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I M A L

**VARIATION OPERATORS ASSOCIATED WITH SEMIGROUPS  
 GENERATED BY HARDY OPERATORS INVOLVING  
 FRACTIONAL LAPLACIANS IN A HALF SPACE**

JORGE J. BETANCOR, ESTEFANÍA DALMASSO, AND PABLO QUIJANO

ABSTRACT. We represent by  $\{W_{\lambda,t}^\alpha\}_{t>0}$  the semigroup generated by  $-\mathbb{L}_\lambda^\alpha$ , where  $\mathbb{L}_\lambda^\alpha$  is a Hardy operator on a half space. The operator  $\mathbb{L}_\lambda^\alpha$  includes a fractional Laplacian and it is defined by

$$\mathbb{L}_\lambda^\alpha = (-\Delta)_{\mathbb{R}_+^d}^{\alpha/2} + \lambda x_d^{-\alpha}, \quad \alpha \in (0, 2], \lambda \geq 0.$$

We prove that, for every  $k \in \mathbb{N}$ , the  $\rho$ -variation operator  $\mathcal{V}_\rho(\{t^k \partial_t^k W_{\lambda,t}^\alpha\})$  is bounded on  $L^p(\mathbb{R}_+^d, w)$  for each  $1 < p < \infty$  and  $w \in A_p(\mathbb{R}_+^d)$ , being  $A_p(\mathbb{R}_+^d)$  the Muckenhoupt  $p$ -class of weights on  $\mathbb{R}_+^d$ .

1. INTRODUCTION

In this paper we consider the non-local Hardy type operator  $\mathbb{L}_\lambda^\alpha$  defined, in a formal way, by

$$\mathbb{L}_\lambda^\alpha := (-\Delta)_{\mathbb{R}_+^d}^{\alpha/2} + \lambda x_d^{-\alpha},$$

on  $\mathbb{R}_+^d := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_d > 0\}$ . Here,  $\alpha \in (0, 2]$  and  $\lambda \geq \lambda_*$  where

$$\lambda_* = -\frac{\Gamma(\frac{\alpha+1}{2})}{\pi} \left( \Gamma\left(\frac{\alpha+1}{2}\right) - \frac{2^{\alpha-1}\sqrt{\pi}}{\Gamma(1-\frac{\alpha}{2})} \right).$$

In order to give a precise definition of the operator  $\mathbb{L}_\lambda^\alpha$  we consider the following quadratic forms. We define, for every  $u, v \in C_c^1(\mathbb{R}_+^d)$  (the space of continuously differentiable functions having compact support on  $\mathbb{R}_+^d$ ), when  $\alpha \in (0, 2)$ ,

$$Q_\lambda^\alpha(u, v) = \frac{1}{2} \mathcal{A}(d, \alpha) \int_{\mathbb{R}_+^d \times \mathbb{R}_+^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} dx dy + \lambda \int_{\mathbb{R}_+^d} \frac{u(x)v(x)}{x_d^\alpha} dx,$$

where  $\mathcal{A}(d, \alpha) := \frac{-\alpha}{2^{1+\alpha}\pi^{d/2}} \frac{\Gamma((d-\alpha)/2)}{\Gamma(1+\frac{\alpha}{2})}$ , and, when  $\alpha = 2$ ,

$$Q_\lambda^2(u, v) = \int_{\mathbb{R}_+^d} \nabla u(x) \overline{\nabla v(x)} dx + \lambda \int_{\mathbb{R}_+^d} \frac{u(x)v(x)}{x_d^2} dx.$$

By using the classical Hardy inequality when  $\alpha = 2$  and those proved by Bogdan and Dyda ([7, Theorem 1]), we have that for every  $\alpha \in (0, 2]$ ,

$$Q_\lambda^\alpha(u, u) \geq 0, \quad u \in C_c^1(\mathbb{R}_+^d),$$

provided that  $\lambda \geq \lambda_*$ . Note that Hardy inequalities proved in [7, Theorem 1] are sharp.

According to Friedrichs's Extension Theorem, the quadratic form  $Q_\lambda^\alpha$  defines a self-adjoint and non-negative operator  $L_\lambda^\alpha$  in  $L^2(\mathbb{R}_+^d)$  for which  $C_c^1(\mathbb{R}_+^d)$  is a form core. When  $f$  is smooth enough in  $\mathbb{R}_+^d$ ,  $\mathbb{L}_\lambda^\alpha f = L_\lambda^\alpha f$ . The operator  $-L_\lambda^\alpha$  generates

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in  $L^2(\mathbb{R}_+^d)$  a contractive analytic  $C_0$ -semigroup of angle  $\frac{\pi}{2}$  ([42, Corollary 9]). We denote by  $\{W_{\lambda,z}^\alpha\}_{\operatorname{Re} z > 0}$  the semigroup generated by  $-L_\lambda^\alpha$  in  $L^2(\mathbb{R}_+^d)$ . From [3, Theorem 3.1], for every  $z \in \mathbb{C}$  with  $\operatorname{Re} z > 0$ ,  $W_{\lambda,z}^\alpha$  is an integral operator whose kernel, denoted by  $W_{\lambda,z}^\alpha(x, y)$ , is analytic in  $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$  for every  $x, y \in \mathbb{R}_+^d$ . In Section 2 we establish some estimates involving the kernel  $W_{\lambda,z}^\alpha$  that will be useful to prove the main results.

In [16] and [17] we can find how the operators  $L_0^\alpha$  appear related to censored processes and the two-sided estimates for the heat kernels associated with the semigroup generated by  $-L_0^\alpha$  (see [17, Theorem 1.1] and [16, Theorem 1.1]).

We will now define the variation operator. Let  $\rho > 0$  and suppose  $\{a_t\}_{t>0}$  is a set of complex numbers. We define the  $\rho$ -variation  $\mathcal{V}(\{a_t\}_{t>0})$  as follows

$$\mathcal{V}_\rho(\{a_t\}_{t>0}) = \sup_{0 < t_0 < \dots < t_n, n \in \mathbb{N}} \left( \sum_{j=0}^{n-1} |a_{t_{j+1}} - a_{t_j}|^\rho \right)^{1/\rho}.$$

Assume that, for some  $p \in (1, \infty)$ ,  $T_t$  is a bounded operator on  $L^p(\Omega, \mu)$  for every  $t > 0$ , where  $(\Omega, \mu)$  is a measure space. The variation operator  $\mathcal{V}_\rho(\{T_t\}_{t>0})$  is defined by

$$\mathcal{V}_\rho(\{T_t\}_{t>0})(f)(x) := \mathcal{V}(\{T_t(f)(x)\}_{t>0}), \quad f \in L^p(\Omega, \mu).$$

In order to have measurability of the function  $\mathcal{V}_\rho(\{T_t\}_{t>0})(f)$  we need to have some continuity property in the time variable for  $T_t(f)$  (see the comments after [14, Theorem 1.2]). Boundedness properties of  $\rho$ -variation operator (also oscillation operators and  $\lambda$ -jump counting function) give us quantitative measures for pointwise convergence of the family  $\{T_t(f)\}_{t>0}$ .

Lépingle ([31]) studied  $\rho$ -variations of martingales. He established strong  $L^p$ -inequalities and weak-type  $(1, 1)$  estimates for the  $\rho$ -variation operator when  $\rho > 2$ . In [24] an example with  $\rho = 2$  shows that  $\rho > 2$  is the best exponent in the above estimate (see also [41]). Variational inequalities for martingales can be seen as an extension of Doob's maximal inequalities.

Bourgain, in a series of papers (see [8, 9, 10]) proved  $L^p$ -variational inequalities in ergodic settings. Bourgain's ideas were the starting point for using the variation operator in the ergodic theory ([26, 28, 29, 34, 43, 44]), probability theory ([1, 36, 40]) and harmonic analysis ([4, 6, 14, 15, 19, 20, 23, 27, 32, 33, 37, 39]).

Our objective is to prove weighted  $L^p$ -inequalities for the  $\rho$ -variation operator  $\mathcal{V}_\rho(\{t^k \partial_t^k W_{\lambda,t}^\alpha\}_{t>0})$  where  $k \in \mathbb{N}$ . Our study is motivated by those ones in [2] and [38] where the Schrödinger operator in  $\mathbb{R}^d$  with inverse square potentials is considered. As it was mentioned,  $\{W_{\lambda,t}^\alpha\}_{t>0}$  can be seen as the semigroup generated by the Hardy operator type in a half space  $\mathbb{L}_\lambda^\alpha$ . Frank and Merz ([22]) recently have proved that the scales of homogeneous Sobolev spaces generated by  $\mathbb{L}_\lambda^\alpha$  and by  $(-\Delta)_{\mathbb{R}_+^d}^\alpha$  are comparable.

Our first result is the following.

**Theorem 1.1.** *Let  $0 < \alpha \leq 2$ ,  $\rho > 2$ ,  $\lambda \geq 0$  and  $1 < p < \infty$ . Then, the  $\rho$ -variation operator  $\mathcal{V}_\rho(\{t^k \partial_t^k W_{\lambda,t}^\alpha\}_{t>0})$  is bounded on  $L^p(\mathbb{R}_+^d)$  for each  $k \in \mathbb{N}$ .*

For the proof of this theorem, we shall start with the case  $\lambda = 0$ . We denote by  $\{\mathbb{W}_t^\alpha\}_{t>0}$  the semigroup of operators generated by  $-(-\Delta)^\alpha$  on  $\mathbb{R}^d$ . We have that

$$0 < W_{0,t}^\alpha(x, y) = W_{0,t}^\alpha(y, x) \leq \mathbb{W}_t^\alpha(x, y), \quad x, y \in \mathbb{R}_+^d, t > 0.$$

Then,

$$\int_{\mathbb{R}_+^d} W_{0,t}^\alpha(x, y) dy = \int_{\mathbb{R}_+^d} W_{0,t}^\alpha(x, y) dx \leq \int_{\mathbb{R}^d} \mathbb{W}_t^\alpha(x, y) dy = 1, \quad x \in \mathbb{R}_+^d, t > 0.$$

We deduce that, for every  $t > 0$ ,  $W_{0,t}^\alpha$  is a contraction in  $L^1(\mathbb{R}_+^d)$  and in  $L^\infty(\mathbb{R}_+^d)$ . According to [30, Corollary 6.1] the  $\rho$ -variation operator  $\mathcal{V}_\rho(\{t^k \partial_t^k W_{0,t}^\alpha\}_{t>0})$  is bounded on  $L^p(\mathbb{R}_+^d)$  for every  $1 < p < \infty$  and  $k \in \mathbb{N}$ .

In a second step, we prove that the  $\rho$ -variation operator

$$\mathcal{V}_\rho(\{t^k \partial_t^k (W_{\lambda,t}^\alpha - W_{0,t}^\alpha)\}_{t>0})$$

is bounded on  $L^p(\mathbb{R}_+^d)$  for every  $1 < p < \infty$  and  $k \in \mathbb{N}$ . In order to do so, we will consider operators that control this  $\rho$ -variation operator.

Suppose that  $F : (0, \infty) \rightarrow \mathbb{C}$  is a differentiable function on  $(0, \infty)$ . For  $n \in \mathbb{N}$ , let  $0 < t_0 < \dots < t_n$ . We have that

$$\left( \sum_{j=0}^{n-1} |F(t_{j+1}) - F(t_j)|^\rho \right)^{1/\rho} \leq \sum_{j=0}^{n-1} \left| \int_{t_j}^{t_{j+1}} F'(s) ds \right| \leq \int_0^\infty |F'(s)| ds.$$

Thus,

$$\mathcal{V}_\rho(\{F(t)\}_{t>0}) \leq \int_0^\infty |F'(s)| ds.$$

From this inequality we deduce that

$$\mathcal{V}_\rho(\{t^k \partial_t^k (W_{\lambda,t}^\alpha - W_{0,t}^\alpha)\}_{t>0})(f)(x) \leq \int_{\mathbb{R}_+^d} f(y) K_\lambda^\alpha(x, y) dy, \quad x \in \mathbb{R}_+^d, \quad (1.1)$$

where

$$K_\lambda^\alpha(x, y) := \int_0^\infty |\partial_t (t^k \partial_t^k (W_{\lambda,t}^\alpha(x, y) - W_{0,t}^\alpha(x, y)))| dt, \quad x, y \in \mathbb{R}_+^d.$$

For  $m \in \mathbb{N}$ , we consider the operator  $T_\lambda^{\alpha, m}$  defined by

$$T_\lambda^{\alpha, m}(f)(x) = \int_{\mathbb{R}_+^d} T_\lambda^{\alpha, m}(x, y) f(y) dy, \quad x \in \mathbb{R}_+^d,$$

with

$$T_\lambda^{\alpha, m}(x, y) = \int_0^\infty |t^m \partial_t^{m+1} (W_{\lambda,t}^\alpha(x, y) - W_{0,t}^\alpha(x, y))| dt, \quad x, y \in \mathbb{R}_+^d.$$

Taking into account (1.1), to see that  $\mathcal{V}_\rho(\{t^k \partial_t^k (W_{\lambda,t}^\alpha - W_{0,t}^\alpha)\}_{t>0})$  is bounded on  $L^p(\mathbb{R}_+^d)$  it will be sufficient to show that the operator  $T_\lambda^{\alpha, m}$  has this property for  $m = 0$  when  $k = 0$ , and for  $m = k - 1$  and  $m = k$  when  $k \geq 1$ .

