ISSN 2451-7100

Publicaciones del IMAL

IMAL Preprints
#2023–0067

Boundedness for fractional integral of the Bi-Harmonic Schrödinger operator

Bruno Bongioanni - Marisa Toschi - Bruno Urrutia

Publication Date: October 23, 2023

Publisher: Instituto de Matemática Aplicada del Litoral IMAL "Dra. Eleonor Harboure" (CCT CONICET Santa Fe – UNL)

https://imal.conicet.gov.ar
https://imal.conicet.gov.ar/preprints-del-imal

Publishing Director: Dra. Estefanía Dalmasso

edalmasso@santafe-conicet.gov.ar

2023

IMAL PREPRINT # 2023-0067

ISSN 2451-7100
Publication date: October 23, 2023

Boundedness for fractional integral of the Bi-Harmonic Schrödinger operator

Bruno Bongioanni[1,2], Marisa Toschi[1,3†], Bruno Urrutia[1,2*†]

[1*]Instituto de Matemática Aplicada del Litoral, Ruta Nacional N° 168 Km 0, Santa Fe, 3000, Santa Fe, Argentina.
[2]Facultad de Ingeniería Química, Universidad Nacional del Litoral, Santiago del Estero 2829, Santa Fe, 3000, Santa Fe, Argentina.
[3]Facultad de Humanidades y Ciencias, Ruta Nacional N° 168 Km 0, Santa Fe, 3000, Santa Fe, Argentina.

*Corresponding author(s). E-mail(s): burrutia@santafe-conicet.gov.ar;
Contributing authors: bbongio@santafe-conicet.gov.ar;
mtoschi@santafe-conicet.gov.ar;
†These authors contributed equally to this work.

Abstract

In this work we obtain boundedness results for the fractional integral operator of the the bi-harmonic Schrödinger operator on weighted Lebesgue and BMO type spaces in $\mathbb{R}^d$ with $d \geq 5$. The techniques are based on some new estimates involving the kernel of the heat semigroup.

Keywords: Bi-Harmonic Operator, Schrödinger Operator, Fractional Integral, BMO Spaces

1 Introduction

Lets consider the bi-harmonic Schrödinger operator on $\mathbb{R}^d$ with $d \geq 5$,

$$\mathcal{L} = (-\Delta)^2 + V^2, \tag{1}$$

IMAL PREPRINT # 2023-0067

ISSN 2451-7100
Publication date: October 23, 2023

where the potential V is non-negative, non-identically zero, and satisfies a reverse Hölder inequality for some $q > d/2$. That means, there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B V(y)^q\,dy\right)^{1/q} \leq \frac{C}{|B|}\int_B V(y)\,dy, \tag{2}$$

for every ball $B \subset \mathbb{R}^d$.

We shall say that $V \in RH_q$ when V satisfies (2).

In the last years, the behaviour of some operators associated to (1) has been studied in many works. For instance, in [6] the authors deal with Hardy spaces and characterizations, giving boundedness results for a higher order Riesz transform and the fractional integral of (1) by making use of those caracterizations. In [7] the authors give boundedness results for a variety of operators related to (1) acting on Lipschitz-type spaces.

Most of the techniques used for the study of the Schrödinger operator $-\Delta + V$ can also be used to develop the theory associated to (1). Some classical works in this subject are [9], [10], [11], [15].

The aim of this article is to give boundedness results for the fractional integral of (1), which we will write as $\mathcal{L}^{-\alpha/4}$. In order to do that, we studied the heat kernel of the bi-harmonic operator, among other kernels, obtaining some useful estimates. These estimates allowed us to categorize $\mathcal{L}^{-\alpha/4}$ into the family of operators given in [5] and in [14], respectively and use the results stated in those articles. The estimates for the kernels are presented in Section 3.

Next, we will state the central results of this work. Theorem 1 and Theorem 3 give boundedness results for $\mathcal{L}^{-\alpha/4}$ which are analogous to the ones stated in Theorem 1.4 in [14] and Theorem 1 in [2], respectively. On the other hand, Theorem 2 deals with the limit case $p = \frac{d}{\alpha}$, in weighted Lebesgue and BMO-type spaces.

Theorem 1. *The fractional integral operator $\mathcal{L}^{-\alpha/4}$ is bounded from BMO_ρ^β to $BMO_\rho^{\alpha+\beta}$, for $0 < \beta < 1$ such that $\alpha + \beta < \min\{1, \delta\}$ and from BMO_ρ to BMO_ρ^α, if $\alpha < \min\{1, \delta\}$.*

Theorem 2. *The fractional integral operator $\mathcal{L}^{-\alpha/4}$ is bounded from $L^{d/\alpha}(w^{d/\alpha})$ into $BMO_\rho(w)$, for every w such that $w^{d/(d-\alpha)} \in A_1^\rho$.*

Theorem 3. *Let $0 < \alpha < d$, $\frac{d}{\alpha} \leq p < \frac{d}{(\alpha-\delta)^+}$ and $w \in RH_{p'} \cap D_\eta$, where $1 \leq \eta < 1 - \frac{\alpha}{d} + \frac{\delta}{d} + \frac{1}{p}$, then the operator $\mathcal{L}^{-\alpha/4}$ is bounded from $L^{p,\infty}(w)$ into $BMO_\rho^{\alpha-d/p}(w)$.*

2 Definitions and auxiliary results

In this section, we will give some important definitions and intermidiate results to fully understand the theorems stated in the previous section.

As in the study of the theory associated to $-\Delta + V$, we will use the well known critical radius function

$$\rho(x) = \sup\left\{r > 0 : \frac{1}{r^{d-2}}\int_{B(x,r)} V \leq 1\right\}, \quad x \in \mathbb{R}^d, \tag{3}$$

IMAL PREPRINT # 2023-0067

ISSN 2451-7100
Publication date: October 23, 2023

which, under our the assumption given in (2), it is easy to check $0 < \rho(x) < \infty$ (see [15]).

The following propositions will be useful for the results to come.

Proposition 4 (See Lemma 1.4 in [15]). *There exist positive constants c_0 and k_0 such that*

$$c_0^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \;\leq\; \rho(y) \;\leq\; c_0\,\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}},$$

for all $x, y \in \mathbb{R}^d$.

Note that in the particular case where $|x-y| \leq \rho(x)$, we have $\rho(x) \simeq \rho(y)$.

In the next lemma and in the rest of this article we shall denote $\delta = 2 - d/q$. Notice that, under the assumption $q > d/2$ we have $\delta > 0$.

Lemma 5. *For $r > 0$ and $x \in \mathbb{R}^d$, there exist constants C and $\eta > 0$ such that*

i) if $r \leq \rho(x)$,

$$\int_{B(x,r)} [V(y)]^2 dy \;\leq\; C\left(\frac{r}{\rho(x)}\right)^{2\delta} r^{d-4};$$

ii) if $r > \rho(x)$,

$$\int_{B(x,r)} [V(y)]^2 dy \;\leq\; C\left(\frac{r}{\rho(x)}\right)^{\eta} [\rho(x)]^{d-4},$$

where η depends on the dimension d and the constant in (2).

The proof for part i) of the previous lemma can be found in [6] (see Lemma 2.6), whereas part ii) can be proved similarly as Lemma 1.8 in [15], by making a few obvious adaptations.

A function w defined on $\mathbb{R}^d$ is called rapidly decaying if for every $N > 0$ there exists a constant C_N such that

$$|w(x)| \leq C_N(1+|x|)^{-N}.$$

Corollary 6. *Let w be a rapidly decaying non-negative function, then there exist constants C and $c > 0$ such that*

$$\int_{\mathbb{R}^d} [V(y)]^2 w_t(x-y)dy \;=\; \begin{cases} \frac{C}{t}\left(\frac{\sqrt[4]{t}}{\rho(x)}\right)^{2\delta} & \text{if } t \leq \rho^4(x), \\ Ct^{-d/4}\left(\frac{\sqrt[4]{t}}{\rho(x)}\right)^{c}[\rho(x)]^{d-4} & \text{if } t > \rho^4(x), \end{cases}$$

where $w_t(x) = t^{-d/4}w(x/\sqrt[4]{t})$.

A proof for the case $t \leq \rho^4(x)$ can be found in [6] (see Lemma 2.7) and the other case can be proved in the same way, using part ii) of Lemma 5. Note that, although the result in the mentioned article was proved for the particular case where $w(x) = |x|^{4/3}$, it can be proved for any rapidly decaying function w as well.

We shall denote the kernels of the heat semi-groups $e^{-t\mathcal{L}}$ and $e^{-t(-\Delta)^2}$, $k_t(x,y)$ and $h_t(x,y)$, respectively, and $q_t(x,y) = k_t(x,y) - h_t(x,y)$, for $t > 0$ and $x, y \in \mathbb{R}^d$. It

ISSN 2451-7100
Publication date: October 23, 2023

is known (see [6]) that q_t can be written as

$$q_t(x,y) \; = \; \int_0^t \int_{\mathbb{R}^d} h_s(x,z)[V(z)]^2 k_{t-s}(z,y)dzds. \tag{4}$$

The fractional integral operator of order $\alpha > 0$, associated to $\mathcal{L}$ can be expressed in terms of its heat semigroup as

$$\mathcal{L}^{-\alpha/4} f(x) = \int_0^\infty e^{-t\mathcal{L}} f(x) t^{\alpha/4} \frac{dt}{t}.$$

Also, we can write this operator in terms of the expression for the heat semigroup $e^{-t\mathcal{L}}$ as follows

$$\mathcal{L}^{-\alpha/4} f(x) = \int_0^\infty \int_{\mathbb{R}^d} k_t(x,y) f(y) t^{\alpha/4} dy \frac{dt}{t} = \int_{\mathbb{R}^d} K_\alpha(x,y) f(y) dy,$$

where

$$K_\alpha(x,y) = \int_0^\infty k_t(x,y) t^{\alpha/4} \frac{dt}{t}.$$

Throughout this paper, we will focus on $\mathcal{L}^{-\alpha/4}$ with α in the range $0 < \alpha < d$.

