)|^2\, d\gamma_\alpha(z)\right)^{1/2} \leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}. \qquad (3.1)
\end{aligned}$$

Suppose there exists $i_0 \in \{1,\dots,n-1\}$ such that $\epsilon_{i_0+1} \leq am(x) < \epsilon_{i_0}$. Thus, for $z \in I$,

$$\begin{aligned}
&\left(\sum_{j=1}^{n-1}|R_{\epsilon_{j+1}}^\alpha(f_2)(z) - R_{\epsilon_j}^\alpha(f_2)(z)|^\rho\right)^{1/\rho}\\
&\leq \left(\sum_{j=1}^{i_0-1}|R_{\epsilon_{j+1}}^\alpha(f_2)(z) - R_{\epsilon_j}^\alpha(f_2)(z)|^\rho + |R_{\epsilon_{i_0}}^\alpha(f_2)(z) - R_{am(x)}^\alpha(f_2)(z)|^\rho\right)^{1/\rho}\\
&\quad + \left(\sum_{j=i_0+1}^{n-1}|R_{\epsilon_{j+1}}^\alpha(f_2)(z) - R_{\epsilon_j}^\alpha(f_2)(z)|^\rho + |R_{\epsilon_{i_0+1}}^\alpha(f_2)(z) - R_{am(x)}^\alpha(f_2)(z)|^\rho\right)^{1/\rho}.
\end{aligned}$$

Then, recalling that $m(z) \leq Cm(x)$ for every $x, z \in I$, where $C > 1$, we obtain

$$\begin{aligned}
&\mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f_2)(z) - \mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f_2)(x)\\
&\leq \sup_{\substack{0<\epsilon_n<\cdots<\epsilon_1\leq am(x)\\ n\in\mathbb{N}}}\left(\sum_{j=1}^{n-1}|R_{\epsilon_{j+1}}^\alpha(f_2)(z) - R_{\epsilon_j}^\alpha(f_2)(z)|^\rho\right)^{1/\rho}\\
&\quad + \sup_{\substack{am(x)\leq\epsilon_n<\cdots<\epsilon_1<Cam(x)\\ n\in\mathbb{N}}}\left(\sum_{j=1}^{n-1}|R_{\epsilon_{j+1}}^\alpha(f_2)(z) - R_{\epsilon_j}^\alpha(f_2)(z)|^\rho\right)^{1/\rho}\\
&\quad - \sup_{\substack{0<\epsilon_n<\cdots<\epsilon_1\leq am(x)\\ n\in\mathbb{N}}}\left(\sum_{j=1}^{n-1}|R_{\epsilon_{j+1}}^\alpha(f_2)(x) - R_{\epsilon_j}^\alpha(f_2)(x)|^\rho\right)^{1/\rho}\\
&\leq \sup_{\substack{am(x)\leq\epsilon_n<\cdots<\epsilon_1<Cam(x)\\ n\in\mathbb{N}}}\sum_{j=1}^{n-1}|R_{\epsilon_{j+1}}^\alpha(f_2)(z) - R_{\epsilon_j}^\alpha(f_2)(z)|
\end{aligned}$$

IMAL PREPRINT # 2023-0063

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$$+ \sup_{\substack{0<\epsilon_n<\cdots<\epsilon_1\leq am(x)\\ n\in\mathbb{N}}} \inf_{\substack{0<\delta_k<\cdots<\delta_1\leq am(x)\\ k\in\mathbb{N}}} \left[\left(\sum_{j=1}^{n-1} |R^\alpha_{\epsilon_{j+1}}(f_2)(z) - R^\alpha_{\epsilon_j}(f_2)(z)|^\rho \right)^{1/\rho} - \left(\sum_{j=1}^{k-1} |R^\alpha_{\delta_{j+1}}(f_2)(x) - R^\alpha_{\delta_j}(f_2)(x)|^\rho \right)^{1/\rho} \right]$$

$$\leq \int_{am(x)<|z-y|<Cam(x)} |R^\alpha(z,y)||f_2(y)|d\gamma_\alpha(y)$$

$$+ \sup_{\substack{0<\epsilon_n<\cdots<\epsilon_1\leq am(x)\\ n\in\mathbb{N}}} \inf_{\substack{0<\delta_k<\cdots<\delta_1\leq am(x)\\ k\in\mathbb{N}}} \left[\left(\sum_{j=1}^{n-1} |R^\alpha_{\epsilon_{j+1}}(f_2)(z) - R^\alpha_{\epsilon_j}(f_2)(z)|^\rho \right)^{1/\rho} - \left(\sum_{j=1}^{k-1} |R^\alpha_{\delta_{j+1}}(f_2)(x) - R^\alpha_{\delta_j}(f_2)(x)|^\rho \right)^{1/\rho} \right]$$

$$:= J_{2,1}(x,z) + J_{2,2}(x,z).$$

If we write

$$R^\alpha(z,y) = e^{\frac{z^2+y^2}{2}}\mathfrak{R}^\alpha(z,y), \quad z,y\in(0,\infty),\ z\neq y,$$

from [43, (3.3) and Proposition 3.1] we know that

$$\mathfrak{R}^\alpha(z,y) \leq \frac{C}{\mathfrak{m}_\alpha(I(z,|z-y|))}, \quad z,y\in(0,\infty),\ z\neq y, \tag{3.2}$$

where $d\mathfrak{m}_\alpha(x) = x^{2\alpha+1}dx$. Therefore, for $x,z\in I$, since $m(x)\leq Cm(z)$ and using [52, (3)], we obtain

$$\begin{aligned} J_{2,1}(x,z) &\leq C\int_{am(x)<|z-y|\leq Cam(x)} |f_2(y)|e^{\frac{z^2-y^2}{2}} \frac{d\mathfrak{m}_\alpha(y)}{\mathfrak{m}_\alpha(I(z,|z-y|))} \\ &\leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)} \int_{am(x)<|z-y|\leq Cam(x)} e^{\frac{(z+y)|z-y|}{2}} \frac{d\mathfrak{m}_\alpha(y)}{\mathfrak{m}_\alpha(I(y,|z-y|))} \\ &\leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)} \int_{am(x)<|z-y|\leq Cam(x)} e^{Cam(x)(z+Cam(x))} \frac{dy}{|z-y|} \\ &\leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)} \int_{am(x)<|z-y|\leq Cam(x)} \frac{dy}{|z-y|} \\ &\leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}. \end{aligned}$$

On the other hand, for every $x,z\in I$ we have

$$\left(\sum_{j=1}^{n-1} |R^\alpha_{\epsilon_{j+1}}(f_2)(z) - R^\alpha_{\epsilon_j}(f_2)(z)|^\rho \right)^{1/\rho} - \left(\sum_{j=1}^{n-1} |R^\alpha_{\epsilon_{j+1}}(f_2)(x) - R^\alpha_{\epsilon_j}(f_2)(x)|^\rho \right)^{1/\rho}$$

$$\leq \left(\sum_{j=1}^{n-1} \left| R^\alpha_{\epsilon_{j+1}}(f_2)(z) - R^\alpha_{\epsilon_j}(f_2)(z) - \left(R^\alpha_{\epsilon_{j+1}}(f_2)(x) - R^\alpha_{\epsilon_j}(f_2)(x) \right) \right|^\rho \right)^{1/\rho}$$

$$\leq \left(\sum_{j=1}^{n-1} \left| \int_{\epsilon_{j+1}<|z-y|<\epsilon_j} (R^\alpha(z,y) - R^\alpha(x,y))f_2(y)d\gamma_\alpha(y) \right.\right.$$

$$+ \left(\int_{\epsilon_{j+1}<|z-y|<\epsilon_j} R^\alpha(x,y)f_2(y)d\gamma_\alpha(y) \right.$$