We will split the operator  $T_\lambda^{\alpha, m}$  into two parts, determined by the sets

$$\begin{aligned} G &= G_1 \cup G_2 \\ &:= \{(x, y, t) \in \mathbb{R}_+^d \times \mathbb{R}_+^d \times (0, \infty) : x_d \vee y_d \leq t^{1/\alpha}\} \\ &\quad \cup \{(x, y, t) \in \mathbb{R}_+^d \times \mathbb{R}_+^d \times (0, \infty) : x_d \vee y_d > t^{1/\alpha}, 2|x - y| \geq x_d \wedge y_d\} \end{aligned} \quad (1.2)$$

and

$$L := (\mathbb{R}_+^d \times \mathbb{R}_+^d \times (0, \infty)) \setminus G. \quad (1.3)$$

Here, the letter  $G$  stands for *global* and the letter  $L$  means *local*.

Hence, for  $m \in \mathbb{N}$ , we decompose  $T_\lambda^{\alpha, m}$  as follows

$$T_\lambda^{\alpha, m} = T_{\lambda, \text{loc}}^{\alpha, m} + T_{\lambda, \text{glob}}^{\alpha, m}$$

where

$$T_{\lambda, \text{loc}/\text{glob}}^{\alpha, m}(f)(x) = \int_{\mathbb{R}_+^d} T_{\lambda, \text{loc}/\text{glob}}^{\alpha, m}(x, y) f(y) dy, \quad x \in \mathbb{R}_+^d,$$

being

$$T_{\lambda, \text{loc}}^{\alpha, m}(x, y) = \int_0^\infty |t^m \partial_t^{m+1} (W_{\lambda, t}^\alpha(x, y) - W_{0, t}^\alpha(x, y))| \chi_L(x, y, t) dt, \quad x, y \in \mathbb{R}_+^d,$$

and

$$T_{\lambda, \text{glob}}^{\alpha, m}(x, y) = T_{\lambda}^{\alpha, m}(x, y) - T_{\lambda, \text{loc}}^{\alpha, m}(x, y), \quad x, y \in \mathbb{R}_+^d.$$

Actually, the cancellation in  $W_{\lambda, t}^\alpha(x, y) - W_{0, t}^\alpha(x, y)$  is only relevant for  $T_{\lambda, \text{loc}}^{\alpha, m}$ ; in the global operator this cancellation does not play any role.

In Section 3 we will prove the  $L^p$ -boundedness of the operators  $T_{\lambda, \text{loc}}^{\alpha, m}$  and  $T_{\lambda, \text{glob}}^{\alpha, m}$  for  $\alpha \in (0, 2)$ , and we shall deal with the case  $\alpha = 2$  in Section 4. We distinguish this two cases since the estimations for the heat kernel  $W_{\lambda, t}^\alpha(x, y)$ ,  $x, y \in \mathbb{R}_+^d$  and  $t > 0$ , are different (see Propositions 2.1 and 2.2).

We now establish the weighted  $L^p$ -inequalities for  $\mathcal{V}_\rho(\{t^k \partial_t^k W_{\lambda, t}^\alpha\}_{t>0})$ . First of all, we recall the definitions of the Muckenhoupt and Reverse Hölder classes in  $\mathbb{R}_+^d$ . A weight in  $\mathbb{R}_+^d$  is a non-negative measurable function defined on  $\mathbb{R}_+^d$ . A weight  $w$  is said to be in  $A_p(\mathbb{R}_+^d)$  with  $1 < p < \infty$  when

$$\sup_B \left( \frac{1}{|B|} \int_B w(x) dx \right) \left( \frac{1}{|B|} \int_B w^{-1/(p-1)}(x) dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls  $B$  in  $\mathbb{R}_+^d$ .

If  $1 < q < \infty$ , we say that a weight  $w$  belongs to the class  $\text{RH}_q(\mathbb{R}_+^d)$  if there exists  $C > 0$  such that

$$\left( \frac{1}{|B|} \int_B w^q(x) dx \right)^{1/q} \leq C \frac{1}{|B|} \int_B w(x) dx$$

for every ball  $B$  in  $\mathbb{R}_+^d$ .

The  $p$ -Lebesgue space with weight  $w$  is denoted by  $L^p(\mathbb{R}_+^d, w)$ .

**Theorem 1.2.** *Let  $0 < \alpha \leq 2$ ,  $\rho > 2$ ,  $\lambda \geq 0$ ,  $k \in \mathbb{N}$  and  $1 < p < \infty$ . Then, the  $\rho$ -variation operator  $\mathcal{V}_\rho(\{t^k \partial_t^k W_{\lambda, t}^\alpha\}_{t>0})$  is bounded on  $L^p(\mathbb{R}_+^d, w)$  provided that  $w \in A_p(\mathbb{R}_+^d)$ .*

Throughout this paper  $c$  and  $C$  will always represent positive constants that may vary on each occurrence. The expression  $a \lesssim b$  will indicate that  $a \leq Cb$  for some positive constant  $C$ , whilst  $a \sim b$  means that both  $a \lesssim b$  and  $b \lesssim a$  hold.

## 2. HEAT KERNEL ESTIMATES

Our goal in this section is to establish estimates for the time derivatives of the heat kernel  $W_{\lambda, t}^\alpha$  that will be useful in the sequel.

We define the exponent  $\mathfrak{p}$  as in [22]. We consider

$$M_\alpha = \begin{cases} \alpha & \text{if } \alpha \in (0, 2), \\ \infty & \text{if } \alpha = 2, \end{cases}$$

and, for every  $\alpha \in (0, 2]$ , we define

$$\Omega_\alpha : (-1, M_\alpha) \longrightarrow \mathbb{R}$$

$$s \longmapsto \Omega_\alpha(s) = \frac{1}{\pi} \left( \Gamma(\alpha) \sin \frac{\pi\alpha}{2} + \Gamma(1+s)\Gamma(\alpha-s) \sin \frac{\pi(2s-\alpha)}{2} \right).$$

We understand  $\Omega_2(s) = s(s-1)$ ,  $s \in (-1, \infty)$ , and  $\Omega_1(s) = \frac{1}{\pi}(1 - \pi s \cot(\pi s))$ ,  $s \in (-1, 1)$ . The main properties of the function  $\Omega_\alpha$  can be found in [22, Appendix A]. For every  $\lambda \geq \lambda_*$  there exists a unique  $\mathfrak{p} \in [\frac{\alpha-1}{2}, M_\alpha)$  such that  $\Omega_\alpha(\mathfrak{p}) = \lambda$ . Note that  $\mathfrak{p}$  depends on  $\lambda$  and  $\alpha$  but we write  $\mathfrak{p}$  instead of  $\mathfrak{p}(\lambda, \alpha)$ . We have that

- (a)  $\mathfrak{p} = \max\{\alpha - 1, 0\} := (\alpha - 1)_+$ , when  $\lambda = 0$ ;
- (b)  $\mathfrak{p} > (\alpha - 1)_+$ , when  $\lambda > 0$ ;
- (c)  $\mathfrak{p} = \frac{1}{2}(1 + \sqrt{1 + 4\lambda})$ , when  $\alpha = 2$ .

For simplicity, we shall denote with  $\gamma = (\alpha - 1)_+$  for any  $\alpha \in (0, 2]$ .

The following pointwise bounds for the heat kernel  $W_{\lambda,t}^\alpha$ ,  $t > 0$ , were established in [22, Theorem 9 and Theorem 10].

**Proposition 2.1.**

- (a) Let  $\alpha \in (0, 2)$  and  $\lambda \geq 0$ . Then, for every  $x, y \in \mathbb{R}_+^d$  and  $t > 0$ ,

$$W_{\lambda,t}^\alpha(x, y) \sim \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^\mathfrak{p} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^\mathfrak{p} t^{-d/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}.$$

- (b) Let  $\lambda \geq -1/4$ . Then, for every  $x, y \in \mathbb{R}_+^d$  and  $t > 0$ ,

$$W_{\lambda,t}^\alpha(x, y) \asymp \left(1 \wedge \frac{x_d}{t^{1/2}}\right)^\mathfrak{p} \left(1 \wedge \frac{y_d}{t^{1/2}}\right)^\mathfrak{p} t^{-d/2} e^{-\frac{c|x-y|^2}{t}}.$$

Here the symbol  $\asymp$  means the same as  $\sim$  but admitting different values of  $c$  in the exponential function in the upper and lower bounds.

Our next objective is to extend the upper bounds in Proposition 2.1 to the complex plane. Note first that from (a) we deduce that

$$W_{\lambda,t}^\alpha(x, y) \lesssim \left(\frac{x_d}{x_d + t^{1/\alpha}}\right)^\mathfrak{p} \left(\frac{y_d}{y_d + t^{1/\alpha}}\right)^\mathfrak{p} t^{-d/\alpha} \left(\frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|}\right)^{d+\alpha}, \quad (2.1)$$

for every  $x, y \in \mathbb{R}_+^d$  and  $t > 0$ .

Since the function  $\phi(s) = \frac{s}{s+a}$ ,  $s > -a$ , is increasing for every  $a > 0$ , it follows that

$$W_{\lambda,t}^\alpha(x, y) \lesssim \left(\frac{|x|}{|x| + t^{1/\alpha}}\right)^\mathfrak{p} \left(\frac{|y|}{|y| + t^{1/\alpha}}\right)^\mathfrak{p} t^{-d/\alpha} \left(\frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|}\right)^{d+\alpha}, \quad (2.2)$$

for every  $x, y \in \mathbb{R}_+^d$  and  $t > 0$ .

By using [13, Proposition 2.2(a)] we obtain that if  $\alpha \in (0, 2)$ ,  $\lambda \geq 0$  and  $\epsilon \in (0, 1)$ , for every  $z \in \mathbb{C}$ ,  $|\arg(z)| < \epsilon\pi/4$ , and  $x, y \in \mathbb{R}_+^d$ ,

$$W_{\lambda,z}^\alpha(x, y) \lesssim \left(\frac{|x|}{|x| + |z|^{1/\alpha}}\right)^\mathfrak{p} \left(\frac{|y|}{|y| + |z|^{1/\alpha}}\right)^\mathfrak{p} t^{-d/\alpha} \left(\frac{|z|^{1/\alpha}}{|z|^{1/\alpha} + |x-y|}\right)^{(d+\alpha)(1-\epsilon)}.$$

By using the Cauchy integral formula we deduce that, if  $\alpha \in (0, 2)$ ,  $\lambda \geq 0$  and  $k \in \mathbb{N}$ ,

$$|t^k \partial_t^k W_{\lambda,t}^\alpha(x, y)| \lesssim \left(\frac{|x|}{|x| + t^{1/\alpha}}\right)^\mathfrak{p} \left(\frac{|y|}{|y| + t^{1/\alpha}}\right)^\mathfrak{p} t^{-d/\alpha} \left(\frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|}\right)^{(d+\alpha)(1-\epsilon)},$$

for every  $x, y \in \mathbb{R}_+^d$  and  $t > 0$ .

In a similar way we get that if  $\alpha = 2$ ,  $\lambda \geq -1/4$  and  $k \in \mathbb{N}$

$$|t^k \partial_t^k W_{\lambda,t}^\alpha(x, y)| \lesssim \left(\frac{|x|}{|x| + t^{1/2}}\right)^\mathfrak{p} \left(\frac{|y|}{|y| + t^{1/2}}\right)^\mathfrak{p} t^{-d/2} e^{-\frac{c|x-y|^2}{t}} \quad (2.3)$$

for every  $x, y \in \mathbb{R}_+^d$  and  $t > 0$ .

However we need to improve (2.2) and (2.3) by replacing  $|x|$  and  $|y|$  by  $x_d$  and  $y_d$  respectively.

**Proposition 2.2.**

(a) Let  $\alpha \in (0, 2)$ ,  $\lambda \geq 0$ ,  $\epsilon \in (0, 1)$  and  $k \in \mathbb{N}$ . For every  $x, y \in \mathbb{R}_+^d$  and  $t > 0$

$$|t^k \partial_t^k W_{\lambda,t}^\alpha(x, y)| \lesssim \left( \frac{x_d}{x_d + t^{1/\alpha}} \right)^p \left( \frac{y_d}{y_d + t^{1/\alpha}} \right)^p t^{-d/\alpha} \left( \frac{t^{1/\alpha}}{t^{1/\alpha} + |x - y|} \right)^{(d+\alpha)(1-\epsilon)}.$$

(b) Let  $\lambda \geq -1/4$  and  $k \in \mathbb{N}$ . For every  $x, y \in \mathbb{R}_+^d$  and  $t > 0$ ,

$$|t^k \partial_t^k W_{\lambda,t}^2(x, y)| \lesssim \left( \frac{x_d}{x_d + t^{1/2}} \right)^p \left( \frac{y_d}{y_d + t^{1/2}} \right)^p t^{-d/2} e^{-\frac{c|x-y|^2}{t}}.$$

*Proof.* We prove (a) first. According to (2.2) and proceeding as in the proof of [12, Proposition 3.4] (see also [13, Proposition 2.2(a)]) we deduce that

$$|w(x_d, z) W_{\lambda,z}^\alpha(x, y) w(y_d, z)| \leq \frac{C}{|z|^{d/\alpha}}, \quad (2.4)$$

for  $x, y \in \mathbb{R}_+^d$  and  $z \in \mathbb{C}$ ,  $|\arg(z)| \leq \pi/4$ , where

$$w(s, z) = \left( \frac{s}{s + z^{1/\alpha}} \right)^{-p},$$

for  $s > 0$  and  $z \in \mathbb{C}$ ,  $\arg(z) \leq \pi/2$ . Indeed, since the operator  $L_\lambda^\alpha$  is self-adjoint and non-negative, for every  $s > 0$ , the operator  $W_{\lambda, is}^\alpha$  is contractive in  $L^2(\mathbb{R}_+^d)$ . In order to prove our claim it is sufficient to see that

$$\|W_{\lambda,t}^\alpha\|_{L^1(\mathbb{R}_+^d, w^{-1}) \hookrightarrow L^2(\mathbb{R}_+^d)} + \|W_{\lambda,t}^\alpha\|_{L^2(\mathbb{R}_+^d) \hookrightarrow wL^\infty(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2\alpha}}, \quad t > 0,$$

where  $wL^\infty(\mathbb{R}^d) = \{f \text{ measurable in } \mathbb{R}^d : wf \in L^\infty(\mathbb{R}^d)\}$ .