Next, we will give some definitions for classes of weights. These weights w are non-negative, locally integrable functions. The following classes depending on the critical radius function ρ, were defined in [4].

Given $p > 1$, a weight w belongs to the class $A_p^{\rho,\theta}$, for some $\theta \geq 0$ if there exists a constant C such that

$$\frac{1}{|B|} \left(\int_B w \right)^{1/p} \left(\int_B w^{-\frac{1}{p-1}} \right)^{1/p'} \; \leq \; C \left(1 + \frac{r}{\rho(x)} \right)^\theta,$$

for every ball $B = B(x,r)$.

As for the case $p = 1$, a weight w belongs to the class $A_1^{\rho,\theta}$, for some $\theta \geq 0$ if there exists a constant C such that

$$\frac{1}{|B|} \int_B w \; \leq \; C \left(1 + \frac{r}{\rho(x)} \right)^\theta \inf_B w,$$

for every ball $B = B(x,r)$. In this last inequality, the infimum is the essential infimum with respect to the Lebesgue measure.

The classes A_p^ρ, for $p \geq 1$ are defined as $A_p^\rho = \bigcup_{\theta \geq 0} A_p^{\rho,\theta}$.

Given a critical radius function ρ, for $q > 1$, $RH_q^\rho = \bigcup_{\theta \geq 0} RH_q^{\rho,\theta}$, where $RH_q^{\rho,\theta}$ is the class of weights w such that there exists a constant C such that

$$\left(\frac{1}{|B|} \int_B w^q \right)^{1/q} \leq C \left(\frac{1}{|B|} \int_B w \right) \left(1 + \frac{r}{\rho(x)} \right)^\theta,$$

IMAL PREPRINT # 2023-0067

ISSN 2451-7100
Publication date: October 23, 2023

for every ball $B = B(x,r)$.

Given a critical radius function ρ, for $\eta \geq 1$, $D_\eta^\rho = \bigcup_{\theta\geq 0} D_\eta^{\rho,\theta}$, where $D_\eta^{\rho,\theta}$ is the class of weights w such that there exists a constant C such that

$$w(tB) \leq Ct^{d\eta} w(B) \left(1 + \frac{r}{\rho(x)}\right)^\theta,$$

for every ball $B = B(x,r)$ and $t \geq 1$.

Notice that in both classes of weights, if $\theta = 0$, we obtain the classic reverse Hölder RH_q and doubling D_η spaces, respectively. This gives us the inclusions $RH_q \subseteq RH_q^\rho$ and $D_\eta \subseteq D_\eta^\rho$.

For $p > 1$ and a weight w, $L^{p,\infty}(w)$ is the space of measurable functions f such that

$$[f]_{p,w} = \left(\sup_{t>0} t^p \left|\left\{x : \frac{|f(x)|}{w(x)} > t\right\}\right|\right)^{1/p} < \infty.$$

Analogously, the weighted Lebesgue spaces $L^p(w)$ are defined as the set of measureable functions f such that

$$\|f\|_{L^p(w)} = \int_{\mathbb{R}^d} \left|\frac{f(x)}{w(x)}\right|^p dx < \infty.$$

The estimates for the fractional integral of $\mathcal{L}$ are given in certain BMO-type spaces, which were first introduced in [3]. The so called $BMO_\rho^\gamma(w)$ spaces, for $\gamma \geq 0$ and a weight w, are defined as the set of locally integrable functions f in $\mathbb{R}^d$, satisfying the following conditions

$$\int_B |f - f_B| \leq Cw(B)|B|^{\gamma/d}, \quad \text{with } f_B = \frac{1}{|B|}\int_B f, \tag{5}$$

and

$$\int_B |f| \leq Cw(B)|B|^{\gamma/d}, \quad \text{if } R \geq \rho(x), \tag{6}$$

for every ball $B = B(x,R)$, with $x \in \mathbb{R}^d$ and $R > 0$. If $\gamma = 0$, we denote the space by $BMO_\rho(w)$ and if $w = 1$, we shall write BMO_ρ^γ.

3 Estimates for the kernels

In this section we give smoothness estimates, first for the function q_t and then for the kernel k_t, as a consequence of the first one. These estimates are interesting in itself and also will allow us to show the main results in this work.

Regardless of the many technical steps contained in the proofs of these estimates, we decided to include them, hoping it will help the reader. We start presenting some known estimates of the size of h_t, which is controlled exponentially. In fact (see section 5.2 in [8]), there exist constants C and $c > 0$, such that

$$|h_t(x,y)| \;\leq\; Ct^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}. \tag{7}$$

ISSN 2451-7100
Publication date: October 23, 2023

As to k_t, in [6] (see Theorem 2.5 there) the authors showed that for every $N \in \mathbb{N}$, there exist constants C_N and $c > 0$, where c is independent of N, such that

$$|k_t(x,y)| \;\le\; C_N t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \left(1 + \frac{\sqrt{t}}{\rho^2(x)} + \frac{\sqrt{t}}{\rho^2(y)}\right)^{-N}. \tag{8}$$

Also, in that same work (see Theorem 2.8 there), it is proved that there exist constants C and $c > 0$ such that

$$|q_t(x,y)| \;\le\; C t^{-d/4} \left(\frac{\sqrt[4]{t}}{\rho(x)}\right)^{2\delta} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}. \tag{9}$$

Next, we will give some smoothness estimates. By the Mean Value Theorem and making use of size estimates for the derivatives of the bi-harmonic heat kernel, which can be found in [13], we get the following smoothness estimate for h_t.

Lemma 7. *There exist constants C and $c > 0$, such that*

$$|h_t(x+h,y) - h_t(x,y)| \le C|h| t^{-\frac{d+1}{4}} \left(1 + \frac{|x-y|}{\sqrt[4]{t}}\right)^{-\frac{d-1}{3}} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}, \tag{10}$$

for every $h, x, y \in \mathbb{R}^d$ such that $|h| \le |x-y|/4$.

Lemma 8. *Given ε, $0 < \varepsilon < \min\{1, \delta\}$, there exist constants C and $c > 0$, such that*

$$|q_t(x,y+h) - q_t(x,y)| \le C \left(\frac{|h|}{\rho(x)}\right)^{\varepsilon} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}, \tag{11}$$

for every $h, x, y \in \mathbb{R}^d$ such that $|h| \le |x-y|/4$, $|h| < \tilde{C}\rho(y)$, for some constant $\tilde{C} > 0$, with c independent of ε.

Proof. We will only show the result in case $|h| \le \rho(x)$. Otherwise, the statement easily follows from estimates (7) and (8).

Because of the simetry of q_t and expression (4), we have

$$\begin{aligned}
|q_t(x,y+h) - q_t(x,y)| \;&\le\; \int_0^t \int_{\mathbb{R}^d} |h_s(y+h,z) - h_s(y,z)|\; [V(z)]^2 k_{t-s}(z,x)\,dz\,ds \\
&= \left(\int_0^{t/2}\int_{\mathbb{R}^d} + \int_{t/2}^t \int_{\mathbb{R}^d}\right) |h_s(y+h,z) - h_s(y,z)|\; [V(z)]^2 k_{t-s}(z,x)\,dz\,ds \\
&= A + B.
\end{aligned}$$

ISSN 2451-7100
Publication date: October 23, 2023

We will start with A. Using (8), we can obtain another estimate for the integrand. Then we will split the integral as shown below. For each $N > 0$,

$$\begin{aligned} A &\le C_N \int_0^{t/2} \int_{\mathbb{R}^d} |h_s(y+h,z) - h_s(y,z)| \, [V(z)]^2 \\ &\quad \times \left(1 + \frac{\sqrt{t-s}}{\rho^2(x)} + \frac{\sqrt{t-s}}{\rho^2(z)}\right)^{-N} (t-s)^{-d/4} e^{-\frac{c|z-x|^{4/3}}{(t-s)^{1/3}}} dzds \\ &= C_N \left(\int_0^{t/2} \int_{4|h|<|z-y|\le|x-y|/2} + \int_0^{t/2} \int_{\substack{|z-y|\le 4|h| \\ |z-y|\le|x-y|/2}} + \int_0^{t/2} \int_{|z-y|>|x-y|/2} \right) \\ &\quad |h_s(y+h,z) - h_s(y,z)| \, [V(z)]^2 \left(1 + \frac{\sqrt{t-s}}{\rho^2(x)} + \frac{\sqrt{t-s}}{\rho^2(z)}\right)^{-N} \\ &\quad \times (t-s)^{-d/4} e^{-\frac{c|z-x|^{4/3}}{(t-s)^{1/3}}} dzds \\ &= A_1 + A_2 + A_3. \end{aligned}$$