IMAL PREPRINT # 2023-0063
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$$
\begin{aligned}
&\quad - \int_{\epsilon_{j+1}<|x-y|<\epsilon_j} R^\alpha(x,y) f_2(y) d\gamma_\alpha(y)\Big)\Big|^\rho\Big)^{1/\rho} \\
&\leq \sum_{j=1}^{n-1} \left| \int_{\epsilon_{j+1}<|z-y|<\epsilon_j} (R^\alpha(z,y) - R^\alpha(x,y)) f_2(y) d\gamma_\alpha(y) \right| \\
&\quad + \left(\sum_{j=1}^{n-1} \left| \int_0^\infty R^\alpha(x,y) \left(\chi_{\{\epsilon_{j+1}<|x-y|<\epsilon_j\}}(y) \right.\right.\right. \\
&\quad - \left.\left.\left. \chi_{\{\epsilon_{j+1}<|z-y|<\epsilon_j\}}(y) \right) f_2(y) d\gamma_\alpha(y) \right|^\rho \right)^{1/\rho}.
\end{aligned}
$$

Now, by taking supremum, we get

$$
\begin{aligned}
J_{2,2}(x,z) &\leq \int_{(0,\infty)\setminus 4I} |R^\alpha(z,y) - R^\alpha(x,y)||f(y)| d\gamma_\alpha(y) \\
&\quad + \sup_{\substack{0<\epsilon_n<\cdots<\epsilon_1\leq am(x)\\ n\in\mathbb{N}}} \left(\sum_{j=1}^{n-1} \left(\int_0^\infty |R^\alpha(x,y)| \left| \chi_{\{\epsilon_{j+1}<|x-y|<\epsilon_j\}}(y) \right.\right.\right. \\
&\quad - \left.\left.\left. \chi_{\{\epsilon_{j+1}<|z-y|<\epsilon_j\}}(y) \right| |f_2(y)| d\gamma_\alpha(y) \right)^\rho \right)^{1/\rho} \\
&:= J_{2,2,1}(x,z) + J_{2,2,2}(x,z).
\end{aligned}
$$

Since (see [4, § 4.3])

$$
\sup_{I\in\mathcal{B}_a} \sup_{x,z\in I} \int_{(0,\infty)\setminus 4I} |R^\alpha(z,y) - R^\alpha(x,y)||f(y)| d\gamma_\alpha(y) < \infty,
$$

it follows that

$$
J_{2,2,1}(x,z) \leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}, \quad x,z\in I.
$$

In order to estimate $J_{2,2,2}$ we adapt a procedure developed in [7]. From (3.2), for $x,z\in I$, we obtain

$$
\begin{aligned}
J_{2,2,2}(x,z) &\leq C \sup_{\substack{0<\epsilon_n<\cdots<\epsilon_1\leq am(x)\\ n\in\mathbb{N}}} \left(\sum_{j=1}^{n-1} \left(\int_0^\infty \frac{e^{\frac{x^2-y^2}{2}}}{\mathfrak{m}_\alpha(I(y,|x-y|))} \right.\right. \\
&\quad \times \left.\left. \left| \chi_{\{\epsilon_{j+1}<|x-y|<\epsilon_j\}}(y) - \chi_{\{\epsilon_{j+1}<|z-y|<\epsilon_j\}}(y) \right| |f_2(y)| d\mathfrak{m}_\alpha(y) \right)^\rho \right)^{1/\rho}.
\end{aligned}
$$

Let us observe that, if $|x-y|\leq am(x)$, then

$$
x^2 - y^2 \leq |x-y||x+y| \leq am(x)(am(x)+2x) \leq C.
$$

Also, if $|z-y|\leq am(x)$, then

$$
|x-y| \leq 2r_0 + am(x) \leq 2am(x_0) + am(x).
$$

Since $x\in I\in\mathcal{B}_a$, $m(x_0)\leq Cm(x)$ so $|x-y|\leq Cam(x)$, and thus $x^2-y^2\leq C$, provided that $|z-y|\leq am(x)$.

This fact together with [52, (3)] lead to

$$
\begin{aligned}
J_{2,2,2}(x,z) &\leq \sup_{\substack{0<\epsilon_n<\cdots<\epsilon_1\leq am(x)\\ n\in\mathbb{N}}} \left(\sum_{j=1}^{n-1} \left(\int_0^\infty \frac{|f_2(y)|}{|x-y|} \left| \chi_{\{\epsilon_{j+1}<|x-y|<\epsilon_j\}}(y) \right.\right.\right. \\
&\quad - \left.\left.\left. \chi_{\{\epsilon_{j+1}<|z-y|<\epsilon_j\}}(y) \right| dy \right)^\rho \right)^{1/\rho}, \quad x,z\in I.
\end{aligned}
$$

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Let us take $0<\epsilon_n<\cdots<\epsilon_1\leq am(x)$ and $j\in\{1,\dots,n-1\}$. Then

$$\begin{aligned}
&\int_0^\infty \frac{|f_2(y)|}{|x-y|}\left|\chi_{\{\epsilon_{j+1}<|x-y|<\epsilon_j\}}(y)-\chi_{\{\epsilon_{j+1}<|z-y|<\epsilon_j\}}(y)\right|dy\\
&\leq C\left(\int_0^\infty \frac{|f_2(y)|}{|x-y|}\chi_{\{\epsilon_{j+1}<|x-y|<\epsilon_j\}}(y)\chi_{\{\epsilon_{j+1}<|x-y|<\epsilon_{j+1}+2r_0\}}(y)dy\right.\\
&\quad+\int_0^\infty \frac{|f_2(y)|}{|x-y|}\chi_{\{\epsilon_{j+1}<|x-y|<\epsilon_j\}}(y)\chi_{\{\epsilon_j<|z-y|<\epsilon_j+2r_0\}}(y)dy\\
&\quad+\int_0^\infty \frac{|f_2(y)|}{|z-y|}\chi_{\{\epsilon_{j+1}<|z-y|<\epsilon_j\}}(y)\chi_{\{\epsilon_{j+1}<|z-y|<\epsilon_{j+1}+2r_0\}}(y)dy\\
&\quad\left.+\int_0^\infty \frac{|f_2(y)|}{|z-y|}\chi_{\{\epsilon_{j+1}<|z-y|<\epsilon_j\}}(y)\chi_{\{\epsilon_j<|x-y|<\epsilon_j+2r_0\}}(y)dy\right)\\
&=\sum_{l=1}^4 J_{2,2,2}^{j,l}(x,z),\quad x,z\in I.
\end{aligned}$$

For the above estimate, we have taken into account that, if $\chi_{\{\epsilon_{j+1}<|x-y|\leq\epsilon_j\}}(y)-\chi_{\{\epsilon_{j+1}<|z-y|\leq\epsilon_j\}}(y)\neq 0$, then $\chi_{\{\epsilon_{j+1}<|x-y|\leq\epsilon_j\}}(y)\chi_{\{\epsilon_{j+1}<|z-y|\leq\epsilon_j\}}(y)=0$, with $y\in(0,\infty)$ and $x,\ z\in I$. Since $f_2(y)=0$ for $y\in 4I$, it follows that $J_{2,2,2}^{j,l}=0$ when $l=1,3$, $z\in I$ and $r_0\geq\epsilon_{j+1}$. Also, $J_{2,2,2}^{j,l}(x,z)=0$ when $l=2,4$, $z\in I$ and $r_0\geq\epsilon_j$.