Suppose  $f \in L^2(\mathbb{R}_+^d)$ . According to (2.1) and using [12, equation (18)] we get

$$\begin{aligned} |W_{\lambda,t}^\alpha(f)(x)| w(x_d, t) &\lesssim \int_{\mathbb{R}_+^d} \left( \frac{y_d}{y_d + t^{1/\alpha}} \right)^p \frac{t}{t^{1/\alpha} + |x - y|^{d+\alpha}} |f(y)| dy \\ &\lesssim \left( \int_{\mathbb{R}_+^d} \left| \left( \frac{|y|}{|y| + t^{1/\alpha}} \right)^p \frac{t}{t^{1/\alpha} + |x - y|^{d+\alpha}} \right|^2 dy \right)^{1/2} \|f\|_{L^2(\mathbb{R}_+^d)} \\ &\lesssim t^{-\frac{d}{2\alpha}} \|f\|_{L^2(\mathbb{R}_+^d)}, \end{aligned}$$

for  $x \in \mathbb{R}_+^d$  and  $t > 0$ .

In a similar way we can see that

$$\|W_{\lambda,t}^\alpha f\|_{L^2(\mathbb{R}_+^d)} \lesssim t^{-\frac{d}{2\alpha}} \|f\|_{L^1(\mathbb{R}_+^d, w^{-1})},$$

for  $f \in L^1(\mathbb{R}_+^d, w^{-1})$  and  $t > 0$ . We conclude that (2.4) holds.

On the other hand, according to (2.1) it follows that

$$\begin{aligned} |w(x_d, t) W_{\lambda,t}^\alpha(x, y) w(y_d, t)| &\lesssim t^{-d/\alpha} \left( \frac{t^{1/\alpha}}{t^{1/\alpha} + |x - y|} \right)^{d+\alpha} \\ &\lesssim t^{-d/\alpha} \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{-d-\alpha} \\ &\lesssim t^{-d/\alpha} \left( 1 + \left( \frac{|x - y|^\alpha}{t} \right)^{1/\alpha} \right)^{-d-\alpha}, \end{aligned}$$

for  $x, y \in \mathbb{R}^d$  and  $t > 0$ .

Let us define the function  $b(x) = (1 + s^{1/\alpha})^{-d-\alpha}$ ,  $s > 0$ . It is clear that  $b$  is bounded and decreasing in  $(0, \infty)$ . Then, by [21, Proposition 3.3], for each  $\epsilon \in (0, 1)$  and  $\theta \in (0, \epsilon\pi/2)$  there exists  $C > 0$  such that

$$|W_{\lambda,z}^\alpha(x, y)| \leq C \left| \frac{x_d}{x_d + z} \right|^p \left| \frac{y_d}{y_d + z} \right|^p (\operatorname{Re}(z))^{-d/\alpha} b \left( \frac{|x - y|^\alpha}{|z|} \right)^{1-\epsilon},$$

for  $x, y \in (0, \infty)$  and  $z \in \mathbb{C}$  such that  $|\arg(z)| \leq \theta$ .

If  $z \in \mathbb{C}$  and  $|\arg(z)| \leq \theta$ , with  $0 < \theta < \pi/2$ , then

$$|a + z| = \sqrt{(a + \operatorname{Re}(z))^2 + (\operatorname{Im}(z))^2} \geq \sqrt{a^2 + |z|^2} \gtrsim a + |z|, \quad a > 0.$$

We conclude that for each  $\epsilon \in (0, 1)$  and  $\theta \in (0, \epsilon\pi/2)$

$$|W_{\lambda,z}^\alpha(x, y)| \lesssim \left( \frac{x_d}{x_d + |z|} \right)^p \left( \frac{y_d}{y_d + |z|} \right)^p |z|^{-d/\alpha} \left( \frac{|z|^{1/\alpha}}{|z|^{1/\alpha} + |x - y|} \right)^{(d+\alpha)(1-\epsilon)}, \quad (2.5)$$

where  $z \in \mathbb{C}$  and  $|\arg(z)| \leq \theta$ . Note that  $\operatorname{Re}(z) \simeq |z|$  for  $z \in \mathbb{C}$  with  $|\arg(z)| \leq \theta$ .

Estimates like (2.5) can also be proved by using the methods in [35, Theorem 2.1].

The Cauchy integral formula allow us to obtain (a). In order to prove (b) we can proceed in a similar way. We can also prove it using Davies method as in the proof of [18, Theorem 3.4.8] and the Cauchy integral formula.  $\square$

### 3. PROOF OF THEOREM 1.1 FOR $\alpha \in (0, 2)$

**3.1. Global part.** We define, for  $k \in \mathbb{N}$ ,  $k \geq 1$ ,

$$J_\lambda^{\alpha,k}(x, y) = \int_0^\infty |t^{k-1} \partial_t^k (W_{\lambda,t}^\alpha(x, y) - W_{0,t}^\alpha(x, y))| \chi_G(x, y, t) dt, \quad x, y \in \mathbb{R}_+^d, \quad (3.1)$$

where  $G$  is as in (1.2).

By using Proposition 2.2 (a) we get

$$\begin{aligned} |J_\lambda^{\alpha,k}(x, y)| &\leq \int_0^\infty (|t^{k-1} \partial_t^k W_{\lambda,t}^\alpha(x, y)| + |t^{k-1} \partial_t^k W_{0,t}^\alpha(x, y)|) \chi_G(x, y, t) dt \\ &\lesssim \int_0^\infty \frac{1}{t^{1+d/\alpha}} \left( \frac{x_d}{x_d + t^{1/\alpha}} \right)^\gamma \left( \frac{y_d}{y_d + t^{1/\alpha}} \right)^\gamma \\ &\quad \times \left( \frac{t^{1/\alpha}}{t^{1/\alpha} + |x - y|} \right)^{(d+\alpha)(1-\epsilon)} \chi_G(x, y, t) dt, \quad x, y \in \mathbb{R}_+^d, \end{aligned}$$

where  $0 < \epsilon < 1$  and, as we set before,  $\gamma = (\alpha - 1)_+$ . Here, we can actually use that the power  $p$  can be replaced by any non-negative number less or equal than  $p$  in the estimate given in Proposition 2.2 (a).

We shall split the last integral over  $G_1$  and  $G_2$ , and denote

$$\begin{aligned} S_{\epsilon,j}^\alpha(x, y) &= \int_0^\infty \frac{1}{t^{1+d/\alpha}} \left( \frac{x_d}{x_d + t^{1/\alpha}} \right)^\gamma \left( \frac{y_d}{y_d + t^{1/\alpha}} \right)^\gamma \\ &\quad \times \left( \frac{t^{1/\alpha}}{t^{1/\alpha} + |x - y|} \right)^{(d+\alpha)(1-\epsilon)} \chi_{G_j}(x, y, t) dt, \quad j = 1, 2. \end{aligned}$$

From the definition of  $G_1$  we can write

$$\begin{aligned} S_{\epsilon,1}^\alpha(x, y) &= \int_{(x_d \vee y_d)^\alpha}^\infty \left( \frac{x_d}{x_d + t^{1/\alpha}} \right)^\gamma \left( \frac{y_d}{y_d + t^{1/\alpha}} \right)^\gamma \left( \frac{t^{1/\alpha}}{t^{1/\alpha} + |x - y|} \right)^{(d+\alpha)(1-\epsilon)} \frac{dt}{t^{1+d/\alpha}} \end{aligned}$$



If we suppose that  $x, y \in \mathbb{R}_+^d$  with  $|x - y| \leq x_d \vee y_d$ , we get

$$S_{\epsilon,1}^\alpha(x, y) \lesssim \int_{(x_d \vee y_d)^\alpha}^\infty \frac{(x_d y_d)^\gamma}{t^{1+d/\alpha+2q/\alpha}} dt \leq C \frac{(x_d y_d)^\gamma}{(x_d \vee y_d)^{d+2q}}, \quad 0 < \epsilon < 1.$$

On the other hand, if  $|x - y| > x_d \vee y_d$ ,

$$\begin{aligned} S_{\epsilon,1}^\alpha(x, y) &= \int_{(x_d \vee y_d)^\alpha}^{|x-y|^\alpha} \left( \frac{x_d}{x_d + t^{1/\alpha}} \right)^\gamma \left( \frac{y_d}{y_d + t^{1/\alpha}} \right)^\gamma \left( \frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|} \right)^{(d+\alpha)(1-\epsilon)} \frac{dt}{t^{1+d/\alpha}} \\ &\quad + \int_{|x-y|^\alpha}^\infty \left( \frac{x_d}{x_d + t^{1/\alpha}} \right)^\gamma \left( \frac{y_d}{y_d + t^{1/\alpha}} \right)^\gamma \left( \frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|} \right)^{(d+\alpha)(1-\epsilon)} \frac{dt}{t^{1+d/\alpha}} \\ &\lesssim \int_0^{|x-y|^\alpha} \frac{(x_d y_d)^\gamma}{t^{1+d/\alpha+2q/\alpha}} \left( \frac{t^{1/\alpha}}{|x-y|} \right)^{(d+\alpha)(1-\epsilon)} dt + \int_{|x-y|^\alpha}^\infty \frac{(x_d y_d)^\gamma}{t^{1+d/\alpha+2q/\alpha}} dt \\ &\lesssim (x_d y_d)^\gamma \left( \frac{1}{|x-y|^{(d+\alpha)(1-\epsilon)}} \int_0^{|x-y|^\alpha} t^{\frac{(d+\alpha)(1-\epsilon)}{\alpha} - 1 - \frac{d}{\alpha} - \frac{2q}{\alpha}} dt \right. \\ &\quad \left. + \int_{|x-y|^\alpha}^\infty \frac{dt}{t^{1+d/\alpha+2q/\alpha}} \right) \\ &\lesssim \frac{(x_d y_d)^\gamma}{|x-y|^{d+2q}}, \end{aligned}$$

provided that  $(d+\alpha)(1-\epsilon) - d - 2\gamma > 0$ , that is, whenever  $0 < \epsilon < (\alpha - 2\gamma)/(d+\alpha)$ . Note that this is possible since  $\gamma = (\alpha - 1)_+$  implies  $\alpha - 2\gamma > 0$ .

We conclude that

$$S_{\epsilon,1}^\alpha(x, y) \lesssim \frac{(x_d y_d)^\gamma}{(|x-y| \vee x_d \vee y_d)^{d+2\gamma}}, \quad x, y \in \mathbb{R}_+^d$$

when  $0 < \epsilon < (\alpha - 2\gamma)/(d+\alpha)$ .

For  $j = 2$ , we take into account that  $|x - y| \geq (x_d \wedge y_d)/2$ , so we have

$$\begin{aligned} S_{\epsilon,2}^\alpha(x, y) &\lesssim \int_0^{(x_d \vee y_d)^\alpha} \left( \frac{x_d \wedge y_d}{x_d \wedge y_d + t^{1/\alpha}} \right)^\gamma \left( \frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|} \right)^{(d+\alpha)(1-\epsilon)} \frac{dt}{t^{1+d/\alpha}} \\ &= \int_0^{(x_d \wedge y_d)^\alpha} \left( \frac{x_d \wedge y_d}{x_d \wedge y_d + t^{1/\alpha}} \right)^\gamma \left( \frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|} \right)^{(d+\alpha)(1-\epsilon)} \frac{dt}{t^{1+d/\alpha}} \\ &\quad + \int_{(x_d \wedge y_d)^\alpha}^{(x_d \vee y_d)^\alpha} \left( \frac{x_d \wedge y_d}{x_d \wedge y_d + t^{1/\alpha}} \right)^\gamma \left( \frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|} \right)^{(d+\alpha)(1-\epsilon)} \frac{dt}{t^{1+d/\alpha}} \\ &\lesssim \int_0^{(x_d \wedge y_d)^\alpha} \left( \frac{t^{1/\alpha}}{|x-y|} \right)^{(d+\alpha)(1-\epsilon)} \frac{dt}{t^{1+d/\alpha}} \\ &\quad + \int_{(x_d \wedge y_d)^\alpha}^{(x_d \vee y_d)^\alpha} \left( \frac{x_d \wedge y_d}{t^{1/\alpha}} \right)^\gamma \left( \frac{t^{1/\alpha}}{|x-y|} \right)^{(d+\alpha)(1-\epsilon)} \frac{dt}{t^{1+d/\alpha}} \\ &\lesssim \int_0^{(x_d \wedge y_d)^\alpha} \frac{t^{(d+\alpha)(1-\epsilon)/\alpha - 1 - d/\alpha}}{|x-y|^{(d+\alpha)(1-\epsilon)}} dt \\ &\quad + (x_d \wedge y_d)^\gamma \int_0^{(x_d \vee y_d)^\alpha} \frac{t^{\frac{(d+\alpha)(1-\epsilon)}{\alpha} - 1 - \frac{d}{\alpha} - \frac{\gamma}{\alpha}}}{|x-y|^{(d+\alpha)(1-\epsilon)}} dt \\ &\lesssim \frac{(x_d \wedge y_d)^{(d+\alpha)(1-\epsilon) - d}}{|x-y|^{(d+\alpha)(1-\epsilon)}} + \frac{(x_d \wedge y_d)^\gamma (x_d \vee y_d)^{(d+\alpha)(1-\epsilon) - d - \gamma}}{|x-y|^{(d+\alpha)(1-\epsilon)}} \end{aligned}$$

$$\lesssim \frac{1}{|x-y|^{(d+\alpha)(1-\epsilon)}} (x_d \vee y_d)^{(d+\alpha)(1-\epsilon)-d-\gamma} (x_d \wedge y_d)^\gamma$$

provided that  $(d+\alpha)(1-\epsilon)-d-\gamma > 0$ , i.e.,  $0 < \epsilon < (\alpha-\gamma)/(d+\alpha)$ . Actually, this is possible since the restriction obtained in the case  $j=1$  implies this one.