Now, applying (7), we have

$$\begin{aligned} A_1 &\le C_N |h| \left(1 + \frac{\sqrt{t}}{\rho^2(x)}\right)^{-N} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \\ &\quad \times \int_0^t s^{-1/4} \int_{4|h|<|z-y|\le|x-y|/2} s^{-d/4} e^{-\frac{c|z-y|^{4/3}}{s^{1/3}}} [V(z)]^2 dzds \\ &\le C_N |h|^{\varepsilon} \left(1 + \frac{\sqrt{t}}{\rho^2(x)}\right)^{-N} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \\ &\quad \times \int_0^t \int_{\mathbb{R}^d} s^{-\frac{d}{4}-\frac{\varepsilon}{4}} e^{-\frac{c|z-y|^{4/3}}{s^{1/3}}} \left(\frac{|z-y|}{\sqrt[4]{s}}\right)^{1-\varepsilon} [V(z)]^2 dzds \qquad (12) \\ &= C_N |h|^{\varepsilon} \left(1 + \frac{\sqrt{t}}{\rho^2(x)}\right)^{-N} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \\ &\quad \times \left(\int_0^{\rho^4(y)} \int_{\mathbb{R}^d} + \int_{\rho^4(y)}^t \int_{\mathbb{R}^d} \right) \\ &\qquad s^{-\frac{d}{4}-\frac{\varepsilon}{4}} e^{-\frac{c|z-y|^{4/3}}{s^{1/3}}} \left(\frac{|z-y|}{\sqrt[4]{s}}\right)^{1-\varepsilon} [V(z)]^2 dzds. \end{aligned}$$

ISSN 2451-7100
Publication date: October 23, 2023

For the first integral, by using Corollary 6, we get

$$\begin{aligned}\int_0^{\rho^4(y)} s^{-\frac{\varepsilon}{4}} \int_{\mathbb{R}^d} s^{-\frac{d}{4}} e^{-\frac{c|z-y|^{4/3}}{s^{1/3}}} \left(\frac{|z-y|}{\sqrt[4]{s}}\right)^{1-\varepsilon} [V(z)]^2 dz ds \\ \leq C_N \int_0^{\rho^4(y)} s^{-\frac{\varepsilon}{4}} \left(\frac{\sqrt[4]{s}}{\rho(y)}\right)^{2\delta} \frac{ds}{s} \\ = C_N \rho(y)^{-\varepsilon}.\end{aligned} \tag{13}$$

As to the second integral, also using Corollary 6, we have

$$\begin{aligned}\int_{\rho^4(y)}^{t} s^{-\frac{\varepsilon}{4}} \int_{\mathbb{R}^d} s^{-\frac{d}{4}} e^{-\frac{c|z-y|^{4/3}}{s^{1/3}}} \left(\frac{|z-y|}{\sqrt[4]{s}}\right)^{1-\varepsilon} [V(z)]^2 dz ds \\ \leq C_N \int_{\rho^4(y)}^{t} s^{-\frac{\varepsilon}{4}-\frac{d}{4}} \left(\frac{\sqrt[4]{s}}{\rho(y)}\right)^{C} [\rho(y)]^{d-4} ds \\ \leq C_N \left(\frac{\sqrt[4]{t}}{\rho(y)}\right)^{C} [\rho(y)]^{d-4} \int_{\rho^4(y)}^{\infty} s^{-\frac{\varepsilon}{4}-\frac{d}{4}} ds \\ \leq C_N \rho(y)^{-\varepsilon} \left(1+\frac{\sqrt{t}}{\rho^2(y)}\right)^{\frac{C}{2}}.\end{aligned} \tag{14}$$

Combining results from (12), (13) and (14), yields

$$A_1 \leq C_N \left(\frac{|h|}{\rho(y)}\right)^{\varepsilon} \left(1+\frac{\sqrt{t}}{\rho^2(x)}\right)^{-N+\frac{C}{2}} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}.$$

Here, we have assumed $\rho^4(y) \leq t$. In case $t < \rho^4(y)$ it is enough to bound the integral in the interval $0 < s \leq \rho^4(y)$.

If $|x-y| \leq \rho(x)$, by Proposition 4, we have $\rho(x) \simeq \rho(y)$, so we get to the estimate by choosing N large enough. Otherwise, if $|x-y| > \rho(x)$, also using Proposition 4, we have

$$\begin{aligned}A_1 &\leq C_N \left(\frac{|h|}{\rho(x)}\right)^{\varepsilon} \left(1+\frac{\sqrt{t}}{\rho^2(x)}\right)^{-N+\frac{C}{2}} \left(1+\frac{|x-y|}{\rho(x)}\right)^{k_0\varepsilon} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \\ &\leq C_N \left(\frac{|h|}{\rho(x)}\right)^{\varepsilon} \left(1+\frac{\sqrt{t}}{\rho^2(x)}\right)^{-N+\frac{C}{2}} \left(\frac{|x-y|}{\rho(x)}\right)^{k_0\varepsilon} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \\ &\leq C_N \left(\frac{|h|}{\rho(x)}\right)^{\varepsilon} \left(1+\frac{\sqrt{t}}{\rho^2(x)}\right)^{\frac{C}{2}+\frac{k_0\varepsilon}{2}-N} \left(\frac{|x-y|}{\sqrt[4]{t}}\right)^{k_0\varepsilon} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \\ &\leq C_N \left(\frac{|h|}{\rho(x)}\right)^{\varepsilon} \left(1+\frac{\sqrt{t}}{\rho^2(x)}\right)^{\frac{C}{2}+\frac{k_0\varepsilon}{2}-N} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}.\end{aligned} \tag{15}$$

ISSN 2451-7100
Publication date: October 23, 2023

again, obtaining the wanted bound by choosing N large enough.

Next, we will deal with A_2. Using (7), we get

$$\begin{aligned} A_2 \ \le\ & C_N \left(1+\frac{\sqrt{t}}{\rho^2(x)}\right)^{-N} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{s^{1/3}}} \\ & \times \int_0^{t/2} \int_{\substack{|z-y|\le 4|h| \\ |z-y|\le |x-y|/2}} s^{-d/4} (e^{-\frac{c|z-y|^{4/3}}{s^{1/3}}} + e^{-\frac{c|z-y-h|^{4/3}}{s^{1/3}}}) [V(z)]^2 dz ds. \end{aligned} \tag{16}$$

To bound the integral with the first term in (16), we make a change of variables, separate the integration region into dyadic anulli and use Lemma 5, obtaining

$$\begin{aligned} & \int_{\substack{|z-y|\le 4|h| \\ |z-y|\le |x-y|/2}} [V(z)]^2 \int_0^{t/2} s^{-d/4} e^{-\frac{c|z-y|^{4/3}}{s^{1/3}}} ds dz \\ & \le \int_{\substack{|z-y|\le 4|h| \\ |z-y|\le |x-y|/2}} |z-y|^{-d+4} [V(z)]^2 \int_0^\infty u^{\frac{3}{4}d-4} e^{-u} du dz \\ & \le \sum_{k=0}^\infty \int_{4^{-k-1}|h|<|z-y|/4\le 4^{-k}|h|} |z-y|^{-d+4} [V(z)]^2 dz \\ & \le \sum_{k=0}^\infty (4^{-k-1}|h|)^{-d+4} \int_{|z-y|/4\le 4^{-k}|h|} [V(z)]^2 dz \\ & \le \sum_{k=0}^\infty \left(\frac{4^{-k+1}|h|}{\rho(y)}\right)^{2\delta} = C\left(\frac{|h|}{\rho(y)}\right)^{2\delta}. \end{aligned} \tag{17}$$

Given that $\delta > 0$, this last series converges.

Note that if $|z-y| \le 4|h|$, then $|z-y-h| \le 5|h|$, so we can bound the integral with the second term in (16) in the same way as the first one.

From equations (16) and (17), we get

$$A_2 \ \le\ C_N \left(\frac{|h|}{\rho(y)}\right)^{2\delta} \left(1+\frac{\sqrt{t}}{\rho^2(x)}\right)^{-N} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{s^{1/3}}}.$$

Once again, replying the strategy used in equation (15), we get to the required estimate.

Now, we will deal with A_3. We know that $|h| \le \frac{|x-y|}{4} \le \frac{|z-y|}{2}$ thereby, we can use the smoothness estimate for the kernel of the bilaplacian operator, given in Lemma (7), obtaining

$$\begin{aligned} A_3 \ \le\ & C_N |h|^\varepsilon \left(1+\frac{\sqrt{t}}{\rho^2(x)}\right)^{-N} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{s^{1/3}}} \\ & \times \int_0^t s^{-\varepsilon/4} \int_{\mathbb{R}^d} s^{-d/4} e^{-\frac{c|z-y|^{4/3}}{s^{1/3}}} \left(\frac{|y-z|}{\sqrt[4]{s}}\right)^{1-\varepsilon} [V(z)]^2 dz ds. \end{aligned}$$

ISSN 2451-7100
Publication date: October 23, 2023

Which is the same bound we found for A_1 in (12). Therefore, we have

$$A_3 \ \leq \ C_N \left(\frac{|h|}{\rho(x)}\right)^{\varepsilon} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}.$$

Now we will move on to bound integral B. From using estimates (7), (8) and splitting the integral, we get

$$\begin{aligned} B \ &\leq \ C_N \int_{t/2}^{t} \int_{\mathbb{R}^d} |h| s^{-\frac{d+1}{4}} e^{-\frac{c|z-\xi|^{4/3}}{s^{1/3}}} \, [V(z)]^2 \\ &\quad \times \left(1 + \frac{\sqrt{t-s}}{\rho^2(x)} + \frac{\sqrt{t-s}}{\rho^2(z)}\right)^{-N} (t-s)^{-d/4} e^{-\frac{c|z-x|^{4/3}}{(t-s)^{1/3}}} dz ds \\ &= \int_{t/2}^{t} \int_{|z-y|\leq|x-y|/2} + \int_{t/2}^{t} \int_{|z-y|>|x-y|/2} \\ &= B_1 + B_2. \end{aligned}$$