If $z\in I$ and $y\notin 4I$, then $2|x-y|\geq|z-y|\geq\frac{1}{2}|x-y|$. Hölder inequality leads to

$$J_{2,2,2}^{j,l}(x,z)\leq C\left(\int_0^\infty \chi_{\{\epsilon_{j+1}<|x-y|<\epsilon_j\}}(y)\left(\frac{|f_2(y)|}{|x-y|}\right)^2dy\right)^{1/2}r_0^{1/2},\quad z\in I,\ l=1,2;$$

$$J_{2,2,2}^{j,l}(x,z)\leq C\left(\int_0^\infty \chi_{\{\epsilon_{j+1}<|z-y|<\epsilon_j\}}(y)\left(\frac{|f_2(y)|}{|z-y|}\right)^2dy\right)^{1/2}r_0^{1/2},\quad z\in I,\ l=3,4.$$

We obtain

$$\begin{aligned}
&\left(\sum_{j=1}^{n-1}\left|\sum_{l=1}^4 J_{2,2,2}^{j,l}(x,z)\right|^\rho\right)^{1/\rho}\\
&\leq Cr_0^{1/2}\left[\left(\sum_{j=1}^{n-1}\left(\int_0^\infty \chi_{\{\epsilon_{j+1}<|x-y|<\epsilon_j\}}(y)\left(\frac{|f_2(y)|}{|x-y|}\right)^2dy\right)^{\rho/2}\right)^{1/\rho}\right.\\
&\quad\left.+\left(\sum_{j=1}^{n-1}\left(\int_0^\infty \chi_{\{\epsilon_{j+1}<|z-y|<\epsilon_j\}}(y)\left(\frac{|f_2(y)|}{|z-y|}\right)^2dy\right)^{\rho/2}\right)^{1/\rho}\right]\\
&\leq Cr_0^{1/2}\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}\left[\left(\sum_{j=1}^{n-1}\int_{(0,\infty)\setminus 4I}\chi_{\{\epsilon_{j+1}<|x-y|<\epsilon_j\}}(y)\frac{dy}{|x-y|^2}\right)^{1/2}\right.\\
&\quad\left.+\left(\sum_{j=1}^{n-1}\int_{(0,\infty)\setminus 4I}\chi_{\{\epsilon_{j+1}<|z-y|<\epsilon_j\}}(y)\frac{dy}{|z-y|^2}\right)^{1/2}\right]\\
&\leq Cr_0^{1/2}\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}\left(\int_{(0,\infty)\setminus 4I}\frac{dy}{|x-y|^2}+\int_{(0,\infty)\setminus 4I}\frac{dy}{|z-y|^2}\right)^{1/2}\\
&\leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}.
\end{aligned}$$

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We conclude that $J_{2,2,2}(x,z) \leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}$. By putting together the above estimates we obtain

$$\mathcal{V}_{\rho,a}\left(\{R^\alpha_\epsilon\}_{\epsilon>0}\right)(f_2)(z) - \mathcal{V}_{\rho,a}\left(\{R^\alpha_\epsilon\}_{\epsilon>0}\right)(f_2)(x) \leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}, \quad x,z \in I.$$

Here, $C>0$ does not depend on x, $z \in I$, so it follows that

$$J_2 \leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}.$$

We now estimate J_3. Note first that if $|x-y|<r_0$, then $|y-x_0|<2r_0$, so it is clear that

$$\int_{\epsilon_1<|x-y|<\epsilon_2} R^\alpha(x,y) f_2(y)\,d\gamma_\alpha(y) = 0, \quad 0<\epsilon_1<\epsilon_2\leq r_0.$$

Suppose that $r_0 \leq am(x)$. We have, for any $x \in I$, that

$$\begin{aligned}
&\mathcal{V}_{\rho,a}\left(\{R^\alpha_\epsilon\}_{\epsilon>0}\right)(f_2)(x)\\
&\leq \sup_{0<\epsilon_n<\dots<\epsilon_1\leq r_0,\ n\in\mathbb{N}} \left(\sum_{j=1}^{n-1} |R^\alpha_{\epsilon_{j+1}}(f_2)(x) - R^\alpha_{\epsilon_j}(f_2)(x)|^\rho\right)^{1/\rho}\\
&\quad + \sup_{r_0<\epsilon_n<\dots<\epsilon_1\leq am(x),\ n\in\mathbb{N}} \left(\sum_{j=1}^{n-1} |R^\alpha_{\epsilon_{j+1}}(f_2)(x) - R^\alpha_{\epsilon_j}(f_2)(x)|^\rho\right)^{1/\rho}\\
&= \sup_{r_0<\epsilon_n<\dots<\epsilon_1\leq am(x),\ n\in\mathbb{N}} \left(\sum_{j=1}^{n-1} |R^\alpha_{\epsilon_{j+1}}(f_2)(x) - R^\alpha_{\epsilon_j}(f_2)(x)|^\rho\right)^{1/\rho}\\
&\leq \mathcal{V}_{\rho,a}\left(\{R^\alpha_\epsilon\}_{\epsilon>0}\right)(f)(x)\\
&\quad + \sup_{r_0<\epsilon_n<\dots<\epsilon_1\leq am(x),\ n\in\mathbb{N}} \left(\sum_{j=1}^{n-1} |R^\alpha_{\epsilon_{j+1}}(f_1)(x) - R^\alpha_{\epsilon_j}(f_1)(x)|^\rho\right)^{1/\rho}.
\end{aligned}$$

Since, for every $y \in 4I$, $|x-y| \leq 5r_0 \leq 5am(x)$, by using again (3.2) and [52, (3)], we deduce that

$$\begin{aligned}
&\sup_{r_0<\epsilon_n<\dots<\epsilon_1\leq am(x),\ n\in\mathbb{N}} \left(\sum_{j=1}^{n-1} |R^\alpha_{\epsilon_{j+1}}(f_1)(x) - R^\alpha_{\epsilon_j}(f_1)(x)|^\rho\right)^{1/\rho}\\
&\qquad \leq C\int_{|x-y|>r_0,\ y\in 4I} \frac{e^{\frac{x^2+y^2}{2}}|f(y)|}{\mathfrak{m}_\alpha(I(y,|x-y|))}\,d\gamma_\alpha(y)\\
&\qquad \leq C\int_{|x-y|>r_0,\ y\in 4I} \frac{e^{\frac{x^2-y^2}{2}}|f(y)|}{|x-y|}\,dy\\
&\qquad \leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)} \int_{|x-y|>r_0,\ y\in 4I} \frac{dy}{|x-y|}\\
&\qquad \leq C\frac{\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}}{r_0}\int_{4I} dy\\
&\qquad \leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)},
\end{aligned}$$

that is, $J_3 \leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}$ for the case $r_0 \leq am(x)$.

When $r_0 > am(x)$,

$$\mathcal{V}_{\rho,a}\left(\{R^\alpha_\epsilon\}_{\epsilon>0}\right)(f_2)(x) = 0,$$

so $J_3 \leq 0$ in this case.

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We conclude that

$$J_3 \leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}.$$

By combining the above estimates, since the constant $C > 0$ does not depend on $x \in (0,\infty)$ or $I \in \mathcal{B}_a(x)$, we get

$$\|\mathcal{M}_a^\alpha (\mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f)) - \mathcal{V}_{\rho,a}(\{R_\epsilon^\alpha\}_{\epsilon>0})(f)\|_{L^\infty(0,\infty),\gamma_\alpha} \leq C\|f\|_{L^\infty(0,\infty),\gamma_\alpha}.$$

Thus the proof is finished.

3.2. **Local oscillation operators.** Theorem 1.1 for oscillation operators can be proved by using the procedure developed in the previous section for the variation operator, so we give a sketch of the proof.

According to [9, Theorem 1.3], the oscillation operator $\mathcal{O}(\{R_\epsilon^\alpha\}_{\epsilon>0}, \{t_j\}_{j\in\mathbb{Z}})$ is bounded on $L^2((0,\infty),\gamma_\alpha)$. This property implies that $\mathcal{O}_a(\{R_\epsilon^\alpha\}_{\epsilon>0}, \{t_j\}_{j\in\mathbb{Z}})(f) \in L^1((0,\infty),\gamma_\alpha)$ for every $f \in L^\infty((0,\infty),\gamma_\alpha)$.