Recall that the above estimates for  $S_{\epsilon,j}^\alpha$ ,  $j=1,2$ , are valid for any non-negative power less or equal to  $\mathfrak{p}=\gamma$ ; in particular, for power zero. Thus, in this case we get

$$\left| J_\lambda^{\alpha,k}(x,y) \right| \lesssim \frac{1}{(|x-y| \vee x_d \vee y_d)^d} + \chi_{\{|x-y| \geq \frac{x_d \wedge y_d}{2}\}}(x,y) \frac{(x_d \vee y_d)^{(d+\alpha)(1-\epsilon)-d}}{|x-y|^{(d+\alpha)(1-\epsilon)}}$$

whenever  $0 < \epsilon < \alpha/(d+\alpha)$ .

Fix now  $0 < \epsilon < \alpha/(d+\alpha)$ . We define now  $\alpha' = (d+\alpha)(1-\epsilon)-d$ . Hence (see [22, p. 31]),

$$\frac{1}{(|x-y| \vee x_d \vee y_d)^d} \lesssim \frac{(|x-y| \vee x_d \vee y_d)^{\alpha'}}{(|x-y| \vee (x_d \wedge y_d))^{d+\alpha'}}$$

and

$$\frac{(x_d \vee y_d)^{\alpha'}}{|x-y|^{d+\alpha'}} \chi_{\{|x-y| \geq \frac{x_d \wedge y_d}{2}\}}(x,y) \lesssim \frac{(|x-y| \vee x_d \vee y_d)^{\alpha'}}{(|x-y| \vee (x_d \wedge y_d))^{d+\alpha'}}$$

Therefore, by proceeding as in the proof of [22, Proposition 19, Step 1], we deduce that

$$\int_{\mathbb{R}^{d-1}} \left| J_\lambda^{\alpha,k}(x,y) \right| dy' \lesssim \frac{(x_d \vee y_d)^{\alpha'}}{(|x_d - y_d| \vee (x_d \wedge y_d))^{\alpha'+1}},$$

being  $y' = (y_1, \dots, y_{d-1})$ .

By considering  $w_\beta(x,y) = \left(\frac{x_d}{y_d}\right)^\beta$ ,  $x,y \in \mathbb{R}_+^d$  and  $\beta > 0$ , we get, as in [22, Proposition 19, Step 2], that for any  $1 < p < \infty$ ,

$$\begin{aligned} \sup_{x \in \mathbb{R}_+^d} \int_{\mathbb{R}_+^d} w_\beta(x,y)^{1/p} \left| J_\lambda^{\alpha,k}(x,y) \right| dy &< \infty, \quad 0 < \beta/p < 1, \\ \sup_{x \in \mathbb{R}_+^d} \int_{\mathbb{R}_+^d} w_\beta(x,y)^{-1/p'} \left| J_\lambda^{\alpha,k}(x,y) \right| dx &< \infty, \quad 0 < \beta/p' < 1. \end{aligned}$$

According to Schur's test with weights (see, for instance, [25, Lemma 3.3]), we can conclude that the integral operator defined by

$$J_\lambda^{\alpha,k}(f)(x) = \int_{\mathbb{R}_+^d} J_\lambda^{\alpha,k}(x,y) f(y) dy, \quad x \in \mathbb{R}_+^d,$$

is bounded on  $L^p(\mathbb{R}_+^d)$  for every  $1 < p < \infty$ .

**3.2. Local part.** According to Duhamel formula we can write, for every  $x,y \in \mathbb{R}_+^d$  and  $t > 0$ , that

$$\begin{aligned} D_{\lambda,t}^{\alpha,0}(x,y) &:= \lambda \int_0^t \int_{\mathbb{R}_+^d} W_{0,t-s}^\alpha(x,z) z_d^{-\alpha} W_{\lambda,s}^\alpha(z,y) dz ds \\ &= \lambda \int_0^{t/2} \int_{\mathbb{R}_+^d} W_{0,t-s}^\alpha(x,z) z_d^{-\alpha} W_{\lambda,s}^\alpha(z,y) dz ds \\ &\quad + \lambda \int_0^{t/2} \int_{\mathbb{R}_+^d} W_{0,s}^\alpha(x,z) z_d^{-\alpha} W_{\lambda,t-s}^\alpha(z,y) dz ds. \end{aligned} \tag{3.2}$$

We have that  $D_{\lambda,t}^{\alpha,0}(x,y) = t^{-d/\alpha} F(t^{-1/\alpha}x, t^{-1/\alpha}y)$ ,  $x, y \in \mathbb{R}_+^d$ ,  $t > 0$  (see comments after [22, Theorem 9]) for certain smooth function  $F : \mathbb{R}_+^d \times \mathbb{R}_+^d \rightarrow \mathbb{R}$ . Thus, for every  $k \in \mathbb{N}$ , there exists an smooth function  $F_k : \mathbb{R}_+^d \times \mathbb{R}_+^d \rightarrow \mathbb{R}$  for which

$$t^k \partial_t^k D_{\lambda,t}^{\alpha,0}(x,y) = t^{-d/\alpha} F_k(t^{-1/\alpha}x, t^{-1/\alpha}y), \quad x, y \in \mathbb{R}_+^d, t > 0. \quad (3.3)$$

Given  $k \in \mathbb{N}$ , from (3.2) we can write

$$\begin{aligned} \partial_t^k D_{\lambda,t}^{\alpha,0}(x,y) &= \sum_{j=0}^{k-1} a_{j,k} \int_{\mathbb{R}_+^d} \partial_t^j W_{0,t/2}^\alpha(x,z) z_d^{-\alpha} \partial_t^{k-1-j} W_{\lambda,t/2}^\alpha(z,y) dz \\ &\quad + \lambda \int_0^{t/2} \int_{\mathbb{R}_+^d} \partial_t^k W_{0,t-s}^\alpha(x,z) z_d^{-\alpha} W_{\lambda,s}^\alpha(z,y) dz ds \\ &\quad + \lambda \int_0^{t/2} \int_{\mathbb{R}_+^d} W_{0,s}^\alpha(x,z) z_d^{-\alpha} \partial_t^k W_{\lambda,t-s}^\alpha(z,y) dz ds, \quad x, y \in \mathbb{R}_+^d, t > 0, \end{aligned}$$

where  $a_{j,k} \in \mathbb{R}$  for each  $j = 1, \dots, k-1$ .

Assume now that  $x, y \in \mathbb{R}_+^d$  with  $|x-y| \leq (x_d \wedge y_d)/2$  and  $0 < t^{1/\alpha} \leq x_d \vee y_d$ . Taking into account the scaling property (3.3), we are going to estimate

$$\partial_t^k D_{\lambda,t}^{\alpha,0}(x,y) \Big|_{t=1}.$$

We define, for every  $j = 1, \dots, k-1$ , the integrals involved in the above summation as

$$H_j(x,y,t) = \int_{\mathbb{R}_+^d} \partial_t^j W_{0,t/2}^\alpha(x,z) z_d^{-\alpha} \partial_t^{k-1-j} W_{\lambda,t/2}^\alpha(z,y) dz, \quad x, y \in \mathbb{R}_+^d, t > 0,$$

and we decompose each one of them as follows

$$\begin{aligned} H_j(x,y,t) &= \int_{\{z \in \mathbb{R}_+^d : z_d > x_d/2\}} \partial_t^j W_{0,t/2}^\alpha(x,z) z_d^{-\alpha} \partial_t^{k-1-j} W_{\lambda,t/2}^\alpha(z,y) dz \\ &\quad + \int_{\{z \in \mathbb{R}_+^d : z_d \leq x_d/2\}} \partial_t^j W_{0,t/2}^\alpha(x,z) z_d^{-\alpha} \partial_t^{k-1-j} W_{\lambda,t/2}^\alpha(z,y) dz \\ &:= H_{j,1}(x,y,t) + H_{j,2}(x,y,t), \quad x, y \in \mathbb{R}_+^d, t > 0. \end{aligned}$$

We notice first that, since  $x_d \sim y_d$  (because  $|x-y| \leq (x_d \wedge y_d)/2$ ), when  $t = 1$  we have  $x_d \sim y_d \gtrsim 1$ .

By using Proposition 2.2(a), for  $\epsilon \in (0, 1)$  we have

$$\begin{aligned} |H_{j,1}(x,y,1)| &\lesssim x_d^{-\alpha} \int_{\mathbb{R}_+^d} \left( \frac{x_d}{x_d+1} \right)^{(\alpha-1)_+} \left( \frac{z_d}{z_d+1} \right)^{(\alpha-1)_+} \left( \frac{1}{1+|x-z|} \right)^{(d+\alpha)(1-\epsilon)} \\ &\quad \times \left( \frac{y_d}{y_d+1} \right)^{(\alpha-1)_+} \left( \frac{z_d}{z_d+1} \right)^{(\alpha-1)_+} \left( \frac{1}{1+|y-z|} \right)^{(d+\alpha)(1-\epsilon)} dz. \end{aligned}$$

We define  $\alpha' = \alpha(1-\epsilon) - d\epsilon$  and consider  $0 < \epsilon < \alpha/(\alpha+d)$ . Then,  $0 < \alpha' < \alpha < 2$ , and we get

$$\begin{aligned} |H_{j,1}(x,y,1)| &\lesssim x_d^{-\alpha} \int_{\mathbb{R}_+^d} \left( \frac{x_d}{x_d+1} \right)^{(\alpha'-1)_+} \left( \frac{z_d}{z_d+1} \right)^{2(\alpha'-1)_+} \left( \frac{y_d}{y_d+1} \right)^{(\alpha'-1)} \\ &\quad \times \left( \frac{1}{1+|x-z|} \right)^{d+\alpha'} \left( \frac{1}{1+|y-z|} \right)^{d+\alpha'} dz. \end{aligned}$$

According to [22, Theorem 9], we have

$$\begin{aligned} |H_{j,1}(x, y, 1)| &\lesssim x_d^{-\alpha} \int_{\mathbb{R}_+^d} W_{0,1}^{\alpha'}(x, z) W_{0,1}^{\alpha'}(z, y) dz \\ &= C x_d^{-\alpha} W_{0,2}^{\alpha'}(x, y) \\ &\lesssim x_d^{-\alpha} \left( \frac{x_d}{x_d + 1} \right)^{(\alpha'-1)_+} \left( \frac{y_d}{y_d + 1} \right)^{(\alpha'-1)} \left( \frac{1}{1 + |x - y|} \right)^{d+\alpha'}. \end{aligned}$$

We now estimate  $H_{j,2}$ . If  $z_d \leq x_d/2$ , since  $|x - y| < (x_d \wedge y_d)/2$ , we deduce that  $|y_d - z_d| \sim y_d \sim x_d$  and  $|x_d - z_d| \sim x_d$ . Then, if  $z_d \leq x_d/2$ , we have  $|x - z| \sim |x' - z'| + x_d$  and  $|y - z| \sim |y' - z'| + x_d$ . By [22, Lemma 22] with  $N = d - 1 \geq 1$ ,  $\beta = \alpha(1 - \epsilon) + 1 - d\epsilon$  and  $r = s = 1 + x_d$  we get that whenever  $z_d \leq x_d/2$ ,

$$\begin{aligned} &\int_{\mathbb{R}^{d-1}} \left| \partial_t^j W_{0,t/2}^\alpha(x, z) \Big|_{t=1} \right| \left| \partial_t^{k-1-j} W_{\lambda,t/2}^\alpha(z, y) \Big|_{t=1} \right| dz' \\ &\lesssim \left( \frac{x_d}{x_d + 1} \right)^{(\alpha-1)_+} \left( \frac{z_d}{z_d + 1} \right)^{(\alpha-1)_++p} \left( \frac{y_d}{y_d + 1} \right)^p \\ &\quad \times \int_{\mathbb{R}^{d-1}} \left( \frac{1}{1 + |x' - z'| + x_d} \right)^{(d+\alpha)(1-\epsilon)} \left( \frac{1}{1 + |y' - z'| + x_d} \right)^{(d+\alpha)(1-\epsilon)} dz' \\ &\lesssim \left( \frac{x_d}{x_d + 1} \right)^{(\alpha-1)_+} \left( \frac{z_d}{z_d + 1} \right)^{(\alpha-1)_++p} \left( \frac{y_d}{y_d + 1} \right)^p \frac{1}{x_d^{\alpha(1-\epsilon)+1-d\epsilon}} \\ &\quad \times \frac{1}{(1 + x_d + |y' - x'|)^{(d+\alpha)(1-\epsilon)}} \\ &\lesssim \left( \frac{x_d}{x_d + 1} \right)^{(\alpha-1)_+} \left( \frac{z_d}{z_d + 1} \right)^{(\alpha-1)_++p} \left( \frac{y_d}{y_d + 1} \right)^p \frac{1}{x_d^{\alpha(1-\epsilon)+1-d\epsilon+(d+\alpha)(1-\epsilon)}}. \end{aligned}$$