We will start with B_1. After a change of variables,

$$\begin{aligned} B_1 \ &\leq \ C_N |h| t^{-\frac{d+1}{4}} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \\ &\quad \times \int_0^t \left(1 + \frac{\sqrt{s}}{\rho^2(x)}\right)^{-N} \int_{\mathbb{R}^d} s^{-d/4} e^{-\frac{c|z-x|^{4/3}}{s^{1/3}}} [V(z)]^2 dz ds \\ &\leq \ C_N |h|^{\varepsilon} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \left(\frac{|x-y|}{\sqrt[4]{t}}\right)^{1-\varepsilon} \\ &\quad \times \int_0^t \int_{\mathbb{R}^d} \left(1 + \frac{\sqrt{s}}{\rho^2(x)}\right)^{-N} s^{-d/4-\frac{\varepsilon}{4}} e^{-\frac{c|z-x|^{4/3}}{s^{1/3}}} [V(z)]^2 dz ds \qquad (18) \\ &\leq \ C_N |h|^{\varepsilon} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \left(\frac{|x-y|}{\sqrt[4]{t}}\right)^{1-\varepsilon} \\ &\quad \times \left(\int_0^{\rho^4(x)} \int_{\mathbb{R}^d} + \int_{\rho^4(x)}^{t} \int_{\mathbb{R}^d}\right) \\ &\qquad \left(1 + \frac{\sqrt{s}}{\rho^2(x)}\right)^{-N} s^{-d/4-\frac{\varepsilon}{4}} e^{-\frac{c|z-x|^{4/3}}{s^{1/3}}} [V(z)]^2 dz ds. \end{aligned}$$

For the first integral, using Corollary 6, we get

$$\begin{aligned} \int_0^{\rho^4(x)} s^{-\frac{\varepsilon}{4}} \int_{\mathbb{R}^d} s^{-d/4} e^{-\frac{c|z-x|^{4/3}}{s^{1/3}}} [V(z)]^2 dz ds \ &\leq \ \int_0^{\rho^4(x)} s^{-\frac{\varepsilon}{4}} \left(\frac{\sqrt[4]{s}}{\rho(x)}\right)^{2\delta} \frac{ds}{s} \qquad (19) \\ &\leq \ C_N \rho(x)^{-\varepsilon}. \end{aligned}$$

As to the second integral, also using Corollary 6, we have

ISSN 2451-7100
Publication date: October 23, 2023

$$\begin{aligned}\int_{\rho^4(x)}^{t}\left(1+\frac{\sqrt{s}}{\rho^2(x)}\right)^{-N} s^{-\frac{\varepsilon}{4}}\int_{\mathbb{R}^d} s^{-d/4}e^{-\frac{c|z-x|^{4/3}}{s^{1/3}}}[V(z)]^2dzds \\ \le \int_{\rho^4(x)}^{t}\left(1+\frac{\sqrt{s}}{\rho^2(x)}\right)^{-N+\frac{C}{2}} s^{-\frac{\varepsilon}{4}-\frac{d}{4}}[\rho(x)]^{d-4}ds \\ \le [\rho(x)]^{d-4}\int_{\rho^4(x)}^{\infty} s^{-\frac{\varepsilon}{4}-\frac{d}{4}}ds \\ \le C\rho(x)^{-\varepsilon}.\end{aligned} \tag{20}$$

Here, we have chosen N large enough. From equations (18), (19) and (20), we reach the following estimate for B_1

$$B_1 \le C_N\left(\frac{|h|}{\rho(x)}\right)^{\varepsilon} t^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}\left(\frac{|x-y|}{\sqrt[4]{t}}\right)^{1-\varepsilon} \le C_N\left(\frac{|h|}{\rho(x)}\right)^{\varepsilon} t^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}.$$

Here, we have assumed $\rho^4(y) \le t$. In case $t < \rho^4(y)$ it is enough to bound the integral in the interval $0 < s \le \rho^4(y)$.

Finally, we will deal with B_2. Making use of equation (7), we can get to the same bound that we obtained for B_1.

$$\begin{aligned}B_2 \le C_N|h|t^{-\frac{d+1}{4}}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \\ \times\int_0^t\left(1+\frac{\sqrt{s}}{\rho^2(x)}\right)^{-N}\int_{\mathbb{R}^d} s^{-d/4}e^{-\frac{c|z-x|^{4/3}}{s^{1/3}}}[V(z)]^2dzds.\end{aligned}$$

Which means we can bound B_2 just as we bounded B_1. □

Proposition 9. *Given ε, $0<\varepsilon<\min\{1,\delta\}$ and $M>0$, there exist constants C and $c>0$ such that for $|h|<\sqrt[4]{t}$, we have*

$$|k_t(x,y+h)-k_t(x,y)| \le C\left(\frac{|h|}{\sqrt[4]{t}}\right)^{\varepsilon} t^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}\left(1+\frac{\sqrt{t}}{\rho^2(x)}+\frac{\sqrt{t}}{\rho^2(y)}\right)^{-M},$$

where c is independent of ε and M.

Proof. We shall begin by proving the result for the case $\sqrt[4]{\frac{t}{2}} \le |h| < \sqrt[4]{t}$. According to Proposition 4, there exist constants $C>0$ and $k_0>0$ such that

$$\frac{1}{\rho^2(y+h)} \ge \frac{C}{\rho^2(y)}\left(1+\frac{\sqrt{t}}{\rho^2(x)}+\frac{\sqrt{t}}{\rho^2(y)}\right)^{-\frac{k_0}{1+k_0}},$$

ISSN 2451-7100
Publication date: October 23, 2023

thus,

$$\begin{aligned}
&\left(1+\frac{\sqrt{t}}{\rho^2(x)}+\frac{\sqrt{t}}{\rho^2(y+h)}\right)^{-N} \\
&\qquad \leq C\left(1+\frac{\sqrt{t}}{\rho^2(x)}+\frac{\sqrt{t}}{\rho^2(y)}\left(1+\frac{\sqrt{t}}{\rho^2(x)}+\frac{\sqrt{t}}{\rho^2(y)}\right)^{\frac{k_0}{1+k_0}}\right)^{-N} \\
&\qquad \leq C\left(1+\frac{\sqrt{t}}{\rho^2(x)}+\frac{\sqrt{t}}{\rho^2(y)}\right)^{-N\varepsilon},
\end{aligned}$$

where $\varepsilon = \dfrac{1}{1+k_0}$.

Using this last inequality and (8), we obtain

$$|k_t(x,y+h)-k_t(x,y)| \leq Ct^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}\left(1+\frac{\sqrt{t}}{\rho^2(x)}+\frac{\sqrt{t}}{\rho^2(y)}\right)^{-N\varepsilon}.$$

Given that $|h| \simeq \sqrt[4]{t}$ in this case, we can easily reach the result.

On the other hand, if $|h| \leq \dfrac{|x-y|}{4}$, also using (8) we get

$$|k_t(x,y+h)-k_t(x,y)| \;\leq\; Ct^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}\left(1+\frac{\sqrt{t}}{\rho^2(x)}+\frac{\sqrt{t}}{\rho^2(y)}\right)^{-M}. \tag{21}$$

If $|h| > \rho(y)$, for $L = M - \frac{\varepsilon}{2}$,

$$\begin{aligned}
|k_t(x,y+h)-k_t(x,y)| \;&\leq\; Ct^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}\left(1+\frac{\sqrt{t}}{\rho^2(x)}+\frac{\sqrt{t}}{\rho^2(y)}\right)^{-L}\left(\frac{\rho(y)}{\sqrt[4]{t}}\right)^{\varepsilon} \\
&\leq\; C\left(\frac{|h|}{\sqrt[4]{t}}\right)^{\varepsilon} t^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}\left(1+\frac{\sqrt{t}}{\rho^2(x)}+\frac{\sqrt{t}}{\rho^2(y)}\right)^{-L}.
\end{aligned}$$

On the contrary, if $|h| \leq \rho(y)$, using equation (7) and Lemma 8,

$$\begin{aligned}
|k_t(x,y+h)-k_t(x,y)| \;&\leq\; |h_t(x,y+h)-h_t(x,y)| + |q_t(x,y+h)-q_t(x,y)| \\
&\leq\; C\frac{|h|}{\sqrt[4]{t}}\, t^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \;+\; C\left(\frac{|h|}{\sqrt[4]{t}}\right)^{\varepsilon}\left(\frac{\sqrt{t}}{\rho^2(x)}\right)^{\varepsilon/2} t^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \\
&\leq\; C\left(\frac{|h|}{\sqrt[4]{t}}\right)^{\varepsilon} t^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}\left(1+\frac{\sqrt{t}}{\rho^2(x)}\right)^{\varepsilon/2}.
\end{aligned}$$

This last inequality combined with (21) give us the result.