In order to prove our result, it is sufficient to find a positive constant C such that, for every $f \in L^\infty((0,\infty),\gamma_\alpha)$,

$$\begin{aligned}\|\mathcal{M}_a^\alpha(\mathcal{O}_a(\{R_\epsilon^\alpha\}_{\epsilon>0}, \{t_j\}_{j\in\mathbb{Z}})(f)) - \mathcal{O}_a(\{R_\epsilon^\alpha\}_{\epsilon>0}, \{t_j\}_{j\in\mathbb{Z}})(f)\|_{L^\infty((0,\infty),\gamma_\alpha)}& \\ \leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}.& \end{aligned} \tag{3.3}$$

Fix $f \in L^\infty((0,\infty),\gamma_\alpha)$ and let $x, x_0, r_0 \in (0,\infty)$ such that $I = I(x_0,r_0) \in \mathcal{B}_a(x)$. We write $f = f\chi_{4I} + f\chi_{(0,\infty)\setminus 4I} := f_1 + f_2$.

$$\begin{aligned}&\frac{1}{\gamma_\alpha(I)}\int_I \mathcal{O}_a(\{R_\epsilon^\alpha\}_{\epsilon>0}, \{t_j\}_{j\in\mathbb{Z}})(f)(z)d\gamma_\alpha(z) - \mathcal{O}_a(\{R_\epsilon^\alpha\}_{\epsilon>0}, \{t_j\}_{j\in\mathbb{Z}})(f)(x)\\ &\leq \frac{1}{\gamma_\alpha(I)}\int_I \mathcal{O}_a(\{R_\epsilon^\alpha\}_{\epsilon>0}, \{t_j\}_{j\in\mathbb{Z}})(f_1)(z)d\gamma_\alpha(z)\\ &\quad + \frac{1}{\gamma_\alpha(I)}\int_I \left[\mathcal{O}_a(\{R_\epsilon^\alpha\}_{\epsilon>0}, \{t_j\}_{j\in\mathbb{Z}})(f_2)(z) - \mathcal{O}_a(\{R_\epsilon^\alpha\}_{\epsilon>0}, \{t_j\}_{j\in\mathbb{Z}})(f_2)(x)\right] d\gamma_\alpha(z)\\ &\quad + \mathcal{O}_a(\{R_\epsilon^\alpha\}_{\epsilon>0}, \{t_j\}_{j\in\mathbb{Z}})(f_2)(x) - \mathcal{O}_a(\{R_\epsilon^\alpha\}_{\epsilon>0}, \{t_j\}_{j\in\mathbb{Z}})(f)(x)\\ &:= J_1 + J_2 + J_3.\end{aligned}$$

It is immediate from the L^2-boundedness of $\mathcal{O}(\{R_\epsilon^\alpha\}_{\epsilon>0}, \{t_j\}_{j\in\mathbb{Z}})$ ([9, Theorem 1.3]) that

$$J_1 \leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}.$$

We now estimate the integrand of J_2. For certain $C > 1$, we have

$$\begin{aligned}&\mathcal{O}_a(\{R_\epsilon^\alpha\}_{\epsilon>0},\{t_j\}_{j\in\mathbb{Z}})(f_2)(z) - \mathcal{O}_a(\{R_\epsilon^\alpha\}_{\epsilon>0}, \{t_j\}_{j\in\mathbb{Z}})(f_2)(x)\\ &\qquad \leq \left(\sum_{\substack{j\in\mathbb{Z}\\ t_j\leq Cam(x)}} \sup_{t_{j-1}\leq\epsilon_{j-1}<\epsilon_j\leq t_j} \left|R_{\epsilon_{j-1}}^\alpha(f_2)(z) - R_{\epsilon_j}^\alpha(f_2)(z)\right|^2\right)^{1/2}\\ &\qquad\quad - \left(\sum_{\substack{j\in\mathbb{Z}\\ t_j\leq am(x)}} \sup_{t_{j-1}\leq\epsilon_{j-1}<\epsilon_j\leq t_j} \left|R_{\epsilon_{j-1}}^\alpha(f_2)(x) - R_{\epsilon_j}^\alpha(f_2)(x)\right|^2\right)^{1/2}.\end{aligned}$$

We define

$$j_0(x) = \max\{j \in \mathbb{Z} : t_j \leq am(x)\} \tag{3.4}$$

and also, provided that $t_{j_0(x)+1} \leq Cam(x)$, we consider

$$j_1(x) = \max\{j \in \mathbb{Z} : j > j_0(x), t_j \leq Cam(x)\}. \tag{3.5}$$

IMAL PREPRINT # 2023-0063

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Thus, when $t_{j_0(x)+1} > Cam(x)$, we can write

$$\begin{aligned}
&\mathcal{O}_a(\{R^\alpha_\epsilon\}_{\epsilon>0},\{t_j\}_{j\in\mathbb{Z}})(f_2)(z)-\mathcal{O}_a(\{R^\alpha_\epsilon\}_{\epsilon>0},\{t_j\}_{j\in\mathbb{Z}})(f_2)(x)\\
&\leq \left(\sum_{\substack{j\in\mathbb{Z}\\ j\leq j_0(x)}} \sup_{t_{j-1}\leq \epsilon_{j-1}<\epsilon_j\leq t_j}\left|R^\alpha_{\epsilon_{j-1}}(f_2)(z)-R^\alpha_{\epsilon_j}(f_2)(z)\right|^2\right)^{1/2}\\
&\quad -\left(\sum_{\substack{j\in\mathbb{Z}\\ j\leq j_0(x)}} \sup_{t_{j-1}\leq \epsilon_{j-1}<\epsilon_j\leq t_j}\left|R^\alpha_{\epsilon_{j-1}}(f_2)(x)-R^\alpha_{\epsilon_j}(f_2)(x)\right|^2\right)^{1/2}\\
&\leq \left(\sum_{\substack{j\in\mathbb{Z}\\ j\leq j_0(x)}} \sup_{t_{j-1}\leq \epsilon_{j-1}<\epsilon_j\leq t_j}|D(x,z)|^2\right)^{1/2} := \tilde{J}_2(x,z),
\end{aligned}$$

where

$$D(x,z):=R^\alpha_{\epsilon_{j-1}}(f_2)(z)-R^\alpha_{\epsilon_j}(f_2)(z)-\left(R^\alpha_{\epsilon_{j-1}}(f_2)(x)-R^\alpha_{\epsilon_j}(f_2)(x)\right).$$

On the other hand, if $t_{j_0(x)+1}\leq Cam(x)$, we get

$$\begin{aligned}
&\mathcal{O}_a(\{R^\alpha_\epsilon\}_{\epsilon>0},\{t_j\}_{j\in\mathbb{Z}})(f_2)(z)-\mathcal{O}_a(\{R^\alpha_\epsilon\}_{\epsilon>0},\{t_j\}_{j\in\mathbb{Z}})(f_2)(x)\\
&\quad\leq \tilde{J}_2(x,z)+\left(\sum_{\substack{j\in\mathbb{Z}\\ j_0(x)<j\leq j_1(x)}} \sup_{t_{j-1}\leq \epsilon_{j-1}<\epsilon_j\leq t_j}\left|R^\alpha_{\epsilon_{j-1}}(f_2)(z)-R^\alpha_{\epsilon_j}(f_2)(z)\right|^2\right)^{1/2}\\
&\quad\leq \tilde{J}_2(x,z)+\int_{\frac{a}{\rho}m(x)\leq|z-y|\leq Cam(x)}|R^\alpha(z,y)||f_2(y)|d\gamma_\alpha(y),
\end{aligned}$$

where in the last inequality we have used that $t_{j_0(x)}\leq am(x)\leq t_{j_0(x)+1}\leq \rho t_{j_0(x)}$ with $\rho>1$.