From [22, p. 26] we also have that

$$\int_0^{x_d/2} \left( \frac{z_d}{z_d + 1} \right)^{(\alpha-1)_++p} \frac{dz_d}{z_d^\alpha} \sim \mathbf{1}_{\alpha \geq 1} + (\ln(1 + x_d)) \mathbf{1}_{\alpha=1} + x_d^{1-\alpha} \mathbf{1}_{\alpha < 1}.$$

Then,

$$\begin{aligned} &\frac{1}{x_d^{\alpha(1-\epsilon)+1-d\epsilon+(d+\alpha)(1-\epsilon)}} \int_0^{x_d/2} \left( \frac{z_d}{z_d + 1} \right)^{(\alpha-1)_++p} \frac{dz_d}{z_d^\alpha} \\ &\lesssim \frac{1}{x_d^{\alpha+(d+\alpha)(1-\epsilon)}} \frac{1}{x_d^{1-(d+\alpha)\epsilon}} (\mathbf{1}_{\alpha \geq 1} + (\ln(1 + x_d)) \mathbf{1}_{\alpha=1} + x_d^{1-\alpha} \mathbf{1}_{\alpha < 1}) \\ &\lesssim \frac{1}{x_d^{\alpha+(d+\alpha)(1-\epsilon)}} \lesssim \frac{1}{x_d^\alpha} \left( \frac{1}{1 + |x - y|} \right)^{(d+\alpha)(1-\epsilon)}, \end{aligned}$$

provided that  $0 < \epsilon < \alpha/(d + \alpha)$  and  $x_d \geq 1$ .

We consider now

$$\begin{aligned} H_k(x, y, t) &= \int_0^{t/2} \int_{\{z \in \mathbb{R}_+^d : z_d > x_d/2\}} \partial_t^k W_{0,t-s}^\alpha(x, z) z_d^{-\alpha} W_{\lambda,s}^\alpha(z, y) dz ds \\ &\quad + \int_0^{t/2} \int_{\{z \in \mathbb{R}_+^d : z_d \leq x_d/2\}} \partial_t^k W_{0,t-s}^\alpha(x, z) z_d^{-\alpha} W_{\lambda,s}^\alpha(z, y) dz ds \\ &:= H_{k,1}(x, y, t) + H_{k,2}(x, y, t), \quad x, y \in \mathbb{R}_+^d, t > 0. \end{aligned}$$

First, we estimate both terms using (2.1) and Proposition 2.2(a). Recalling that  $\mathfrak{p} \geq (\alpha - 1)_+$ , for  $\ell = 1, 2$  we get

$$\begin{aligned}
 & |H_{k,\ell}(x, y, 1)| \\
 & \lesssim \int_0^{1/2} \int_{\mathbb{R}_+^d} (1-s)^{-k-d/\alpha} \left( \frac{x_d}{(1-s)^{1/\alpha} + x_d} \right)^{(\alpha-1)_+} \left( \frac{z_d}{(1-s)^{1/\alpha} + z_d} \right)^{(\alpha-1)_+} \\
 & \quad \times \left( \frac{(1-s)^{1/\alpha}}{(1-s)^{1/\alpha} + |x-z|} \right)^{(d+\alpha)(1-\epsilon)} z_d^{-\alpha} \left( \frac{y_d}{s^{1/\alpha} + y_d} \right)^{\mathfrak{p}} \left( \frac{z_d}{s^{1/\alpha} + z_d} \right)^{\mathfrak{p}} \frac{1}{s^{d/\alpha}} \\
 & \quad \times \left( \frac{s^{1/\alpha}}{s^{1/\alpha} + |z-y|} \right)^{(d+\alpha)(1-\epsilon)} dz ds. \tag{3.4}
 \end{aligned}$$

We define, as before,  $\alpha' = \alpha(1-\epsilon) - d\epsilon$  for  $0 < \epsilon < \alpha/(d+\alpha)$ , so  $0 < \alpha' < \alpha < 2$ . Thus, from Proposition 2.1(a) with  $\lambda = 0$ , we have

$$\begin{aligned}
 |H_{k,1}(x, y, 1)| & \lesssim \frac{1}{x_d^\alpha} \int_0^{1/2} \int_{\mathbb{R}_+^d} \left( \frac{x_d}{(1-s)^{1/\alpha} + x_d} \right)^{(\alpha'-1)_+} \left( \frac{z_d}{(1-s)^{1/\alpha} + z_d} \right)^{(\alpha'-1)_+} \\
 & \quad \times \frac{1}{s^{d/\alpha}} \left( \frac{y_d}{s^{1/\alpha} + y_d} \right)^{(\alpha'-1)_+} \left( \frac{z_d}{s^{1/\alpha} + z_d} \right)^{(\alpha'-1)_+} \\
 & \quad \times \left( \frac{(1-s)^{1/\alpha}}{(1-s)^{1/\alpha} + |x-z|} \right)^{d+\alpha'} \left( \frac{s^{1/\alpha}}{s^{1/\alpha} + |z-y|} \right)^{d+\alpha'} dz ds \\
 & \lesssim \frac{1}{x_d^\alpha} \int_0^{1/2} \int_{\mathbb{R}_+^d} \left( \frac{x_d}{((1-s)^{\alpha'/\alpha})^{1/\alpha'} + x_d} \right)^{(\alpha'-1)_+} \left( \frac{z_d}{((1-s)^{\alpha'/\alpha})^{1/\alpha'} + z_d} \right)^{(\alpha'-1)_+} \\
 & \quad \times \frac{1}{(s^{\alpha'/\alpha})^{d/\alpha'}} \left( \frac{y_d}{(s^{\alpha'/\alpha})^{1/\alpha'} + y_d} \right)^{(\alpha'-1)_+} \left( \frac{z_d}{(s^{\alpha'/\alpha})^{1/\alpha'} + z_d} \right)^{(\alpha'-1)_+} \\
 & \quad \times \left( \frac{((1-s)^{\alpha'/\alpha})^{1/\alpha'}}{((1-s)^{\alpha'/\alpha})^{1/\alpha'} + |x-z|} \right)^{d+\alpha'} \left( \frac{(s^{\alpha'/\alpha})^{1/\alpha'}}{(s^{\alpha'/\alpha})^{1/\alpha'} + |z-y|} \right)^{d+\alpha'} dz ds \\
 & \lesssim \frac{1}{x_d^\alpha} \int_0^{1/2} W_{0,(1-s)^{\alpha'/\alpha}}^{\alpha'}(x, z) W_{0,s^{\alpha'/\alpha}}^{\alpha'}(z, y) dz ds \\
 & \lesssim \frac{1}{x_d^\alpha} \int_0^{1/2} W_{0,(1-s)^{\alpha'/\alpha} + s^{\alpha'/\alpha}}^{\alpha'}(x, y) dz ds \\
 & \lesssim \frac{1}{x_d^\alpha} \int_0^{1/2} \left( \frac{x_d}{((1-s)^{\alpha'/\alpha} + s^{\alpha'/\alpha})^{1/\alpha'} + x_d} \right)^{(\alpha'-1)_+} \\
 & \quad \times \left( \frac{y_d}{((1-s)^{\alpha'/\alpha} + s^{\alpha'/\alpha})^{1/\alpha'} + y_d} \right)^{(\alpha'-1)_+} \\
 & \quad \times \left( \frac{((1-s)^{\alpha'/\alpha} + s^{\alpha'/\alpha})^{1/\alpha'}}{((1-s)^{\alpha'/\alpha} + s^{\alpha'/\alpha})^{1/\alpha'} + |x-y|} \right)^{\alpha'+d} ds \\
 & \lesssim \frac{1}{x_d^\alpha} \int_0^{1/2} \left( \frac{x_d}{2^{-1/\alpha} + x_d} \right)^{(\alpha'-1)_+} \left( \frac{y_d}{2^{-1/\alpha} + y_d} \right)^{(\alpha'-1)_+} \left( \frac{2^{1/\alpha}}{2^{1/\alpha} + |x-y|} \right)^{\alpha'+d} ds \\
 & \lesssim \frac{1}{x_d^\alpha} \left( \frac{x_d}{1+x_d} \right)^{(\alpha'-1)_+} \left( \frac{y_d}{1+y_d} \right)^{(\alpha'-1)_+} \left( \frac{1}{1+|x-y|} \right)^{\alpha'+d}.
 \end{aligned}$$

In order to estimate  $H_{k,2}$ , we notice first that from [22, Lemma 22],

$$\begin{aligned} & \int_{\mathbb{R}^{d-1}} \left( \frac{(1-s)^{1/\alpha}}{(1-s)^{1/\alpha} + |x-z|} \right)^{(d+\alpha)(1-\epsilon)} \left( \frac{s^{1/\alpha}}{s^{1/\alpha} + |z-y|} \right)^{(d+\alpha)(1-\epsilon)} dz' \\ & \lesssim \int_{\mathbb{R}^{d-1}} \left( \frac{(1-s)^{1/\alpha}}{(1-s)^{1/\alpha} + x_d + |x' - z'|} \right)^{(d+\alpha)(1-\epsilon)} \\ & \quad \times \left( \frac{s^{1/\alpha}}{s^{1/\alpha} + x_d + |z' - y'|} \right)^{(d+\alpha)(1-\epsilon)} dz' \\ & \lesssim \frac{((1-s)s)^{(d+\alpha)(1-\epsilon)/\alpha}}{x_d^{\alpha(1-\epsilon)-d\epsilon+1}} \frac{1}{(x_d + |x' - y'|)^{(d+\alpha)(1-\epsilon)}}, \end{aligned}$$

provided that  $z_d \leq x_d/2$ .

Then, by taking  $0 < \epsilon < (\alpha - \mathfrak{p})/(d + \alpha)$  (which is possible since  $\mathfrak{p} < \alpha$ ), from (3.4) we get

$$\begin{aligned} |H_{k,2}(x, y, 1)| & \lesssim \int_0^{1/2} \int_0^{x_d/2} \left( \frac{x_d}{(1-s)^{1/\alpha} + x_d} \right)^{(\alpha-1)_+} \left( \frac{z_d}{(1-s)^{1/\alpha} + z_d} \right)^{(\alpha-1)_+} \\ & \quad \times z_d^{-\alpha} \left( \frac{y_d}{s^{1/\alpha} + y_d} \right)^{\mathfrak{p}} \left( \frac{z_d}{s^{1/\alpha} + z_d} \right)^{\mathfrak{p}} \frac{1}{s^{d/\alpha}} \\ & \quad \times \frac{((1-s)s)^{(d+\alpha)(1-\epsilon)/\alpha}}{x_d^{\alpha(1-\epsilon)-d\epsilon+1}} \frac{1}{(x_d + |x' - y'|)^{(d+\alpha)(1-\epsilon)}} dz_d ds \\ & \lesssim \int_0^{x_d/2} \left( \int_0^{1/2} \left( \frac{z_d}{(1-s)^{1/\alpha} + z_d} \right)^{(\alpha-1)_+} \left( \frac{z_d}{s^{1/\alpha} + z_d} \right)^{\mathfrak{p}} \right. \\ & \quad \times \left. \frac{((1-s)s)^{(d+\alpha)(1-\epsilon)/\alpha} s^{-d/\alpha}}{(x_d + |x' - y'|)^{(d+\alpha)(1-\epsilon)}} ds \right) \frac{z_d^{-\alpha}}{x_d^{\alpha(1-\epsilon)-d\epsilon+1}} dz_d \\ & \lesssim \int_0^{x_d/2} \frac{z_d^{-\alpha}}{x_d^{\alpha(1-\epsilon)-d\epsilon+1+(d+\alpha)(1-\epsilon)}} (1 \wedge z_d)^{\mathfrak{p}+(\alpha-1)_+} dz_d. \end{aligned}$$

Here, we have used that, for the values of  $\epsilon$  indicated above,

$$\begin{aligned} s^{(d+\alpha)(1-\epsilon)/\alpha-d/\alpha} \left( 1 \wedge \frac{z_d}{s^{1/\alpha}} \right)^{\mathfrak{p}} & = \left( s^{(1-\epsilon(d+\alpha)/\alpha)/\mathfrak{p}} \wedge \left( s^{(1-\epsilon(d+\alpha)/\alpha)/\mathfrak{p}-1/\alpha} z_d \right)^{\mathfrak{p}} \right)^{\mathfrak{p}} \\ & \lesssim \left( 1 \wedge s^{(\alpha-\mathfrak{p}-\epsilon(d+\alpha))/(\alpha\mathfrak{p})} z_d \right)^{\mathfrak{p}} \leq (1 \wedge z_d)^{\mathfrak{p}}. \end{aligned}$$

Finally, since  $0 < \epsilon < \alpha/(d + \alpha)$ , by proceeding as in page 11 we get

$$|H_{k,2}(x, y, 1)| \lesssim \frac{1}{x_d^\alpha} \left( \frac{1}{1 + |x - y|} \right)^{(d+\alpha)(1-\epsilon)}, \quad x_d \geq 1.$$

In a similar way, we can also see that

$$|\tilde{H}_k(x, y, 1)| \lesssim \frac{1}{x_d^\alpha} \left( \frac{1}{1 + |x - y|} \right)^{(d+\alpha)(1-\epsilon)}, \quad x_d \geq 1,$$

where

$$\tilde{H}_k(x, y, t) = \int_0^{t/2} \int_{\mathbb{R}_+^d} W_{0,s}^\alpha(x, z) z_d^{-\alpha} \partial_t^k W_{\lambda,t-s}^\alpha(z, y) dz ds.$$

By putting together all of the above estimates, and using the scaling property (3.3), we obtain

$$\left| t^{k-1} \partial_t^k D_{\lambda,t}^{\alpha,0}(x, y) \right| \lesssim \frac{t^{-d/\alpha}}{(x_d \vee y_d)^\alpha} \left( \frac{t^{1/\alpha}}{t^{1/\alpha} + |x - y|} \right)^{(d+\alpha)(1-\epsilon)},$$

for  $0 < \epsilon < (\alpha - \mathfrak{p})/(d + \alpha)$ .