ISSN 2451-7100
Publication date: October 23, 2023

At last, we need to show the result for $\frac{|x-y|}{4} < |h| \leq \sqrt[4]{\frac{t}{2}}$. Using the semi-group property, we get the following integral representation for the difference

$$\begin{aligned} |k_t(x, y+h) - k_t(x,y)| &\leq \int_{\mathbb{R}^d} k_{t/2}(x,z)|k_{t/2}(z,y+h) - k_{t/2}(z,y)|dz \\ &= \int_{|z-y|\leq 4|h|} + \int_{|z-y|>4|h|} = S_1 + S_2. \end{aligned}$$

First, we will bound S_1. Using (8), we have

$$\begin{aligned} S_1 &\leq Ct^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \left(\frac{1}{\sqrt[4]{t}}\right)^d \left(1 + \frac{\sqrt{t}}{\rho^2(x)}\right)^{-N} \int_{|z-y|\leq 4|h|} dz \\ &\leq Ct^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \left(\frac{|h|}{\sqrt[4]{t}}\right)^\varepsilon \left(1 + \frac{\sqrt{t}}{\rho^2(x)}\right)^{-N}. \end{aligned} \tag{22}$$

As the result has already been proved for the case $|h| < \frac{|z-y|}{4}$, we shall use it, along with (8) to bound S_2, obtaining

$$\begin{aligned} S_2 &\leq C\int_{|z-y|>4|h|} k_{t/2}(x,z) \left(\frac{|h|}{\sqrt[4]{t}}\right)^\varepsilon t^{-d/4}e^{-\frac{c|z-y|^{4/3}}{t^{1/3}}} dz \\ &\leq C\left(\frac{|h|}{\sqrt[4]{t}}\right)^\varepsilon t^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \left(1 + \frac{\sqrt{t}}{\rho^2(x)}\right)^{-N} \int_{\mathbb{R}^d} t^{-d/4}e^{-\frac{c|x-z|^{4/3}}{t^{1/3}}} dz. \end{aligned} \tag{23}$$

Where the last integral is finite.

From equations (22) and (23), we get

$$|k_t(x,y+h) - k_t(x,y)| \leq C\left(\frac{|h|}{\sqrt[4]{t}}\right)^\varepsilon t^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \left(1 + \frac{\sqrt{t}}{\rho^2(x)}\right)^{-N}.$$

Finally, using the fact that $|h| \leq \sqrt[4]{\frac{t}{2}}$ and following the same steps we used before, we can show

$$\left(1 + \frac{\sqrt{t}}{\rho^2(x)}\right)^{-N} \leq C\left(1 + \frac{\sqrt{t}}{\rho^2(x)} + \frac{\sqrt{t}}{\rho^2(y)}\right)^{-N\varepsilon}.$$

Which concludes the proof of the result. □

4 Proof of Theorem 1

In order to prove the central results of this work, we will need to study some families of fractional integral operators concerning a critical radius function, which were first introduced in [1]. In that article, the authors define the spaces $S(\rho,\infty,\gamma)$, along with

ISSN 2451-7100
Publication date: October 23, 2023

other spaces of singular integral operators, called $S(\rho, s)$. For the purposes of this article, it will be relevant to state the definition of the spaces mentioned first.

Given a critical radius function ρ and γ, $0 \leq \gamma < d$, we say that a kernel K belongs to $S(\rho, \infty, \gamma)$ if it satisfies both of the following conditions.

i) For every $N > 0$ there exists a constant C_N such that

$$|K(x,y)| \leq \frac{C_N}{|x-y|^{d-\gamma}} \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N}, \tag{24}$$

for every $x, y \in \mathbb{R}^d$.

ii) There exist constants C and $\lambda > 0$ such that

$$|K(x,y) - K(z,y)| \leq C \frac{|x-z|^\lambda}{|x-y|^{d-\gamma+\lambda}}, \tag{25}$$

for every $x, y, z \in \mathbb{R}^d$ such that $|x-z| \leq |x-y|/2$.

On the other hand, we say that K belongs to $S_0(\rho, \infty, \gamma)$ if it satisfies equation (24) and a stronger smoothness condition stated below.

iii) For $M > 0$ and $0 < \lambda < 1$ there exists a constant C such that

$$|K(x,y) - K(z,y)| \leq C \frac{|x-z|^\lambda}{|x-y|^{d-\gamma+\lambda}} \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-M}, \tag{26}$$

for every $x, y, z \in \mathbb{R}^d$ such that $|x-z| \leq |x-y|/2$.

Next, we shall state the following lemma, which will be used in the proof of Theorem 1.

Lemma 10. *For every $0 < \varepsilon < \min\{1, \delta\}$, there exists a constant C_ε such that*

$$|k_t(x,y) - k_t(x,z)| \leq C_\varepsilon \frac{|y-z|^\varepsilon}{|x-y|^{d+\varepsilon}},$$

for $x, y, z \in \mathbb{R}^d$ such that $|x-y| > 2|y-z|$.

Proof. If $|y-z| \leq \sqrt[4]{t}$, using Proposition 9 we get

$$\begin{aligned} |k_t(x,y) - k_t(x,z)| &\leq C \left(\frac{|y-z|}{\sqrt[4]{t}}\right)^\varepsilon t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} \left(1 + \frac{\sqrt{t}}{\rho^2(x)} + \frac{\sqrt{t}}{\rho^2(y)}\right)^{-M} \\ &\leq C|y-z|^\varepsilon t^{-\varepsilon/4} \left(1 + \frac{|x-y|}{\sqrt[4]{t}}\right)^{-N} t^{-d/4} \\ &\leq C \frac{|y-z|^\varepsilon}{|x-y|^{d+\varepsilon}}, \end{aligned}$$

ISSN 2451-7100
Publication date: October 23, 2023

by choosing $N = d + \varepsilon$. Conversely, if $|y - z| > \sqrt[4]{t}$, using (8) and the fact that $|x - y| \simeq |x - z|$, we have

$$\begin{aligned} |k_t(x,y) - k_t(x,z)| &\leq Ct^{-d/4}[e^{-c\frac{|x-y|^{4/3}}{t^{1/3}}} + e^{-c\frac{|x-z|^{4/3}}{t^{1/3}}}] \\ &\leq Ct^{-d/4}e^{-c\frac{|x-y|^{4/3}}{t^{1/3}}} \\ &\leq C\left(\frac{|y-z|}{\sqrt[4]{t}}\right)^{\varepsilon} t^{-d/4}\left(1 + \frac{|x-y|}{\sqrt[4]{t}}\right)^{-N} \\ &\leq C\left(\frac{|y-z|}{\sqrt[4]{t}}\right)^{\varepsilon} t^{-d/4}\left(\frac{|x-y|}{\sqrt[4]{t}}\right)^{-(d+\varepsilon)} \\ &= C\frac{|y-z|^{\varepsilon}}{|x-y|^{d+\varepsilon}}. \end{aligned}$$

Here, we have chosen $N = d + \varepsilon$ again. □

Proposition 11. *K_α belongs to $S_0(\rho, \infty, \alpha)$.*

Proof. This result can be proved using the same techniques as in the proof of Proposition 8 in [1], using estimate (8) and Lemma 10. □

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. We will make use of Theorem 1.1 and Theorem 1.2 in [14], which give us $T1$-type conditions equivalent to equations (5) and (6). In order to do that, we need to prove that $\mathcal{L}^{-\alpha/4}$ satisfies the hypothesis of those theorems.

First, notice that by Proposition 11 and Theorem 4.2 in [6], along with the fact that the kernel K_α is symmetric (by the self-adjointness of $\mathcal{L}^{-\alpha/4}$), we can assure that $\mathcal{L}^{-\alpha/4}$ is an α-Schrödinger-Calderón-Zygmund operator with any regularity exponent ε such that $0 < \varepsilon < \min\{1, \delta\}$ (the definition for the families of operators previously mentioned can be found in [14]).

Now, it will be enough to show that there exist constants C_1 and C_2 such that the following inequalities hold.

$$\left(\frac{\rho(x_0)}{r}\right)^{\beta} \frac{1}{|B|^{1+\frac{\alpha}{d}}} \int_B |\mathcal{L}^{-\alpha/4}1(y) - (\mathcal{L}^{-\alpha/4}1)_B| dy \leq C_1, \tag{27}$$

$$\log\left(\frac{\rho(x_0)}{r}\right) \frac{1}{|B|^{1+\frac{\alpha}{d}}} \int_B |\mathcal{L}^{-\alpha/4}1(y) - (\mathcal{L}^{-\alpha/4}1)_B| dy \leq C_2, \tag{28}$$

for each ball $B = B(x_0, r)$, $x_0 \in \mathbb{R}^d$ and $0 < r \leq \frac{1}{2}\rho(x_0)$. Notice that by Proposition 11, K_α satisfies (24), so $\mathcal{L}^{-\alpha/4}1$ is well defined for $0 < \alpha < d$.

Although the proof is based on certain techniques used in [14], we will give all the details for a better understanding for the reader, since non-trivial details concerning the bi-harmonic Schrödinger operator may appear.

ISSN 2451-7100
Publication date: October 23, 2023

We shall first show that for these balls

$$\left|\int_{\mathbb{R}^d} k_t(x,y)dx - \int_{\mathbb{R}^d} k_t(x,z)dx\right| \leq C\left(\frac{r}{\rho(x_0)}\right)^{\varepsilon}, \tag{29}$$

for every ε such that $0<\varepsilon<\min\{1,\delta\}$.

Set $B=B(x_0,r)$, where $0<r\leq \frac{1}{2}\rho(x_0)$ and $y,z\in B$.