Notice that we can estimate $\tilde{J}_2(x,z)$ in the following form

$$\begin{aligned}
&\tilde{J}_2(x,z)\\
&\leq \left(\sum_{\substack{j\in\mathbb{Z}\\ j\leq j_0(x)}} \sup_{t_{j-1}\leq \epsilon_{j-1}<\epsilon_j\leq t_j}\left|\int_{\epsilon_{j-1}<|z-y|<\epsilon_j}(R^\alpha(z,y)-R^\alpha(x,y))f_2(y)d\gamma_\alpha(y)\right.\right.\\
&\quad\left.\left.+\int_0^\infty\left(\chi_{\{\epsilon_{j-1}<|z-y|<\epsilon_j\}}(y)-\chi_{\{\epsilon_{j-1}<|x-y|<\epsilon_j\}}(y)\right)R^\alpha(x,y)f_2(y)d\gamma_\alpha(y)\right|^2\right)^{1/2}\\
&\leq \int_{(0,\infty)\setminus 4I}|R^\alpha(z,y)-R^\alpha(x,y)||f_2(y)|d\gamma_\alpha(y)\\
&\quad+\left(\sum_{\substack{j\in\mathbb{Z}\\ j\leq j_0(x)}}\left(\sup_{t_{j-1}\leq \epsilon_{j-1}<\epsilon_j\leq t_j}\int_0^\infty\left|\chi_{\{\epsilon_{j-1}<|z-y|<\epsilon_j\}}(y)-\chi_{\{\epsilon_{j-1}<|x-y|<\epsilon_j\}}(y)\right|\right.\right.\\
&\quad\quad\left.\left.\times\ |R^\alpha(x,y)||f_2(y)|d\gamma_\alpha(y)\right)^2\right)^{1/2}.
\end{aligned}$$

At this point, we can proceed as in the proof of the corresponding result for variation operators $\mathcal{V}_{\rho,a}$, by using Hölder's inequality with an exponent $s\in(1,2)$

IMAL PREPRINT # 2023-0063
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instead of applying it with exponent 2. In this way, we deduce that

$$J_2 \leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}.$$

In order to study J_3, we first recall that

$$\int_{\epsilon_1<|x-y|<\epsilon_2} R^\alpha(x,y)f_2(y)d\gamma_\alpha(y), \quad 0<\epsilon_1<\epsilon_2\leq r_0.$$

Then, if $r_0 \geq am(x)$, we obtain

$$\mathcal{O}_a(\{R^\alpha_\epsilon\}_{\epsilon>0},\{t_j\}_{j\in\mathbb{Z}})(f_2)(x)=0. \tag{3.6}$$

Suppose now that $r_0 < am(x)$ and define $j_0(x)$ as in (3.4). If $t_{j_0(x)} \leq r_0$, we again have (3.6). If not, we define $j_1 = \max\{j\in\mathbb{Z} : t_{j_1}\leq r_0\}$. Then

$$\begin{aligned}
&\mathcal{O}_a(\{R^\alpha_\epsilon\}_{\epsilon>0},\{t_j\}_{j\in\mathbb{Z}})(f_2)(x)\\
&\quad=\left(\sum_{j=j_1+1}^{j_0(x)}\sup_{t_{j-1}\leq\epsilon_{j-1}<\epsilon_j\leq t_j}|R^\alpha_{\epsilon_{j-1}}(f_2)(x)-R^\alpha_{\epsilon_j}(f_2)(x)|^2\right)^{1/2}\\
&\quad\leq \mathcal{O}_a(\{R^\alpha_\epsilon\}_{\epsilon>0},\{t_j\}_{j\in\mathbb{Z}})(f)(x)\\
&\qquad+\left(\sum_{j=j_1+1}^{j_0(x)}\sup_{t_{j-1}\leq\epsilon_{j-1}<\epsilon_j\leq t_j}|R^\alpha_{\epsilon_{j-1}}(f_1)(x)-R^\alpha_{\epsilon_j}(f_1)(x)|^2\right)^{1/2}.
\end{aligned}$$

Since $t_{j_1}\leq r_0\leq t_{j+1}\leq \rho t_{j_1}$, it follows that

$$\begin{aligned}
&\left(\sum_{j=j_1+1}^{j_0(x)}\sup_{t_{j-1}\leq\epsilon_{j-1}<\epsilon_j\leq t_j}|R^\alpha_{\epsilon_{j-1}}(f_1)(x)-R^\alpha_{\epsilon_j}(f_1)(x)|^2\right)^{1/2}\\
&\quad\leq \sum_{j=j_1+1}^{j_0(x)}\sup_{t_{j-1}\leq\epsilon_{j-1}<\epsilon_j\leq t_j}|R^\alpha_{\epsilon_{j-1}}(f_1)(x)-R^\alpha_{\epsilon_j}(f_1)(x)|\\
&\quad\leq C\int_{|x-y|>r_0/\rho}\frac{e^{\frac{x^2+y^2}{2}}|f(y)|}{\mathfrak{m}_\alpha(I(y,|x-y|))}d\gamma_\alpha(y)\\
&\quad\leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)},
\end{aligned}$$

where we have used again the bound given in (3.2) and [52, (3)].

We conclude that

$$J_3\leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}.$$

By putting together all of the above estimates, we get (3.3) and the proof of Theorem 1.1 for local oscillation operators is completed.

3.3. **Local maximal Riesz transform.** We firstly prove that $R^\alpha_{*,a}$ is bounded on $L^p((0,\infty),\gamma_\alpha)$ for every $1<p<\infty$. In order to do so, we need to decompose, for every $\epsilon>0$, the truncated integral R^α_ϵ into two parts, called local and global parts (see [47]).

For every $\tau>0$, we consider the sets

$$L_\tau=\left\{(x,y,s)\in(0,\infty)\times(0,\infty)\times(-1,1):\sqrt{q_-(x,y,s)}\leq\frac{a(1+\alpha)\tau}{1+x+y}\right\}$$

and

$$G_\tau=((0,\infty)\times(0,\infty)\times(-1,1))\setminus L_\tau.$$

Here, and in the sequel, we denote $q_\pm(x,y,s)=x^2+y^2\pm 2xys$, for $x,y\in(0,\infty)$ and $s\in(-1,1)$.

IMAL PREPRINT # 2023-0063

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We choose a function $\varphi \in C^\infty((0,\infty)\times(0,\infty)\times(-1,1))$ such that $0 \le \varphi \le 1$,

$$\varphi(x,y,s) = \begin{cases} 1, & (x,y,s) \in L_1, \\ 0, & (x,y,s) \in G_2, \end{cases}$$

and

$$|\partial_x \varphi(x,y,s)| + |\partial_y \varphi(x,y,s)| \le \frac{C}{\sqrt{q_-(x,y,s)}}, \quad x,y \in (0,\infty), s \in (-1,1).$$

We define, for each $\epsilon > 0$ and $x \in (0,\infty)$

$$R_\epsilon^{\alpha,\mathrm{loc}}(f)(x) = \int_{|x-y|>\varepsilon,\ y\in(0,\infty)} R^{\alpha,\mathrm{loc}}(x,y) f(y) d\gamma_\alpha(y),$$
$$R_\epsilon^{\alpha,\mathrm{glob}}(f)(x) = R_\epsilon^{\alpha}(f)(x) - R_\epsilon^{\alpha,\mathrm{loc}}(f)(x),$$

where

$$R^{\alpha,\mathrm{loc}}(x,y) = \int_{-1}^{1} R^{\alpha,\mathrm{loc}}(x,y,s)\Pi_\alpha(s)ds, \quad x,y \in (0,\infty)$$

and, for $x,y \in (0,\infty)$, $s \in (-1,1)$,

$$R^{\alpha,\mathrm{loc}}(x,y,s) = -\frac{2}{\sqrt{\pi}} \int_0^\infty \frac{e^{-t(\alpha+2)}(e^{-t}x - ys)}{(1-e^{-2t})^{\alpha+2}} e^{-\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}+y^2} \varphi(x,y,s) \frac{dt}{\sqrt{t}}.$$