Consequently, recalling the definition of the local part  $L$  in (1.3), for certain constant  $C > 0$  we have

$$\begin{aligned}
 & \int_{\mathbb{R}_+^d} \int_0^\infty \left| t^{k-1} \partial_t^k D_{\lambda,t}^{\alpha,0}(x,y) \right| \chi_L(x,y,t) dt dy \\
 & \lesssim \int_{\{y \in \mathbb{R}_+^d : C^{-1}x_d \leq y_d \leq Cx_d\}} \int_0^{(x_d \vee y_d)^\alpha} \frac{t^{-d/\alpha}}{(x_d \vee y_d)^\alpha} \left( \frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|} \right)^{(d+\alpha)(1-\epsilon)} dt dy \\
 & \lesssim \int_0^{(Cx_d)^\alpha} t^{-d/\alpha} \int_{\{y \in \mathbb{R}_+^d : C^{-1}x_d \leq y_d \leq Cx_d\}} x_d^{-\alpha} \left( \frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|} \right)^{(d+\alpha)(1-\epsilon)} dy dt \\
 & \lesssim x_d^{-\alpha} \int_0^{(Cx_d)^\alpha} t^{-d/\alpha} \int_{\mathbb{R}^d} \left( \frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|} \right)^{(d+\alpha)(1-\epsilon)} dy dt \\
 & \lesssim x_d^{-\alpha} \int_0^{(Cx_d)^\alpha} \int_{\mathbb{R}^d} \frac{1}{(1+|z|)^{(d+\alpha)(1-\epsilon)}} dy dt \\
 & \lesssim 1,
 \end{aligned}$$

for each  $x \in \mathbb{R}_+^d$ , provided that  $0 < \epsilon < (\alpha - \mathfrak{p})/(d + \alpha)$ .

By symmetry, we also deduce that

$$\int_{\mathbb{R}_+^d} \int_0^\infty \left| t^{k-1} \partial_t^k D_{\lambda,t}^{\alpha,0}(x,y) \right| \chi_L(x,y,t) dt dx \lesssim 1, \quad y \in \mathbb{R}_+^d,$$

when  $0 < \epsilon < \alpha/(d + \alpha)$ .

Schur's test allows us to conclude that the integral operator

$$\mathbb{J}_\lambda^{\alpha,k}(f)(x) := \int_{\mathbb{R}_+^d} \int_0^\infty \left| t^{k-1} \partial_t^k D_{\lambda,t}^{\alpha,0}(x,y) \right| \chi_L(x,y,t) f(y) dt dy, \quad x \in \mathbb{R}_+^d,$$

is bounded on  $L^p(\mathbb{R}_+^d)$  for every  $1 \leq p \leq \infty$ .

#### 4. PROOF OF THEOREM 1.1 FOR $\alpha = 2$

**4.1. Global part.** For the case  $\alpha = 2$  we know, by Proposition 2.2(b), that

$$\begin{aligned}
 \left| t^k \partial_t^k W_{\lambda,t}^2(x,y) \right| & \lesssim \left( 1 \wedge \frac{x_d}{t^{1/2}} \right)^{\mathfrak{p}} \left( 1 + \frac{y_d}{t^{1/2}} \right)^{\mathfrak{p}} \frac{1}{t^{d/2}} e^{-\frac{c|x-y|^2}{t}} \\
 & \lesssim \left( 1 \wedge \frac{x_d}{t^{1/2}} \right)^{\mathfrak{p}} \left( 1 + \frac{y_d}{t^{1/2}} \right)^{\mathfrak{p}} \frac{1}{t^{d/2}} \left( \frac{t^{1/2}}{t^{1/2} + |x-y|} \right)^{d+2},
 \end{aligned}$$

for any  $x, y \in \mathbb{R}_+^d, t > 0$ .

With this estimate, we can proceed as in the case  $\alpha \in (0, 2)$  (see §3) and prove that, for every  $k \in \mathbb{N}, k \geq 1$ , the operator  $J_\lambda^{2,k}$  given by

$$J_\lambda^{2,k}(f)(x) = \int_{\mathbb{R}_+^d} J_\lambda^{2,k}(x,y) f(y) dy, \quad x \in \mathbb{R}_+^d,$$

whose kernel  $J_\lambda^{2,k}(x,y)$  is as in (3.1) with  $\alpha = 2$ , is bounded on  $L^p(\mathbb{R}_+^d)$  for every  $1 < p < \infty$ .

**4.2. Local part.** When dealing with the local part and  $\alpha = 2$ , we keep the notation used in Section 3.2. We recall that in the local part, we are considering  $x, y \in \mathbb{R}_+^d$  with  $|x-y| \leq (x_d \wedge y_d)/2$  and  $0 < t^{1/\alpha} \leq x_d \vee y_d$ .

Let  $j = 1, \dots, k-1$ . According to Proposition 2.2(b) we get the following estimate

$$\begin{aligned} |H_{j,1}(x, y, 1)| &\lesssim x_d^{-2} \int_{\mathbb{R}_+^d} \left( \frac{x_d}{1+x_d} \right) \left( \frac{z_d}{1+z_d} \right)^{1+p} \left( \frac{y_d}{1+y_d} \right)^p e^{-c(|x-z|^2+|z-y|^2)} dz \\ &\lesssim x_d^{-2} \int_{\mathbb{R}_+^d} \left( \frac{x_d}{1+x_d} \right) \left( \frac{z_d}{1+z_d} \right)^{1+p} \left( \frac{y_d}{1+y_d} \right)^p e^{-\frac{c}{2}|x-y|^2} e^{-\frac{c}{2}|z-y|^2} dz \\ &\lesssim x_d^{-2} \left( \frac{x_d}{1+x_d} \right) \left( \frac{y_d}{1+y_d} \right)^p e^{-\frac{c}{2}|x-y|^2} \int_{\mathbb{R}^d} e^{-\frac{c}{2}|x-z|^2} dz \\ &\lesssim x_d^{-2} e^{-\frac{c}{2}|x-y|^2}. \end{aligned}$$

Therefore, for  $x, y \in \mathbb{R}_+^d$  with  $|x-y| \leq (x_d \wedge y_d)/2$  and  $1 \leq x_d \vee y_d$ , we get

$$|H_{j,1}(x, y, 1)| \lesssim \frac{1}{x_d^2} \left( \frac{1}{1+|x-y|} \right)^{d+\alpha}.$$

Similarly, we get

$$|H_{j,2}(x, y, 1)| \lesssim \frac{1}{x_d^2} \left( \frac{1}{1+|x-y|} \right)^{d+\alpha}.$$

On the other hand, Proposition 2.2(b) leads to

$$\begin{aligned} |H_{k,1}(x, y, 1)| &\lesssim \int_0^{1/2} \int_{\{z \in \mathbb{R}_+^d : z_d > x_d/2\}} (1-s)^{-k-d/2} \left( \frac{x_d}{(1-s)^{1/2} + x_d} \right) \left( \frac{z_d}{(1-s)^{1/2} + z_d} \right) \\ &\quad \times z_d^{-2} \left( \frac{y_d}{s^{1/2} + y_d} \right)^p \left( \frac{z_d}{s^{1/2} + z_d} \right)^p \frac{1}{s^{d/2}} e^{-c \frac{|x-z|^2}{1-s} - c \frac{|z-y|^2}{s}} dz ds \\ &\lesssim x_d^{-2} \int_0^{1/2} \frac{x_d}{(1-s)^{1/2} + x_d} \left( \frac{y_d}{s^{1/2} + y_d} \right)^p e^{-\frac{c}{2} \frac{|x-y|^2}{1-s}} \int_{\mathbb{R}^d} \frac{e^{-\frac{c}{2} \frac{|y-z|^2}{s}}}{s^{d/2}} dz ds \\ &\lesssim x_d^{-2} \int_0^{1/2} \frac{x_d}{(1-s)^{1/2} + x_d} \left( \frac{y_d}{s^{1/2} + y_d} \right)^p e^{-\frac{c}{2} \frac{|x-y|^2}{1-s}} ds \\ &\lesssim x_d^{-2} e^{-\frac{c}{2}|x-y|^2}, \end{aligned}$$

for  $(x, y, 1) \in L$ .

We now study  $H_{k,2}$ . Using again Proposition 2.2(b), it yields, as before,

$$\begin{aligned} |H_{k,2}(x, y, 1)| &\lesssim \int_0^{1/2} \int_{\{z \in \mathbb{R}_+^d : z_d \leq x_d/2\}} (1-s)^{-k-d/2} \left( \frac{x_d}{(1-s)^{1/2} + x_d} \right) \left( \frac{z_d}{(1-s)^{1/2} + z_d} \right) \\ &\quad \times z_d^{-2} \left( \frac{y_d}{s^{1/2} + y_d} \right)^p \left( \frac{z_d}{s^{1/2} + z_d} \right)^p \frac{1}{s^{d/2}} e^{-c \frac{|x-z|^2}{1-s} - c \frac{|z-y|^2}{s}} dz ds. \end{aligned}$$

Note that, for each  $s \in (0, \frac{1}{2})$ , we can write

$$\int_{\mathbb{R}^{d-1}} e^{-c \frac{|x'-z'|^2}{1-s} - c \frac{|y'-z'|^2}{s}} dz' \lesssim \int_{\mathbb{R}^{d-1}} e^{-c \frac{|w|^2}{s}} dw \lesssim s^{(d-1)/2}.$$



Then, for  $s \in (0, \frac{1}{2})$ ,

$$\begin{aligned} & \int_{\{z \in \mathbb{R}_+^d : z_d \leq x_d/2\}} \left( \frac{z_d}{(1-s)^{1/2} + z_d} \right) \left( \frac{z_d}{s^{1/2} + z_d} \right)^{\mathfrak{p}} z_d^{-2} e^{-c \frac{|x-z|^2}{1-s} - c \frac{|z-y|^2}{s}} dz \\ & \lesssim \int_0^{x_d/2} \left( \frac{z_d}{(1-s)^{1/2} + z_d} \right) \left( \frac{z_d}{s^{1/2} + z_d} \right)^{\mathfrak{p}} z_d^{-2} s^{(d-1)/2} e^{-\frac{c}{2} \frac{|x-y|^2}{1-s} - c \frac{|z_d-y_d|^2}{s}} dz_d. \end{aligned}$$

Combining the above estimates, we get

$$\begin{aligned} |H_{k,2}(x, y, 1)| & \lesssim \int_0^{x_d/2} \int_0^{1/2} s^{(d-1)/2} \left( \frac{z_d}{(1-s)^{1/2} + z_d} \right) \left( \frac{z_d}{s^{1/2} + z_d} \right)^{\mathfrak{p}} z_d^{-2} \\ & \quad \times \left( \frac{(1-s)^{1/2}}{(1-s)^{1/2} + x_d + |y' - x'|} \right)^{d+4} ds dz_d \\ & \lesssim \int_0^{x_d/2} \int_0^{1/2} s^{(d-1)/2} \left( 1 \wedge \frac{z_d}{(1-s)^{1/2}} \right) \left( 1 \wedge \frac{z_d}{s^{1/2}} \right)^{\mathfrak{p}} ds \\ & \quad \times \frac{dz_d}{z_d^2} \left( \frac{1}{x_d + |y' - x'|} \right)^{d+4} \\ & \lesssim \int_0^{x_d/2} \int_0^{1/2} s^{(d-1)/2-1/4} (1 \wedge z_d)^{1+1/4} \frac{dz_d}{z_d^2} \frac{1}{x_d^{d+4}} \\ & \lesssim \int_0^1 z_d^{d-3/4} dz_d + \int_1^\infty \frac{dz_d}{z_d^2} \frac{1}{x_d^{d+4}} \\ & \lesssim \frac{1}{x_d^{d+4}} \\ & \lesssim \frac{1}{x_d^2} \left( \frac{1}{1 + |x - y|} \right)^{d+2}, \quad \frac{x_d}{2} \geq 1. \end{aligned}$$

In a similar way, we deduce that

$$|\tilde{H}_k(x, y, 1)| \lesssim \frac{1}{x_d^2} \left( \frac{1}{1 + |x - y|} \right)^{d+2}, \quad x_d \geq 1.$$

We now put all of the above estimates together obtaining

$$\left| \partial_t^k D_\lambda^{2,0}(x, y) \Big|_{t=1} \right| \lesssim \frac{1}{x_d^2} \left( \frac{1}{1 + |x - y|} \right)^{d+2}.$$

Therefore,

$$\left| t^{k-1} \partial_t^k D_\lambda^{2,0}(x, y) \right| \lesssim \frac{1}{t^{d/2}} \left( \frac{t^{1/2}}{t^{1/2} + |x - y|} \right)^{(d+2)(1-\epsilon)} \frac{1}{(x_d \vee y_d)^2},$$

with  $0 < \epsilon < (2 - \mathfrak{p})/(d + 2) = (\alpha - \mathfrak{p})/(d + \alpha)$ .