We will begin analyzing the case $\sqrt[4]{t}\leq 2r$. Since $\int_{\mathbb{R}^d} h_t(x,y)dx = \int_{\mathbb{R}^d} h_t(x,z)dx$ for all $x,y\in\mathbb{R}^d$ (see the expression of h_t in Section 2.2 in [13]), using (9) and the fact that $\rho(y)\simeq\rho(z)\simeq\rho(x_0)$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} k_t(x,y)dx - \int_{\mathbb{R}^d} k_t(x,z)dx\right| \\
&\quad\leq \left|\int_{\mathbb{R}^d} k_t(x,y)dx - \int_{\mathbb{R}^d} h_t(x,y)dx\right| + \left|\int_{\mathbb{R}^d} k_t(x,z)dx - \int_{\mathbb{R}^d} h_t(x,z)dx\right| \\
&\quad\leq \int_{\mathbb{R}^d} |q_t(x,y)|dx + \int_{\mathbb{R}^d} |q_t(x,z)|dx \\
&\quad\leq C\left(\frac{\sqrt[4]{t}}{\rho(x_0)}\right)^{2\delta} \int_{\mathbb{R}^d}\left(t^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} + t^{-d/4}e^{-\frac{c|x-z|^{4/3}}{t^{1/3}}}\right)dx \\
&\quad\leq C\left(\frac{\sqrt[4]{t}}{\rho(x_0)}\right)^{2\delta} \leq C\left(\frac{r}{\rho(x_0)}\right)^{2\delta}.
\end{aligned}$$

On the other hand, if $\sqrt[4]{t}>2r$ and $\sqrt[4]{t}>\rho(x_0)$, using Proposition 9, we get

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} k_t(x,y)dx - \int_{\mathbb{R}^d} k_t(x,z)dx\right| &\leq \int_{\mathbb{R}^d} |k_t(x,y)-k_t(x,z)|dx \\
&\leq C\left(\frac{|y-z|}{\sqrt[4]{t}}\right)^{\varepsilon} \int_{\mathbb{R}^d} t^{-d/4}e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}}dx \\
&\leq C\left(\frac{|y-z|}{\sqrt[4]{t}}\right)^{\varepsilon} \leq C\left(\frac{r}{\rho(x_0)}\right)^{\varepsilon}.
\end{aligned}$$

ISSN 2451-7100
Publication date: October 23, 2023

Finally, to deal with the case $2r < \sqrt[4]{t} \le \rho(x_0)$, we will split the domain of the integral in three, as shown below

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} k_t(x,y)dx - \int_{\mathbb{R}^d} k_t(x,z)dx\right| \\
&\quad = \left|\left(\int_{\mathbb{R}^d} k_t(x,y)dx - \int_{\mathbb{R}^d} h_t(x,y)dx\right) - \left(\int_{\mathbb{R}^d} k_t(x,z)dx - \int_{\mathbb{R}^d} h_t(x,z)dx\right)\right| \\
&\quad = \left|\int_{\mathbb{R}^d} (q_t(x,y) - q_t(x,z))dx\right| \\
&\quad = \left|\int_{|x-y|>C\rho(y)} + \int_{4|y-z|<|x-y|\le C\rho(y)} + \int_{|x-y|\le 4|y-z|}\right| \\
&\quad = |I + II + III|.
\end{aligned}$$

To bound the first integral, using Lemma 10, we have

$$\begin{aligned}
|I| &\le \int_{|x-y|>C\rho(y)} |k_t(x,y) - k_t(x,z)|dx \;\le\; C\int_{|x-y|>C\rho(y)} \frac{|y-z|^\varepsilon}{|x-y|^{d+\varepsilon}}dx \\
&\le Cr^\varepsilon \int_{|x-y|>C\rho(y)} \frac{dx}{|x-y|^{d+\varepsilon}} \;=\; C\left(\frac{r}{\rho(y)}\right)^\varepsilon \;\le\; C\left(\frac{r}{\rho(x_0)}\right)^\varepsilon.
\end{aligned}$$

For the second one, we will use Lemma 8, getting

$$\begin{aligned}
|II| &\le C\int_{4|y-z|<|x-y|\le C\rho(y)} \left(\frac{|y-z|}{\rho(y)}\right)^\varepsilon t^{-d/4} e^{-c\frac{|x-y|^{4/3}}{t^{1/3}}} dx \\
&\le C\left(\frac{|y-z|}{\rho(y)}\right)^\varepsilon \int_{4|y-z|<|x-y|\le C\rho(y)} t^{-d/4} e^{-c\frac{|x-y|^{4/3}}{t^{1/3}}} dx \;\le\; C\left(\frac{r}{\rho(x_0)}\right)^\varepsilon.
\end{aligned}$$

Lastly, to bound the third integral, we will use (9), obtaining

$$\begin{aligned}
|III| &\le C\left(\frac{\sqrt[4]{t}}{\rho(x_0)}\right)^{2\delta} \left(\int_{|x-y|\le 4|y-z|} t^{-d/4} e^{-\frac{c|x-y|^{4/3}}{t^{1/3}}} dx \right. \\
&\qquad \left. + \int_{|x-z|\le 5|y-z|} t^{-d/4} e^{-\frac{c|x-z|^{4/3}}{t^{1/3}}} dx\right) \\
&\le C\left(\frac{\sqrt[4]{t}}{\rho(x_0)}\right)^{2\delta} \int_{|\xi|\le 5\frac{|y-z|}{\sqrt[4]{t}}} t^{-d/4} e^{-\frac{c|\xi|^{4/3}}{t^{1/3}}} d\xi \\
&\le C\left(\frac{\sqrt[4]{t}}{\rho(x_0)}\right)^{2\delta} \left(\frac{|y-z|}{\sqrt[4]{t}}\right)^d \\
&\le C\frac{r^d}{[\rho(x_0)]^{2d}(\sqrt[4]{t})^{d-2\delta}} \;\le\; C\left(\frac{r}{\rho(x_0)}\right)^{2\delta}.
\end{aligned}$$

ISSN 2451-7100
Publication date: October 23, 2023

Hence, we have proved equation (29).

To conclude the proof we will bound the integrands from inequalities i) and ii).

$$\begin{aligned}\mathcal{L}^{-\alpha/4}1(y)-\mathcal{L}^{-\alpha/4}1(z) &= \int_0^\infty \int_{\mathbb{R}^d} (k_t(x,y)-k_t(x,z))dx\, t^{\alpha/4}\frac{dt}{t} \\ &= \left(\int_0^{\rho^4(x_0)} + \int_{\rho^4(x_0)}^\infty\right)\int_{\mathbb{R}^d} (k_t(x,y)-k_t(x,z))dx\, t^{\alpha/4}\frac{dt}{t} \\ &= I + II.\end{aligned}$$

For the first integral, using equation (29) and integrating, we get

$$|I| \le C\left(\frac{r}{\rho(x_0)}\right)^\varepsilon \int_0^{\rho^4(x_0)} t^{\alpha/4}\frac{dt}{t} = C\left(\frac{r}{\rho(x_0)}\right)^\varepsilon \rho^\alpha(x_0).$$

For the second integral, notice $r \le \frac{1}{2}\rho(x_0) \le \frac{1}{2}\sqrt[4]{t}$ in this region. Therefore, using equation (29), we have

$$\left|\int_{\mathbb{R}^d} k_t(x,y)dx - \int_{\mathbb{R}^d} k_t(x,z)dx\right| \le C\left(\frac{r}{\sqrt[4]{t}}\right)^\varepsilon.$$

Making use of this last inequality and then integrating, we obtain

$$|II| \le \int_{\rho^4(x_0)}^\infty t^{\alpha/4}\frac{dt}{t} = Cr^\varepsilon \int_{\rho^4(x_0)}^\infty t^{\frac{\alpha-\varepsilon}{4}}\frac{dt}{t} = C\left(\frac{r}{\rho(x_0)}\right)^\varepsilon \rho^\alpha(x_0).$$

So, we have proved that, for every $0<\varepsilon<\min\{1,\delta\}$, we have

$$|\mathcal{L}^{-\alpha/4}1(y)-\mathcal{L}^{-\alpha/4}1(z)| \le C\left(\frac{r}{\rho(x_0)}\right)^\varepsilon \rho^\alpha(x_0).$$

This last estimate proves both inequalities (27) and (28).

□

5 Proof of Theorem 2

Notice that, by estimate (8), the kernel K_α of the fractional integral $\mathcal{L}^{-\alpha/4}$ is bounded by the kernel of the classical fractional integral. This is

$$K_\alpha(x,y) \le \frac{C}{|x-y|^{d-\alpha}}. \tag{30}$$

As a consequence, the operator $\mathcal{L}^{-\alpha/4}$ is of weak type $(1, d/d-\alpha)$.

ISSN 2451-7100
Publication date: October 23, 2023

The next proposition is similar to Proposición 4.2.4 in [5]. However, the spaces of functions involved there differ from the ones we are considering in this work. Even though both proofs may look similar, they have some tecnichal differences.

Proposition 12. *Let T be an operator of weak type $(1, d/(d-\alpha))$ such that its kernel K belongs to $S(\rho, \infty, \alpha)$, with $0 < \alpha < d$. Then T is bounded from $L^{d/\alpha}(w^{d/\alpha})$ into $BMO_\rho(w)$, for every weight w such that $w^{d/(d-\alpha)} \in A_1^\rho$.*

Proof. Let $B = B(x_0, r)$ and $\tilde{B} = B(x_o, \rho(x_0))$ with $x_0 \in \mathbb{R}^d$ and $r \le \rho(x_0)$. We write $f = f_1 + f_2 + f_3$, with $f_1 = f\chi_{2B}$ and $f_3 = f_{(2\tilde{B})^C}$.