We also consider the maximal operators associated with the above,

$$R_*^{\alpha,\mathrm{loc}}(f) = \sup_{\epsilon>0} \left| R_\epsilon^{\alpha,\mathrm{loc}}(f) \right|, \quad R_*^{\alpha,\mathrm{glob}}(f) = \sup_{\epsilon>0} \left| R_\epsilon^{\alpha,\mathrm{glob}}(f) \right|,$$

which clearly verify

$$R_*^\alpha(f) \le R_*^{\alpha,\mathrm{loc}}(f) + R_*^{\alpha,\mathrm{glob}}(f).$$

According to [9, § 3.1] (see also [47, Proposition 3.1]), we have that

$$R_*^{\alpha,\mathrm{glob}}(f)(x) \le C \int_0^\infty K^\alpha(x,y) f(y) d\gamma_\alpha(y), \quad x \in (0,\infty),$$

where

$$K^\alpha(x,y) = \int_{-1}^{1} K^\alpha(x,y,s)\chi_{G_1}(x,y,s)\Pi_\alpha(s)ds, \quad x,y \in (0,\infty),$$

and, for $x,y \in (0,\infty)$ and $s \in (-1,1)$,

$$K^\alpha(x,y,s) = \begin{cases} 1, & s < 0, \\ \left(\frac{q_+(x,y,s)}{q_-(x,y,s)}\right)^{\frac{\alpha+1}{2}} \exp\left(\frac{x^2+y^2-\sqrt{q_-(x,y,s)q_+(x,y,s)}}{2}\right), & s \ge 0. \end{cases} \tag{3.7}$$

It follows that $R_*^{\alpha,\mathrm{glob}}$ is bounded on $L^p((0,\infty),\gamma_\alpha)$ for every $1 < p < \infty$ (see [9, § 3.1]).

We recall that the measure $\mathfrak{m}_\alpha$ defined in Section 3.1 has the doubling property on $(0,\infty)$. Therefore, by [9, (18) and (19)], $e^{-y^2} R_\alpha^{\mathrm{loc}}(x,y)$, for $x,y \in (0,\infty)$, is an $\mathfrak{m}_\alpha$-standard Calderón-Zygmund kernel, that is, for every $x,y \in (0,\infty)$, $x \neq y$,

$$\left| e^{-y^2} R_\alpha^{\mathrm{loc}}(x,y) \right| \le \frac{C}{\mathfrak{m}_\alpha(I(x,|x-y|))},$$

and

$$\left| \partial_x \left[e^{-y^2} R_\alpha^{\mathrm{loc}}(x,y) \right] \right| + \left| \partial_y \left[e^{-y^2} R_\alpha^{\mathrm{loc}}(x,y) \right] \right| \le \frac{C}{|x-y|\mathfrak{m}_\alpha(I(x,|x-y|))}.$$

If we define the operators $R^{\alpha,\mathrm{loc}}$ and $R^{\alpha,\mathrm{glob}}$ in the obvious way, we can see as above that the later is bounded on $L^p((0,\infty),\gamma_\alpha)$ for every $1 < p < \infty$. Since R^α is also bounded on $L^p((0,\infty),\gamma_\alpha)$ for every $1 < p < \infty$ (see [41, Theorem 13]), we conclude that $R^{\alpha,\mathrm{loc}}$ is bounded on $L^p((0,\infty),\gamma_\alpha)$ for every $1 < p < \infty$. By

IMAL PREPRINT # 2023-0063
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proceeding as in [5, § 2], we deduce that $R^{\alpha,\mathrm{loc}}$ is bounded on $L^p((0,\infty),\mathfrak{m}_\alpha)$ for every $1<p<\infty$. Moreover, since $R^{\alpha,\mathrm{loc}}$ is an $\mathfrak{m}_\alpha$-Calderón-Zygmund operator, $R_*^{\alpha,\mathrm{loc}}$ is bounded on $L^p((0,\infty),\mathfrak{m}_\alpha)$ for every $1<p<\infty$. By using again the arguments given in [5, § 2], we get that $R_*^{\alpha,\mathrm{loc}}$ is bounded on $L^p((0,\infty),\gamma_\alpha)$ for every $1<p<\infty$.

It follows now that R_*^α is bounded on $L^p((0,\infty),\gamma_\alpha)$ for every $1<p<\infty$. Particularly, using this property for $p=2$, for any $f\in L^\infty((0,\infty),\gamma_\alpha)$,

$$\|R_{*,a}^\alpha\|_{L^1((0,\infty),\gamma_\alpha)}\leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)}.$$

We recall that, from (3.2),

$$|R^\alpha(x,y)|\leq C\frac{e^{\frac{x^2+y^2}{2}}}{\mathfrak{m}_\alpha(I(x,|x-y|))},\quad x,y\in(0,\infty),\ x\neq y,$$

and also we can see (as in [4, § 4.3]) that

$$\sup_{I\in\mathcal{B}_a}\sup_{x\in I} r_0\int_{(0,\infty)\setminus 2I}|\partial_x R^\alpha(x,y)|d\gamma_\alpha(y)<\infty.$$

By proceeding as in the proof of [32, Theorem 4.1], it yields

$$\sup_{I\in\mathcal{B}_a}\left\|\mathcal{M}_a^\alpha\left(R_{*,a}^\alpha(f)\right)-R_{*,a}^\alpha(f)\right\|_{L^\infty((0,\infty),\gamma_\alpha)}\leq C\|f\|_{L^\infty((0,\infty),\gamma_\alpha)},$$

meaning that $R_{*,a}^\alpha$ is bounded from $L^\infty((0,\infty),\gamma_\alpha)$ into $\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$.

4. Proof of Theorem 1.2

In this section, we will study $L^\infty((0,\infty),\gamma_\alpha)$-$\mathrm{BLO}_a((0,\infty),\gamma_\alpha)$ estimates for the a-local maximal operator

$$\begin{aligned}Q_{\phi,*,a}^\alpha(f)(x)&=\sup_{0<\epsilon\leq am(x)}|Q_{\phi,\epsilon}^\alpha(f)(x)|\\&=\sup_{0<\epsilon\leq am(x)}\left|\int_{|x-y|>\epsilon,\ y\in(0,\infty)}K_\phi^\alpha(x,y)f(y)d\gamma_\alpha(y)\right|,\end{aligned}$$

for $x\in(0,\infty)$ and $a>0$.

We recall that

$$K_\phi^\alpha(x,y)=-\int_0^\infty\phi(t)\partial_t W_t^\alpha(x,y)dy,\quad x,t,\in(0,\infty),\ x\neq y,$$

being

$$W_t^\alpha(x,y)=\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+1}\int_{-1}^1 e^{-\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}+y^2}\Pi_\alpha(s)ds,\quad x,y,t\in(0,\infty).$$

Firstly, we shall see that $Q_{\phi,*,a}^\alpha$ is bounded on $L^p((0,\infty),\gamma_\alpha)$ for every $1<p<\infty$. We define, for $x,y,t\in(0,\infty)$,

$$W_t^{\alpha,\mathrm{loc}}(x,y)=\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+1}\int_{-1}^1 e^{-\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}+y^2}\varphi(x,y,s)\Pi_\alpha(s)ds,$$

and

$$W_t^{\alpha,\mathrm{glob}}(x,y)=W_t^\alpha(x,y)-W_t^{\alpha,\mathrm{loc}}(x,y).$$

In terms of these, we consider $K_\phi^{\alpha,\mathrm{loc}}$ and $K_\phi^{\alpha,\mathrm{glob}}$ given as K_ϕ^α but with W_t^α replaced by $W_t^{\alpha,\mathrm{loc}}$ and $W_t^{\alpha,\mathrm{glob}}$, respectively. Similarly, we define $Q_{\phi,*,a}^{\alpha,\mathrm{loc}}$ and $Q_{\phi,*,a}^{\alpha,\mathrm{glob}}$ by putting $K_\phi^{\alpha,\mathrm{loc}}$ and $K_\phi^{\alpha,\mathrm{glob}}$ instead of K_ϕ^α, respectively.