## 5. PROOF OF THEOREM 1.2

We proceed as in the proof of [38, Proposition 3.5] and of the weighted property in [2, Theorem 1.1] by using [11, Proposition 2.3] (see also [5, Theorem 6.6]). We define, for every  $t > 0$ ,

$$A_{\lambda,t}^\alpha = (I - W_{\lambda,t}^\alpha)^m,$$

for  $m \in \mathbb{N}$  to be fixed later.

For every  $x \in \mathbb{R}^d$ ,  $r, s > 0$ , let  $B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$  be the usual ball in  $\mathbb{R}^d$ ,  $sB(x, r) = B(x, sr)$  and  $\mathcal{B}(x, r) = B(x, r) \cap \mathbb{R}_+^d$ . Also, let us consider for a ball  $B \subset \mathbb{R}^d$ , the sets  $S_0(B) = B$  and for  $j \in \mathbb{N}$ ,  $j \geq 1$ ,  $S_j(B) = 2^j B \setminus 2^{j-1} B$ . We denote also, for  $B \subset \mathbb{R}^d$  and  $j \in \mathbb{N}$ ,  $S_j(\mathcal{B}) = S_j(B) \cap \mathbb{R}_+^d$ .

From now on, we will assume that  $1 < p \leq q < \infty$ . Suppose that  $\mathcal{B} = B \cap \mathbb{R}_+^d$ , where  $B = B(x_B, r_B)$  with  $x_B \in \mathbb{R}_+^d$ ,  $r_B > 0$  and that  $f$  is a smooth function supported in  $\mathcal{B}$ .

We consider first the case  $\alpha = 2$ . By using [2, (2.17)], from Proposition 2.2 (b) we deduce

$$\left( \frac{1}{|S_j(\mathcal{B})|} \int_{S_j(\mathcal{B})} |A_{\lambda, r_B^2}^2(f)(x)|^q dx \right)^{1/q} \leq C \frac{e^{-c2^{2j}}}{2^{j\alpha/q}} \left( \frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |f(x)|^p dx \right)^{1/p},$$

for  $j \in \mathbb{N}$  where  $C > 0$  does not depend on  $j$ ,  $\beta$  or  $f$ .

On the other hand, the arguments in [2, pages 13 and 14] allow us to obtain that

$$\left( \frac{1}{|S_j(\mathcal{B})|} \int_{S_j(\mathcal{B})} |V_\rho(\{t^k \partial_t^k W_{\lambda, t}^w\}_{t>0})(I - A_{\lambda, r_B^2}^2)(f)(x)|^q dx \right)^{1/q} \lesssim 2^{-j(dq+2m)} f_{q, \mathcal{B}},$$

where  $f_{q, \mathcal{B}} = \left( \frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |f(x)|^q dx \right)^{1/q}$  and the constant does not depend on  $j$ ,  $\mathcal{B}$  or  $f$ .

We choose now  $m \in \mathbb{N}$  such that  $m > d/2$ . Then

$$\sum_{j=1}^{\infty} 2^{-j(d/q+2m-d)} < \infty.$$

According to Theorem 1.1, the  $\rho$ -variation operator  $V_\rho(\{t^k \partial_t^k W_{\lambda, t}^2\}_{t>0})$  is bounded on  $L^q(\mathbb{R}_+^d)$ . By using [11, Proposition 2.3] we conclude that  $V_\rho(\{t^k \partial_t^k W_{\lambda, t}^2\}_{t>0})$  is bounded on  $L^r(\mathbb{R}_+^d, w)$ , provided that  $p < r < q$  and  $w \in A_{r/p}(\mathbb{R}_+^d) \cap \text{RH}_{(q/r)'}(\mathbb{R}_+^d)$ . Taking  $p = 1$  we obtain that  $V_\rho(\{t^k \partial_t^k W_{\lambda, t}^2\}_{t>0})$  is bounded on  $L^r(\mathbb{R}_+^d, w)$ , provided that  $w \in A_r(\mathbb{R}_+^d) \cap \text{RH}_{s'}(\mathbb{R}_+^d)$ ,  $1 < r < s < \infty$ . Since, if  $w \in A_r(\mathbb{R}_+^d)$  with  $1 < r < \infty$ , there exists  $\beta > 1$  such that  $w \in \text{RH}_\beta(\mathbb{R}_+^d)$  we conclude that  $V_\rho(\{t^k \partial_t^k W_{\lambda, t}^2\}_{t>0})$  is bounded on  $L^r(\mathbb{R}_+^d, w)$  for every  $1 < r < \infty$  and  $w \in A_r(\mathbb{R}_+^d)$ .

Assume now  $0 < \alpha < 2$ . According to Proposition 2.2 (a) and by proceeding as in the proof of [7, Theorem 5.1] we deduce that if  $1 \leq p < q < \infty$ ,  $\ell \in \mathbb{N}$  and  $0 < \epsilon < 1$ , there exists  $C > 0$  such that for every ball  $\mathcal{B} = B(x_B, r_B) \cap \mathbb{R}_+^d$ , with  $x_B \in \mathbb{R}_+^d$  and  $r_B > 0$ , every  $t > 0$  and  $j \in \mathbb{N}$ , we have that

$$\begin{aligned} & \left( \frac{1}{|S_j(\mathcal{B})|} \int_{S_j(\mathcal{B})} |t^\ell \partial_t^\ell W_{\lambda, t}^\alpha(f)(x)|^q dx \right)^{1/q} \\ & \leq C \max \left\{ \left( \frac{r_B}{t^{1/\alpha}} \right)^{d/p}, \left( \frac{r_B}{t^{1/\alpha}} \right)^d \right\} \\ & \quad \times \left( 1 + \frac{t^{1/\alpha}}{2^j r_B} \right)^{d/q} \left( 1 + \frac{2^j r_B}{t^{1/\alpha}} \right)^{-(d+\alpha)(1-\epsilon)} \left( \frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |f(x)|^p dx \right)^{1/p}, \end{aligned} \quad (5.1)$$

for every  $f \in L^p(\mathbb{R}_+^d)$  supported in  $\mathcal{B}$ , and

$$\begin{aligned} & \left( \frac{1}{|S_j(\mathcal{B})|} \int_{S_j(\mathcal{B})} |t^\ell \partial_t^\ell W_{\lambda, t}^\alpha(f)(x)|^q dx \right)^{1/q} \\ & \lesssim \max \left\{ \left( \frac{2^\ell r_B}{t^{1/\alpha}} \right)^{d/p}, \left( \frac{2^\ell r_B}{t^{1/\alpha}} \right)^d \right\} \\ & \quad \times \left( 1 + \frac{t^{1/\alpha}}{2^j r_B} \right)^{d/q} \left( 1 + \frac{2^j r_B}{t^{1/\alpha}} \right)^{-(d+\alpha)(1-\epsilon)} \left( \frac{1}{|S_j(\mathcal{B})|} \int_{S_j(\mathcal{B})} |f(x)|^p dx \right)^{1/p}, \end{aligned} \quad (5.2)$$

for every  $f \in L^p(S_j(\mathcal{B}))$ .

According to (1.1) we obtain

$$V_\rho(\{t^k \partial_t^k W_{\lambda,t}^2\}_{t>0})(f)(x) \leq \int_0^\infty |\partial_t(t^k \partial_t^k W_{\lambda,t}^2(f)(x))| dt, \quad x \in \mathbb{R}_+^d.$$

Now, for  $\ell \in \mathbb{R} \setminus \{0\}$ , we consider the operator

$$\mathcal{T}_\ell(f)(x) = \int_0^\infty |t^{\ell-1} \partial_t^\ell W_{\lambda,t}^\alpha(f)(x)| dt, \quad x \in \mathbb{R}^d.$$

Assume that  $j \in \mathbb{N}$  and let  $\mathcal{B} = B(x_B, r_B) \cap \mathbb{R}_+^d$ , with  $x_B \in \mathbb{R}_+^d$  and  $r_B > 0$ . By using Minkowski inequality we get

$$\begin{aligned} & \left( \frac{1}{|S_j(\mathcal{B})|} \int_{S_j(\mathcal{B})} |\mathcal{T}_\ell((I - W_{\lambda,r_B^\alpha}^\alpha)^m(f))(x)|^q dx \right)^{1/q} \\ & \leq \int_0^\infty \left( \frac{1}{|S_j(\mathcal{B})|} \int_{S_j(\mathcal{B})} |t^{\ell-1} \partial_t^\ell W_{\lambda,t}^\alpha((I - W_{\lambda,r_B^\alpha}^\alpha)^m(f))(x)|^q dx \right)^{1/q} dt \\ & \leq \left( \int_0^{r_B^\alpha} + \int_{r_B^\alpha}^\infty \right) \left( \frac{1}{|S_j(\mathcal{B})|} \int_{S_j(\mathcal{B})} |t^{\ell-1} \partial_t^\ell W_{\lambda,t}^\alpha((I - W_{\lambda,r_B^\alpha}^\alpha)^m(f))(x)|^q dx \right)^{1/q} dt \\ & := I_1 + I_2. \end{aligned}$$

We now adapt the arguments in [38, p. 15 and p. 16]. We begin analysing  $I_1$ . First observe that

$$(I - W_{\lambda,r_B^\alpha}^\alpha)^m = \int_0^{r_B^\alpha} \dots \int_0^{r_B^\alpha} \partial_{s_1} \dots \partial_{s_m} W_{\lambda,s_1+\dots+s_m}^\alpha ds_1 \dots ds_m.$$

Then, by (5.1) and calling  $\mathfrak{s} = s_1 + \dots + s_m$  and  $d\mathfrak{s} = ds_1 \dots ds_m$ ,

$$\begin{aligned} I_2 & \leq \int_{r_B^\alpha}^\infty \int_0^{r_B^\alpha} \dots \int_0^{r_B^\alpha} \left( \frac{1}{|S_j(\mathcal{B})|} \int_{S_j(\mathcal{B})} |t^{\ell-1} \partial_t^\ell \partial_{s_1} \dots \partial_{s_m} W_{\lambda,t+\mathfrak{s}}^\alpha(f)(x)|^q dx \right)^{1/q} d\mathfrak{s} dt \\ & \leq \int_{r_B^\alpha}^\infty \int_0^{r_B^\alpha} \dots \int_0^{r_B^\alpha} \frac{1}{(t+\mathfrak{s})^{m+1}} \\ & \quad \times \left( \frac{1}{|S_j(\mathcal{B})|} \int_{S_j(\mathcal{B})} |u^{\ell+m} \partial_u^{\ell+m} W_{\lambda,u}^\alpha(f)(x)|_{u=t+\mathfrak{s}}^q dx \right)^{1/q} d\mathfrak{s} dt \\ & \lesssim f_{q,\mathcal{B}} \left( \int_{r_B^\alpha}^\infty \int_0^{r_B^\alpha} \dots \int_0^{r_B^\alpha} \frac{1}{(t+\mathfrak{s})^{m+1}} \left( \frac{r_B}{(t+\mathfrak{s})^{1/\alpha}} \right)^{d/q} \left( 1 + \frac{(t+\mathfrak{s})^{1/\alpha}}{2^j r_B} \right)^{d/q} \right. \\ & \quad \times \left. \left( 1 + \frac{2^j r_B}{(t+\mathfrak{s})^{1/\alpha}} \right)^{-(d+\alpha)(1-\epsilon)} d\mathfrak{s} dt \right) \\ & \lesssim f_{q,\mathcal{B}} \left( \int_{r_B^\alpha}^{(2^j r_B)^\alpha} \int_0^{r_B^\alpha} \dots \int_0^{r_B^\alpha} \frac{1}{t^{m+1}} \left( \frac{r_B}{t^{1/\alpha}} \right)^{d/q} \left( \frac{t^{1/\alpha}}{2^j r_B} \right)^{d/q} d\mathfrak{s} dt \right. \\ & \quad \left. + \int_{(2^j r_B)^\alpha}^\infty \int_0^{r_B^\alpha} \dots \int_0^{r_B^\alpha} \frac{1}{t^{m+1}} \left( \frac{r_B}{t^{1/\alpha}} \right)^{d/q} \left( \frac{t^{1/\alpha}}{2^j r_B} \right)^{(d+\alpha)(1-\epsilon)} d\mathfrak{s} dt \right) \\ & \lesssim f_{q,\mathcal{B}} \left( 2^{-j(d+\alpha)(1-\epsilon)} \int_{r_B^\alpha}^{(2^j r_B)^\alpha} t^{-m-1-\frac{d}{q\alpha} + \frac{(d+\alpha)(1-\epsilon)}{\alpha}} dt \right. \\ & \quad \left. \times r_B^{\frac{d}{q} + m\alpha - (d+\alpha)(1-\epsilon)} + \int_{(2^j r_B)^\alpha}^\infty 2^{-dj/q} r_B^{\alpha m} t^{-m-1} dt \right) \end{aligned}$$

$$\begin{aligned} &\lesssim f_{q,\mathcal{B}} \left( 2^{-j(d+\alpha)(1-\epsilon)} + 2^{-j(\alpha m+d/q)} \right) \\ &\lesssim f_{q,\mathcal{B}} 2^{-j(d+\alpha)(1-\epsilon)}, \end{aligned}$$

provided that  $m > \frac{(d+\alpha)(1-\epsilon)}{\alpha} - \frac{d}{q^\alpha}$ . Hence, if  $m > \frac{d+\alpha}{\alpha} - \frac{dq}{q^\alpha}$ , by choosing  $0 < \epsilon < 1$  such that  $m > \frac{(d+\alpha)(1-\epsilon)}{\alpha} - \frac{d}{q^\alpha}$  the estimates hold.