Let $0 \le \alpha < d$. Lets start estimating $|Tf_3(x)|$ uniformly for $x \in B$. When $x \in B$ and $y \in (2\tilde{B})^C$, it follows $|x-y| \simeq |x_0 - y|$ and also $\rho(x) \le C\rho(x_0)$. Lets note $\tilde{B}_k = 2^k\tilde{B}$. Then, from equation (24), Hölder's inequality with exponent d/α and considering θ such that $w^{\frac{d}{d-\alpha}} \in A_1^{\rho,\theta}$, we obtain

$$
\begin{aligned}
|Tf_3(x)| &\le C\int_{(2\tilde{B})^C} \frac{|f(y)|}{|x_0-y|^{d-\alpha}}\left(1+\frac{|x_0-y|}{\rho(x_0)}\right)^{-N} dy \\
&\le C(\rho(x_0))^N \int_{(2\tilde{B})^C} \frac{|f(y)|}{|x_0-y|^{d-\alpha+N}} dy \\
&\le C(\rho(x_0))^N \sum_{k=1}^{\infty} \frac{1}{(2^k\rho(x_0))^{d-\alpha+N}} \int_{\tilde{B}_{k+1}} |f(y)| dy \\
&\le C\|f/w\|_{L^{d/\alpha}} \sum_{k=1}^{\infty} \frac{1}{2^{kN}} \left(\frac{1}{|\tilde{B}_{k+1}|}\int_{\tilde{B}_{k+1}} w(y)^{\frac{d}{d-\alpha}} dy\right)^{\frac{d-\alpha}{d}} \\
&\le C\|f/w\|_{L^{d/\alpha}} \sum_{k=1}^{\infty} 2^{-kN} \left(\inf_{\tilde{B}_{k+1}} w\right)\left(1+\frac{2^{k+1}\rho(x_0)}{\rho(x_0)}\right)^{\theta(1-\frac{\alpha}{d})} \\
&\le C\|f/w\|_{L^{d/\alpha}} \inf_B w \sum_{k=1}^{\infty} 2^{-k(N-\theta(1-\alpha/d))} \\
&\le C\frac{w(B)}{|B|}\|f/w\|_{L^{d/\alpha}},
\end{aligned}
$$

where the last inequality follows from taking N large enough.

Now, we will estimate $|Tf_2(x) - c_B|$, where $c_B = Tf_2(x_0)$ and $x \in B$. We shall name $k_0 = \sup\{k : 2^k r < 2\rho(x_0)\}$ and $B_k = 2^k B(x_0, r)$. Given that $|x - x_0| < |y - x_0|/2$ for every $x \in B$ and $y \in (2B)^C$, we will use the smoothness estimate in equation (25) and, as done before, Hölder's inequality with exponent d/α and the fact that the weight w

ISSN 2451-7100
Publication date: October 23, 2023

satisfies $w^{\frac{d}{d-\alpha}} \in A_1^{\rho,\theta}$ for some $\theta > 0$. Then,

$$\begin{aligned}
|Tf_2(x) - c_B| &\leq \int_{2\tilde{B}\setminus 2B} |f(y)||K(x_0,y) - K(x,y)|dy \\
&\leq Cr^\lambda \int_{2\tilde{B}\setminus 2B} \frac{|f(y)|}{|x_0-y|^{d-\alpha+\lambda}} dy \\
&\leq Cr^\lambda \sum_{k=1}^{k_0} \frac{1}{(2^k r)^{d-\alpha+\lambda}} \int_{B_{k+1}} |f(y)|dy \\
&\leq C\|f/w\|_{L^{d/\alpha}} \sum_{k=1}^{k_0} \frac{1}{2^{k\lambda}} \left(\frac{1}{|B_{k+1}|} \int_{B_{k+1}} w(y)^{\frac{d}{d-\alpha}} dy \right)^{\frac{d-\alpha}{d}} \\
&\leq C\|f/w\|_{L^{d/\alpha}} \sum_{k=1}^{k_0} 2^{-k\lambda} \left(\inf_{B_{k+1}} w \right) \left(1 + \frac{2^{k+1} r}{\rho(x_0)} \right)^{\theta(1-\frac{\alpha}{d})} \\
&\leq C\|f/w\|_{L^{d/\alpha}} \inf_B w \sum_{k=1}^{\infty} 2^{-k\lambda} \\
&\leq C \frac{w(B)}{|B|} \|f/w\|_{L^{d/\alpha}},
\end{aligned}$$

where we have used the fact that $(1+2^{k+1}r/\rho(x_0))^\sigma \leq 5^\sigma$, for every $k \leq k_0$ and $\theta > 0$.

Now, lets supose $\alpha > 0$. From the $(1, d/(d-\alpha))$ weak type of the operator T and Kolmorogov's inequality, yields

$$\frac{1}{|B|}\int_B |Tf_1(y)|dy \leq C\frac{|B|^{1-\frac{d-\alpha}{d}}}{|B|}\int_{2B} |f(y)|dy = C\frac{1}{|B|^{1-\alpha/d}}\int_{2B}|f(y)|dy.$$

Then, from Hölder's inequality with exponent d/α and the fact that $w^{\frac{d}{d-\alpha}} \in A_1^\rho$, follows

$$\begin{aligned}
\frac{1}{|B|}\int_B |Tf_1(y)|dy &\leq C\|f/w\|_{L^{d/\alpha}} \left(\frac{1}{|2B|} \int_{2B} w(y)^{\frac{d}{d-\alpha}} dy \right)^{1-\alpha/d} \\
&\leq C\|f/w\|_{L^{d/\alpha}} \left(\inf_{2B} w \right) \left(1 + \frac{2r}{\rho(x_0)} \right)^{\theta(1-\frac{\alpha}{d})} \\
&\leq C\|f/w\|_{L^{d/\alpha}} \left(\inf_B w \right) \\
&\leq C\frac{w(B)}{|B|}\|f/w\|_{L^{d/\alpha}}.
\end{aligned}$$

which concludes the proof.

□

IMAL PREPRINT # 2023-0067

ISSN 2451-7100
Publication date: October 23, 2023

Proof of Theorem 2. The result is a direct consequence of the weak type $(1, d/d-\alpha)$ of $\mathcal{L}^{-\alpha/4}$, Proposition 11 and Proposition 12.

□

6 Proof of Theorem 3

We start this section stating some results that will be usefull. Next proposition characterizes, in the presence of a doubling weight, the space $BMO_\rho^\beta(w)$ reducing the set of balls that in ineqality (6) must be verified to only critical balls.

Proposition 13 (See Corollary 1 in [2])**.** *Given $w \in D_\eta$ for some $\eta \geq 1$, and $\gamma \geq 0$, a function $f \in L^1_{loc}(\mathbb{R}^d)$ belongs to $BMO_\rho^\gamma(w)$ if and only if both of the following conditions hold:*

i) For every ball $B = B(x, R)$, with $x \in \mathbb{R}^d$ and $R < \rho(x)$

$$\int_B |f - f_B| \leq Cw(B)|B|^{\gamma/d}, \quad \text{with } f_B = \frac{1}{|B|}\int_B f.$$

ii) For every $x \in \mathbb{R}^d$,

$$\int_{B(x,\rho(x))} |f| \leq Cw(B(x, \rho(x)))|\rho(x)|^\gamma.$$

The following result gives some control of the average of a function in terms of its weak Lebesgue semi-norm.

Proposition 14 (See Lemma 4.1 in [12])**.** *Let $p > 1$ and w a weight in $RH_{p'}$. There exists a constant C such that, if f is a locally integrable function and B is a ball in $\mathbb{R}^d$ then,*

$$\int_B |f| \leq Cw(B)|B|^{-1/p}[f]_{p,w}.$$

We now give the proof of Theorem 3.

Proof of Theorem 3. First, we will show that given $f \in L^1_{loc}$ and $B = B(x_0, r)$, with $x_0 \in \mathbb{R}^d$,

$$\frac{1}{w(B)}\int_B \mathcal{L}^{-\alpha/4}(|f\chi_{2B}|) \leq C|B|^{\frac{\alpha}{d}-\frac{1}{p}}[f]_{p,w}. \tag{31}$$

ISSN 2451-7100
Publication date: October 23, 2023

In fact, from (30) and Proposition 14, we get

$$\begin{aligned}\frac{1}{w(B)}\int_B \mathcal{L}^{-\alpha/4}(|f\chi_{2B}|) &= \frac{1}{w(B)}\int_B\int_{\mathbb{R}^d} K_\alpha(x,y)|f(y)|\chi_{2B}(y)dydx\\ &\le \frac{1}{w(B)}\int_B\int_{2B}\frac{|f(y)|}{|x-y|^{d-\alpha}}dydx\\ &= \frac{1}{w(B)}\int_{2B}|f(y)|\int_B\frac{dx}{|x-y|^{d-\alpha}}dy\\ &\le C\frac{r^\alpha}{w(B)}\int_{2B}|f(y)|dy\\ &\le C|B|^{\frac{\alpha}{d}-\frac{1}{p}}[f]_{p,w}.\end{aligned}$$

To check that $\mathcal{L}^{-\alpha/4}f$ belongs to $BMO_\rho^{\alpha-d/p}(w)$, by using Proposition 13, we shall prove there exists a constant C such that the following conditions hold

i) For every $x_0\in\mathbb{R}^d$,

$$\frac{1}{w(B(x_0,\rho(x_0)))}\int_{B(x_0,\rho(x_0))}|\mathcal{L}^{-\alpha/4}f| \le C|B(x_0,\rho(x_0))|^{\frac{\alpha}{d}-\frac{1}{p}}[f]_{p,w}. \tag{32}$$

ii) For every ball $B=B(x_0,r)$ with $r<\rho(x_0)$ and some constant C_B

$$\frac{1}{w(B)}\int_B|\mathcal{L}^{-\alpha/4}f(x)-C_B|dx \le C|B|^{\frac{\alpha}{d}-\frac{1}{p}}[f]_{p,w}. \tag{33}$$