IMAL PREPRINT # 2023-0063
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We will first deal with $Q^{\alpha,\text{glob}}_{\phi,*,a}$. Notice that, for $x,y,t\in(0,\infty)$ and $s\in(-1,1)$

$$\partial_t\left[\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+1}\exp\left(-\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}\right)\right]$$
$$=P_{x,y,s}\left(e^{-t}\right)\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+1}\exp\left(-\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}\right),$$

where, for every $x,y\in(0,\infty)$ and $s\in(-1,1)$, $P_{x,y,s}$ is a polynomial whose degree is at most four. Hence,

$$|K^{\alpha,\text{glob}}_{\phi}(x,y)|\leq C\int_{-1}^{1}\sup_{t>0}\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+1}e^{-\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}+y^2}\chi_{L_1^c}(x,y,s)\Pi_\alpha(s)ds$$
$$\leq C\int_{-1}^{1}K^\alpha(x,y,s)\chi_{L_1^c}(x,y,s)\Pi_\alpha(s)ds,\quad x,y\in(0,\infty),$$

being $K^\alpha(x,y,s)$ as in (3.7), for $(x,y,s)\in L_1^c$.

From [9, § 3.1], it follows that the operator whose kernel is the one on the right-hand side is bounded on $L^p((0,\infty),\gamma_\alpha)$ for every $1<p<\infty$, and so will be $Q^{\alpha,\text{glob}}_{\phi,*,a}$.

Furthermore, for every $f\in L^p((0,\infty),\gamma_\alpha)$, $1<p<\infty$,

$$\lim_{\varepsilon\to0^+}\int_{|x-y|>\varepsilon,\ y\in(0,\infty)}K^{\alpha,\text{glob}}_{\phi}(x,y)f(y)d\gamma_\alpha(y)=\int_0^\infty K^{\alpha,\text{glob}}_{\phi}(x,y)f(y)d\gamma_\alpha(y),$$

for a.e. $x\in(0,\infty)$.

We now consider the operators

$$T^{\alpha,\text{loc}}_{M}(f)(x)=\lim_{\varepsilon\to0^+}\left(\Lambda(\varepsilon)f(x)+\int_{|x-y|>\varepsilon,\ y\in(0,\infty)}K^{\alpha,\text{loc}}_{\phi}(x,y)f(y)d\gamma_\alpha(y)\right),$$

and

$$T^{\alpha,\text{glob}}_{M}(f)(x)=\int_0^\infty K^{\alpha,\text{glob}}_{\phi}(x,y)f(y)d\gamma_\alpha(y),$$

for a.e. $x\in(0,\infty)$. Since T^α_M and $T^{\alpha,\text{glob}}_M$ are both bounded on $L^2((0,\infty),\gamma_\alpha)$ ([45, Proposition 3]), also $T^{\alpha,\text{loc}}_M$ is bounded on $L^2((0,\infty),\gamma_\alpha)$. Moreover, for every $f\in L^\infty((0,\infty),\gamma_\alpha)$

$$T^{\alpha,\text{loc}}_{M}(f)(x)=\int_0^\infty K^{\alpha,\text{loc}}_{\phi}(x,y)f(y)d\gamma_\alpha(y),\quad x\notin\text{supp}(f).$$

Let us now consider $\mathbb{K}^\alpha_\phi(x,y):=e^{-y^2}K^{\alpha,\text{loc}}_\phi(x,y)$, for $x,y\in(0,\infty)$. We have

$$\mathbb{K}^\alpha_\phi(x,y)=(\alpha+1)\int_0^\infty\varphi(t)\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+1}\int_{-1}^{1}e^{-\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}}\varphi(x,y,s)\Pi_\alpha(s)dsdt$$
$$-\int_0^\infty\varphi(t)e^{-t(\alpha+1)}\int_{-1}^{1}\partial_t\left[\frac{e^{-\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}}}{(1-e^{-2t})^{\alpha+1}}\right]\varphi(x,y,s)\Pi_\alpha(s)dsdt$$
$$:=\mathbb{K}^\alpha_{\phi,1}(x,y)+\mathbb{K}^\alpha_{\phi,2}(x,y),\quad x,y\in(0,\infty).$$

As in [5, § 7], we can prove that

$$|\mathbb{K}^\alpha_{\phi,2}(x,y)|\leq\frac{C}{\mathfrak{m}_\alpha(I(x,|x-y|))},\quad x,y\in(0,\infty),\ x\neq y,$$

and

$$|\partial_x\mathbb{K}^\alpha_{\phi,2}(x,y)|+|\partial_y\mathbb{K}^\alpha_{\phi,2}(x,y)|\leq\frac{C}{|x-y|\mathfrak{m}_\alpha(I(x,|x-y|))},\quad x,y\in(0,\infty),\ x\neq y.$$

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On the other hand, using [45, (2.6)], i.e., $q_-(e^{-t}x,y,s) \geq q_-(x,y,s) - 2(1-e^{-2t})$, for every $(x,y,s) \in N_1$, and the estimates obtained in [2, p. 12 and Lemma 3.1], we get

(a)

$$\begin{aligned}|\mathbb{K}^\alpha_{\phi,1}(x,y)| &\leq C \int_0^\infty |\varphi(t)| \left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+1} \int_{-1}^1 e^{-\frac{q_-(x,y,s)}{1-e^{-2t}}} \Pi_\alpha(s) ds dt \\ &\leq C \int_0^\infty |\varphi(t)| e^{-t(\alpha+1)} dt \int_{-1}^1 \frac{\Pi_\alpha(s)}{q_-(x,y,s)^{\alpha+1}} ds \\ &\leq \frac{C}{\mathfrak{m}_\alpha(I(x,|x-y|))}, \quad x,y \in (0,\infty),\ x \neq y;\end{aligned}$$

(b) by [4, Lemma 3.4 (E8)],

$$\begin{aligned}|\partial_x \mathbb{K}^\alpha_{\phi,1}(x,y)| &\leq C \int_0^\infty |\varphi(t)| \left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+1} \int_{-1}^1 e^{-\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}} \\ &\quad \times \left[\frac{e^{-t}|e^{-t}x-ys|}{1-e^{-2t}} \varphi(x,y,s) + |\partial_x \varphi(x,y,s)|\right] \Pi_\alpha(s) ds dt \\ &\leq C \int_0^\infty |\varphi(t)| \left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+1} e^{-t} \int_{-1}^1 e^{-\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}} \\ &\quad \times \left[\frac{\sqrt{q_-(e^{-t}x,y,s)}}{1-e^{-2t}} + \frac{1}{\sqrt{q_-(x,y,s)}}\right] \Pi_\alpha(s) ds dt \\ &\leq C \int_0^\infty |\varphi(t)| e^{-t(\alpha+2)} \int_{-1}^1 \left[\frac{e^{-\frac{q_-(x,y,s)}{2(1-e^{-2t})}}}{(1-e^{-2t})^{\alpha+3/2}} \right. \\ &\quad \left. + \frac{e^{-\frac{q_-(x,y,s)}{1-e^{-2t}}}}{(1-e^{-2t})^{\alpha+1}\sqrt{q_-(x,y,s)}}\right] \Pi_\alpha(s) ds dt \\ &\leq C \int_0^\infty |\varphi(t)| e^{-t(\alpha+2)} dt \int_{-1}^1 \frac{\Pi_\alpha(s)}{q_-(x,y,s)^{\alpha+3/2}} ds \\ &\leq \frac{C}{|x-y|\mathfrak{m}_\alpha(I(x,|x-y|))}, \quad x,y \in (0,\infty),\ x \neq y;\end{aligned}$$