We can conclude that

$$I_2 \leq C 2^{-j(d+\alpha)(1-\epsilon)} f_{q,\mathcal{B}}.$$

On the other hand, we can prove by using (5.2) (see [38, p. 15 and p.16]) that

$$I_1 \leq C 2^{-j(d+\alpha)(1-\epsilon)} f_{q,\mathcal{B}},$$

where  $0 < \epsilon < 1$ .

We obtain that

$$\left( \frac{1}{|S_j(\mathcal{B})|} \int_{S_j(\mathcal{B})} |\mathcal{T}_\ell((I - W_{\lambda, r_{\mathcal{B}}}^\alpha)^m)(f)(x)|^q dx \right)^{1/q} \lesssim 2^{-j(d+\alpha)(1-\epsilon)} f_{q,\mathcal{B}},$$

provided that  $0 < \epsilon < 1$  and  $m > \frac{(d+\alpha)(1-\epsilon)}{\alpha} - \frac{d}{q^\alpha}$ .

Let  $\epsilon \in (0, 1)$ . According to (5.1) we deduce that

$$\begin{aligned} &\left( \frac{1}{|S_j(\mathcal{B})|} \int_{S_j(\mathcal{B})} |(I - A_{\lambda, r_{\mathcal{B}}}^\alpha)(f)(x)|^q dx \right)^{1/q} \\ &\lesssim \sum_{i=1}^m \left( \frac{1}{|S_j(\mathcal{B})|} \int_{|S_j(\mathcal{B})|} |W_{\lambda, r_{\mathcal{B}}}^\alpha(f)(x)|^q dx \right)^{1/q} \\ &\lesssim \sum_{i=1}^m \left( \frac{1}{i^{1/\alpha}} \right)^{1/p} \left( 1 + \frac{i^{1/\alpha}}{2^j} \right)^{\alpha/q} \left( 1 + \frac{2^j}{i^{1/\alpha}} \right)^{-(d+\alpha)(1-\epsilon)} \left( \frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |f(x)|^p dx \right)^{1/p} \\ &\lesssim 2^{-j(d+\alpha)(1-\epsilon)} \left( \frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |f(x)|^p dx \right)^{1/p}. \end{aligned}$$

We define  $\alpha_j = 2^{-j(d+\alpha)(1-\epsilon)}$ ,  $j \in \mathbb{N}$ . We have that  $\sum_{j=1}^{\infty} \alpha_j 2^{j\alpha} < \infty$  provided that  $0 < \epsilon < \alpha/(d+\alpha)$ .

By [11, Proposition 2.3] (see also [5, Theorem 6.6]) and by proceeding as in the case  $\alpha = 2$ , we conclude that the operator  $\mathcal{T}_\ell$  is bounded on  $L^p(\mathbb{R}_+^d, w)$  for every  $1 < p < \infty$  and  $w \in A_p(\mathbb{R}_+^d)$ . Hence, the  $\rho$ -variation operator  $\mathcal{V}_\rho \left( \{t^k \partial_t^k W_{\lambda, t}^\alpha\}_{t>0} \right)$  is also bounded on  $L^p(\mathbb{R}_+^d, w)$  for every  $1 < p < \infty$  and  $w \in A_p(\mathbb{R}_+^d)$ .

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## REFERENCES

- [1] AKCOGLU, M. A., JONES, R. L., AND SCHWARTZ, P. O. Variation in probability, ergodic theory and analysis. *Illinois J. Math.* 42, 1 (1998), 154–177.
- [2] ALMEIDA, V., BETANCOR, J. J., AND RODRÍGUEZ-MESA, L. Variation operators associated with the semigroups generated by Schrödinger operators with inverse square potentials. *Anal. Appl. (Singap.)* 21, 2 (2023), 353–383.
- [3] ARENDT, W., AND BUKHVALOV, A. V. Integral representations of resolvents and semigroups. *Forum Math.* 6, 1 (1994), 111–135.
- [4] BELTRAN, D., OBERLIN, R., RONCAL, L., SEEGER, A., AND STOVALL, B. Variation bounds for spherical averages. *Math. Ann.* 382, 1-2 (2022), 459–512.
- [5] BERNICOT, F., AND ZHAO, J. New abstract Hardy spaces. *J. Funct. Anal.* 255, 7 (2008), 1761–1796.
- [6] BETANCOR, J. J., CRESCIMBENI, R., AND TORREA, J. L. The  $\rho$ -variation of the heat semigroup in the Hermitian setting: behaviour in  $L^\infty$ . *Proc. Edinb. Math. Soc. (2)* 54, 3 (2011), 569–585.
- [7] BOGDAN, K., AND DYDA, B. O. The best constant in a fractional Hardy inequality. *Math. Nachr.* 284, 5-6 (2011), 629–638.
- [8] BOURGAIN, J. On the maximal ergodic theorem for certain subsets of the integers. *Israel J. Math.* 61, 1 (1988), 39–72.
- [9] BOURGAIN, J. On the pointwise ergodic theorem on  $L^p$  for arithmetic sets. *Israel J. Math.* 61, 1 (1988), 73–84.
- [10] BOURGAIN, J. Pointwise ergodic theorems for arithmetic sets. *Inst. Hautes Études Sci. Publ. Math.*, 69 (1989), 5–45. With an appendix by the author, Harry Furstenberg, Yitzhak Katznelson and Donald S. Ornstein.
- [11] BUI, T. A., AND BUI, T. Q. Maximal regularity of parabolic equations associated to generalized Hardy operators in weighted mixed-norm spaces. *J. Differential Equations* 303 (2021), 547–574.
- [12] BUI, T. A., AND D’ANCONA, P. Generalized Hardy operators. *Nonlinearity* 36, 1 (2023), 171–198.
- [13] BUI, T. A., AND MERZ, K. Equivalence of Sobolev norms in Lebesgue spaces for Hardy operators in a half-space, 2023. [arXiv:2309.02928](https://arxiv.org/abs/2309.02928).
- [14] CAMPBELL, J. T., JONES, R. L., REINHOLD, K., AND WIERDL, M. Oscillation and variation for the Hilbert transform. *Duke Math. J.* 105, 1 (2000), 59–83.
- [15] CAMPBELL, J. T., JONES, R. L., REINHOLD, K., AND WIERDL, M. Oscillation and variation for singular integrals in higher dimensions. *Trans. Amer. Math. Soc.* 355, 5 (2003), 2115–2137.
- [16] CHEN, Z.-Q., KIM, P., AND SONG, R. Two-sided heat kernel estimates for censored stable-like processes. *Probab. Theory Related Fields* 146, 3-4 (2010), 361–399.
- [17] CHEN, Z.-Q., AND KUMAGAI, T. Heat kernel estimates for stable-like processes on  $d$ -sets. *Stochastic Process. Appl.* 108, 1 (2003), 27–62.
- [18] DAVIES, E. B. *Heat kernels and spectral theory*, vol. 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.
- [19] DI PLINIO, F., DO, Y. Q., AND URALTSEV, G. N. Positive sparse domination of variational Carleson operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 18, 4 (2018), 1443–1458.
- [20] DO, Y., AND LACEY, M. Weighted bounds for variational Fourier series. *Studia Math.* 211, 2 (2012), 153–190.
- [21] DUONG, X. T., AND ROBINSON, D. W. Semigroup kernels, Poisson bounds, and holomorphic functional calculus. *J. Funct. Anal.* 142, 1 (1996), 89–128.
- [22] FRANK, R. L., AND MERZ, K. On Sobolev norms involving Hardy operators in a half-space. *J. Funct. Anal.* 285, 10 (2023), Paper No. 110104.
- [23] GILLESPIE, T. A., AND TORREA, J. L. Dimension free estimates for the oscillation of Riesz transforms. *Israel J. Math.* 141 (2004), 125–144.

- [24] JONES, R. L., AND WANG, G. Variation inequalities for the Fejér and Poisson kernels. *Trans. Amer. Math. Soc.* 356, 11 (2004), 4493–4518.
- [25] KILLIP, R., MIAO, C., VISAN, M., ZHANG, J., AND ZHENG, J. Sobolev spaces adapted to the Schrödinger operator with inverse-square potential. *Math. Z.* 288, 3-4 (2018), 1273–1298.
- [26] KRAUSE, B. Polynomial ergodic averages converge rapidly: Variations on a theorem of Bourgain, 2014. [arXiv:1402.1803](https://arxiv.org/abs/1402.1803).
- [27] KRAUSE, B. Discrete fractional integration operators along the primes, 2019. [arXiv:1905.02767](https://arxiv.org/abs/1905.02767).
- [28] KRAUSE, B. Pointwise ergodic theory: examples and entropy, 2023. [arXiv:2301.01511](https://arxiv.org/abs/2301.01511).
- [29] KRAUSE, B., MIREK, M., AND TAO, T. Pointwise ergodic theorems for non-conventional bilinear polynomial averages. *Ann. of Math. (2)* 195, 3 (2022), 997–1109.
- [30] LE MERDY, C., AND XU, Q. Strong  $q$ -variation inequalities for analytic semigroups. *Ann. Inst. Fourier (Grenoble)* 62, 6 (2012), 2069–2097 (2013).
- [31] LÉPINGLE, D. La variation d'ordre  $p$  des semi-martingales. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 36, 4 (1976), 295–316.
- [32] MA, T., TORREA, J. L., AND XU, Q. Weighted variation inequalities for differential operators and singular integrals in higher dimensions. *Sci. China Math.* 60, 8 (2017), 1419–1442.
- [33] MAS, A., AND TOLSA, X. Variation and oscillation for singular integrals with odd kernel on Lipschitz graphs. *Proc. Lond. Math. Soc. (3)* 105, 1 (2012), 49–86.
- [34] MEHLHOP, NATHAN AND SŁOMIAN, W. Oscillation and jump inequalities for the polynomial ergodic averages along multi-dimensional subsets of primes. *Math. Ann.* (2023).
- [35] MERZ, K. On complex-time heat kernels of fractional Schrödinger operators via Phragmén-Lindelöf principle. *J. Evol. Equ.* 22, 3 (2022), Paper No. 62, 30.
- [36] MIREK, M., STEIN, E. M., AND ZORIN-KRANICH, P. Jump inequalities via real interpolation. *Math. Ann.* 376, 1-2 (2020), 797–819.
- [37] MIREK, M., AND TROJAN, B. Cotlar's ergodic theorem along the prime numbers. *J. Fourier Anal. Appl.* 21, 4 (2015), 822–848.
- [38] NADER, G. Variation and oscillation operators associated to semigroup generated by Schrödinger operator with fractional power. *J. Math. Anal. Appl.* 523, 1 (2023), Paper No. 127000, 26.
- [39] OBERLIN, R., SEEGER, A., TAO, T., THIELE, C., AND WRIGHT, J. A variation norm Carleson theorem. *J. Eur. Math. Soc. (JEMS)* 14, 2 (2012), 421–464.
- [40] PISIER, G., AND XU, Q. H. The strong  $p$ -variation of martingales and orthogonal series. *Probab. Theory Related Fields* 77, 4 (1988), 497–514.
- [41] QIAN, J. The  $p$ -variation of partial sum processes and the empirical process. *Ann. Probab.* 26, 3 (1998), 1370–1383.
- [42] SCHNAUBELT, R. Lecture notes: Evolution equations. <https://www.math.kit.edu/iana3/~schnaubelt/media/evgl-skript.pdf>, 2023.
- [43] WIERDL, M. Pointwise ergodic theorem along the prime numbers. *Israel J. Math.* 64, 3 (1988), 315–336 (1989).
- [44] ZORIN-KRANICH, P. Variation estimates for averages along primes and polynomials. *J. Funct. Anal.* 268, 1 (2015), 210–238.

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