(c) by [4, Lemma 3.4 (E7)] (with x and y interchanged), and proceeding like before,

$$\begin{aligned}|\partial_y \mathbb{K}^\alpha_{\phi,1}(x,y)| &\leq C \int_0^\infty |\varphi(t)| \left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+1} \int_{-1}^1 e^{-\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}} \\ &\quad \times \left[\frac{|y-e^{-t}xs|}{1-e^{-2t}} \varphi(x,y,s) + |\partial_y \varphi(x,y,s)|\right] \Pi_\alpha(s) ds dt \\ &\leq C \int_0^\infty |\varphi(t)| \left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+1} \int_{-1}^1 e^{-\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}} \\ &\quad \times \left[\frac{\sqrt{q_-(e^{-t}x,y,s)}}{1-e^{-2t}} + \frac{1}{\sqrt{q_-(x,y,s)}}\right] \Pi_\alpha(s) ds dt \\ &\leq C \int_0^\infty |\varphi(t)| e^{-t(\alpha+1)} dt \int_{-1}^1 \frac{\Pi_\alpha(s)}{q_-(x,y,s)^{\alpha+3/2}} ds \\ &\leq \frac{C}{|x-y|\mathfrak{m}_\alpha(I(x,|x-y|))}, \quad x,y \in (0,\infty),\ x \neq y.\end{aligned}$$

IMAL PREPRINT # 2023-0063
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All of the above proves that $T_M^{\alpha,\mathrm{loc}}$ is an $\mathfrak{m}_\alpha$-Calderón-Zygmund operator. Therefore, $T_M^{\alpha,\mathrm{loc}}$ is bounded on $L^p((0,\infty),\mathfrak{m}_\alpha)$ for every $1<p<\infty$, which yields $Q_{\phi,*,a}^{\alpha,\mathrm{loc}}$ is also bounded on $L^p((0,\infty),\mathfrak{m}_\alpha)$ for every $1<p<\infty$. The arguments in [5, § 2] allow us to deduce that $Q_{\phi,*,a}^{\alpha,\mathrm{loc}}$ is also bounded on $L^p((0,\infty),\gamma_\alpha)$ for every $1<p<\infty$.

Finally, we conclude that $Q_{\phi,*,a}^{\alpha}$ is bounded on $L^p((0,\infty),\gamma_\alpha)$ for any $1<p<\infty$.

Remark 4.1. We can also prove that $Q_{\phi,*,a}^{\alpha}$ is bounded from $L^1((0,\infty),\gamma_\alpha)$ to $L^{1,\infty}((0,\infty),\gamma_\alpha)$. Actually, it is sufficient at this moment to know that $Q_{\phi,*,a}^{\alpha}$ is bounded on $L^{p_0}((0,\infty),\gamma_\alpha)$ for some $1<p_0<\infty$.

We have proved above that

$$|K_\phi^{\alpha,\mathrm{loc}}(x,y)|\le C\frac{e^{y^2}}{\mathfrak{m}_\alpha(I(x,|x-y|))},\quad x,y\in(0,\infty),\ x\neq y.$$

We also saw that

$$|K_\phi^{\alpha,\mathrm{glob}}(x,y)|\le C\int_{-1}^1 K^\alpha(x,y,s)\chi_{L_1^c}(x,y,s)\Pi_\alpha(s)ds,\quad x,y\in(0,\infty),$$

where $K^\alpha(x,y,s)$ was defined in (3.7). It is easy to see that, for any $(x,y,s)\in L_1^c$,

$$|K^\alpha(x,y,s)|\le C\begin{cases}1, & s\in(-1,0),\\ \frac{\exp\left(\frac{x^2+y^2}{2}\right)}{q_-(x,y,s)^{\alpha+1}}, & s\in[0,1).\end{cases}$$

Moreover, for any fixed constant $c>0$, if $x,y\in(0,\infty)$ with $|x-y|\le cam(x)$ and $s\in(-1,1)$,

$$\begin{aligned}q_-(x,y,s)&=(x-y)^2+2xy(1-s)\le(x-y)^2+4y(|x-y|+y)\\&\le 5(x-y)^2+4y^2+4|x-y|x\le C(1+y^2),\end{aligned}$$

which yields

$$|K^\alpha(x,y,s)|\le\frac{C}{q_-(x,y,s)^{\alpha+1}}\begin{cases}(1+y^2)^{\alpha+1}, & s\in(-1,0),\\ \exp\left(\frac{x^2+y^2}{2}\right), & s\in[0,1),\end{cases}$$

for any $(x,y,s)\in L_1^c$ with $|x-y|\le cam(x)$.

According to [2, Lemma 3.1], we obtain

$$|K_\phi^{\alpha,\mathrm{glob}}(x,y)|\le C\frac{e^{\frac{x^2+y^2}{2}}}{\mathfrak{m}_\alpha(I(x,|x-y|))},\quad x,y\in(0,\infty),0<|x-y|\le cam(x).$$

Hence, we conclude that

$$|K_\phi^{\alpha}(x,y)|\le C\frac{e^{\frac{x^2+y^2}{2}}}{\mathfrak{m}_\alpha(I(x,|x-y|))},\quad x,y\in(0,\infty),0<|x-y|\le cam(x).$$

We are going to prove now that

$$\sup_{I\in\mathcal{B}_1}\sup_{x\in I}\int_{(0,\infty)\setminus 2I}|\partial_x K_\phi^\alpha(x,y)|d\gamma_\alpha(y)<\infty.$$

By partial integration, we have that

$$K_\phi^\alpha(x,y)=\int_0^\infty\phi'(t)W_t^\alpha(x,y)dt,\quad x,y,\in(0,\infty),$$

IMAL PREPRINT # 2023-0063
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and thus,

$$\begin{aligned}\partial_x K_\phi^\alpha(x,y) &= \int_0^\infty \phi'(t)\partial_x W_t^\alpha(x,y)dt \\ &= -2\int_0^\infty \phi'(t)\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+1} \\ &\quad\times \int_{-1}^1 e^{-\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}+y^2}\frac{e^{-t}(e^{-t}x-ys)}{1-e^{-2t}}\Pi_\alpha(s)dsdt, \quad x,y\in(0,\infty).\end{aligned}$$

By [4, Lemma 3.4 (E8)], $|e^{-t}x-ys|\le\sqrt{q_-(e^{-t}x,y,s)}$ for every $x,y\in(0,\infty)$ and $s\in(-1,1)$. Then, using the hypothesis on ϕ',

$$\begin{aligned}|\partial_x K_\phi^\alpha(x,y)| &\le C\int_0^\infty |\phi'(t)|\frac{e^{-t(\alpha+2)}}{(1-e^{-2t})^{\alpha+1}} \\ &\quad\times\int_{-1}^1 e^{-\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}+y^2}\sqrt{q_-(e^{-t}x,y,s)}\Pi_\alpha(s)dsdt \\ &\le C\int_0^\infty \frac{1}{t}\frac{e^{-t(\alpha+2)}}{(1-e^{-2t})^{\alpha+3/2}}\int_{-1}^1 e^{-c\frac{q_-(e^{-t}x,y,s)}{1-e^{-2t}}+y^2}\Pi_\alpha(s)dsdt\end{aligned}$$

for any $x,y\in(0,\infty)$.

Therefore, by [4, Lemma 3.6], there exists $C>0$ such that

$$\sup_{x\in I}\int_{(0,\infty)\setminus 2I}|\partial_x K_\phi^\alpha(x,y)|d\gamma_\alpha(y)\le C$$

for every $I\in\mathcal{B}_1$, as claimed.

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