In order to prove (32), given f and $B(x_0,R)$, with $R=\rho(x_0)$, we split the function as $f=f_1+f_2$, where $f_1=f\chi_{2B}$. We only need to show that the condition holds for f_2. Notice

$$\mathcal{L}^{-\alpha/4}f_2(x)=\int_0^{R^4} e^{-t\mathcal{L}}f_2(x)t^{\alpha/4}\frac{dt}{t}+\int_{R^4}^\infty e^{-t\mathcal{L}}f_2(x)t^{\alpha/4}\frac{dt}{t}.$$

If $x\in B,\ y\in(2B)^C$,

$$|x_0-y|\le|x-x_0|+|x-y|\le r+|x-y|\le 2|x-y|.$$

ISSN 2451-7100
Publication date: October 23, 2023

Then $|x-y| \geq C|x_0 - y|$. Thus, from (8),

$$\begin{aligned}
\left|\int_0^{R^4} e^{-t\mathcal{L}} f_2(y) t^{\alpha/4-1} dt\right| &= \left|\int_0^{R^4} \int_{(2B)^C} k_t(x,y) f(y) t^{\alpha/4-1} dy dt\right| \\
&\leq C \int_0^{R^4} \int_{(2B)^C} t^{-d/4} e^{-c\frac{|x-y|^{4/3}}{t^{1/3}}} |f(y)| t^{\alpha/4-1} dy dt \\
&\leq C \int_0^{R^4} t^{\frac{-d+\alpha}{4}-1} \int_{(2B)^C} \left(\frac{t^{1/3}}{|x-y|^{4/3}}\right)^{\frac{3}{4}M} |f(y)| dy dt \\
&\leq C \int_0^{R^4} t^{\frac{-d+\alpha+M}{4}-1} dt \int_{(2B)^C} \frac{|f(y)|}{|x_0-y|^M} dy.
\end{aligned}$$

To bound the second integral, we split the integration region as follows

$$\begin{aligned}
\int_{(2B)^C} \frac{|f(y)|}{|x_0-y|^M} dy &= \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|f(y)|}{|x_0-y|^M} dy \\
&\leq \frac{1}{R^M} \sum_{k=1}^{\infty} \frac{1}{2^{kM}} \int_{2^{k+1}B} |f(y)| dy \\
&\leq C R^{-M-\frac{d}{p}} [f]_{p,w} \sum_{k=1}^{\infty} w(2^{k+1}B) 2^{-k\left(\frac{d}{p}+M\right)} \\
&\leq C w(B) R^{-M-\frac{d}{p}} [f]_{p,w} \sum_{k=1}^{\infty} 2^{-k\left(\frac{d}{p}+M-d\eta\right)},
\end{aligned} \tag{34}$$

where we have used Proposition 14 and the fact that $w \in D_\eta$. That last series converges by choosing M large enough. Finally,

$$\begin{aligned}
\left|\int_0^{R^4} e^{-t\mathcal{L}} f_2(y) t^{\alpha/4-1} dt\right| &\leq C w(B) R^{-M-\frac{d}{p}} [f]_{p,w} \int_0^{R^4} t^{\frac{-d+\alpha+M}{4}-1} dt \\
&= C w(B) |B|^{\frac{\alpha}{d}-\frac{1}{p}-1} [f]_{p,w}.
\end{aligned}$$

ISSN 2451-7100
Publication date: October 23, 2023

As to the second integral in equation (6), from inequality (8) and Proposition 14, we have

$$\begin{aligned}
\left|\int_{R^4}^{\infty} e^{-t\mathcal{L}}\ f_2(y)\ t^{\alpha/4-1}dt\right| &\leq \int_{R^4}^{\infty}\int_{(2B)^C} k_t(x,y)|f(y)|t^{\alpha/4-1}dydt \\
&\leq C\int_{R^4}^{\infty}\int_{(2B)^C} t^{-d/4}e^{-c\frac{|x-y|^{4/3}}{t^{1/3}}}\left(\frac{\rho(x)}{\sqrt[4]{t}}\right)^{2N}|f(y)|t^{\alpha/4-1}dydt \\
&= C\int_{R^4}^{\infty}\int_{(2B)^C} t^{\frac{\alpha-d-2N}{4}-1}e^{-c\frac{|x-y|^{4/3}}{t^{1/3}}}[\rho(x)]^{2N}|f(y)|dydt \\
&\leq C[\rho(x)]^{2N}\int_{R^4}^{\infty} t^{\frac{\alpha-d-2N}{4}-1}\int_{(2B)^C}\left(\frac{t^{1/3}}{|x-y|^{4/3}}\right)^{\frac{3}{4}M}|f(y)|dydt \\
&\leq CR^{2N}\int_{R^4}^{\infty} t^{\frac{M+\alpha-d-2N}{4}-1}dt\int_{(2B)^C}\frac{|f(y)|}{|x_0-y|^M}dy \\
&\leq Cw(B)R^{2N-\frac{d}{p}-M}[f]_{p,w}R^{M+\alpha-d-2N} \\
&= Cw(B)|B|^{\frac{\alpha}{d}-\frac{1}{p}-1}[f]_{p,w}.
\end{aligned}$$

Given that $\rho(x) \simeq \rho(x_0) = R$, since $x \in B$.

Next, we will prove (33). Let $B = B(x_0, r)$ with $r < \rho(x_0)$. We split f as $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and set

$$C_B = \int_{R^4}^{\infty} e^{-t\mathcal{L}} f_2(x_0)t^{\alpha/2-1}dt.$$

From equation (31), we have

$$\begin{aligned}
\frac{1}{w(B)}\int_B |\mathcal{L}^{\alpha/4}(f) - C_B| &\leq \frac{1}{w(B)}\int_B \mathcal{L}^{\alpha/4}(|f_1|) + \frac{1}{w(B)}\int_B |\mathcal{L}^{\alpha/4}(f_2) - C_B| \\
&\leq C|B|^{\frac{\alpha}{d}-\frac{1}{p}}[f]_{p,w} + \frac{1}{w(B)}\int_B |\mathcal{L}^{\alpha/4}(f_2) - C_B|.
\end{aligned}$$

Also,

$$|\mathcal{L}^{\alpha/4}f_2(x) - C_B| = \left|\int_0^{r^4} e^{-t\mathcal{L}}f_2(x)t^{\alpha/4-1}dt\right| + \left|\int_{r^4}^{\infty} e^{-t\mathcal{L}}f_2(y)t^{\alpha/4-1}dt - C_B\right|.$$

ISSN 2451-7100
Publication date: October 23, 2023

Then, we only need to bound the second integral. Using Proposition 9, we obtain

$$\begin{aligned}\left|\int_{r^4}^{\infty} e^{-t\mathcal{L}} f_2(y)\ t^{\alpha/4}\frac{dt}{t} - C_B\right| &\leq \int_{r^4}^{\infty}\int_{(2B)^C} |k_t(x,y)-k_t(x_0,y)||f(y)|dy t^{\alpha/4}\frac{dt}{t} \\ &\leq C\int_{r^4}^{\infty}\int_{(2B)^C}\left(\frac{|x-x_0|}{\sqrt[4]{t}}\right)^{\delta''} t^{-d/4}e^{-c\frac{|x-y|^{4/3}}{t^{1/3}}}|f(y)|dy t^{\alpha/4}\frac{dt}{t} \\ &\leq Cr^{\delta''}\int_{(2B)^C}|f(y)|\int_{r^4}^{\infty} t^{-\frac{d-\alpha+\delta''}{4}}e^{-c\frac{|x-y|^{4/3}}{t^{1/3}}}\frac{dt}{t}dy \\ &= Cr^{\delta''}\int_{(2B)^C}\frac{|f(y)|}{|x-y|^{d-\alpha+\delta''}}dy\int_0^{\left(\frac{|x-y|}{r}\right)^{4/3}} s^{\frac{3}{4}(d-\alpha+\delta'')}e^{-cs}\frac{ds}{s} \\ &\leq Cr^{\delta''}\int_{(2B)^C}\frac{|f(y)|}{|x-y|^{d-\alpha+\delta''}}dy\int_0^{\infty} s^{\frac{3}{4}(d-\alpha+\delta'')}e^{-cs}\frac{ds}{s} \\ &\leq Cr^{\delta''}\int_{(2B)^C}\frac{|f(y)|}{|x-y|^{d-\alpha+\delta''}}dy.\end{aligned}$$

Finally, by splitting the domain of the integral into dyadic anulli as we did in equation (34), we get

$$\left|\int_{r^4}^{\infty} e^{-t\mathcal{L}} f_2(y) t^{\alpha/4-1}dt - C_B\right| \leq Cw(B)r^{\alpha-d-\frac{d}{p}}[f]_{p,w}.$$

□

Acknowledgement

This research is partially supported by grants from Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) and Universidad Nacional del Litoral (UNL), Argentina.

The authors are grateful to Eleonor Harboure who suggested the problem. Despite she is not with us anymore, her memory lives in the essence of this work.

References

[1] B. Bongioanni, A. Cabral, and E. Harboure. Lerner's inequality associated to a critical radius function and applications. *J. Math. Anal. Appl.*, 407(1):35–55, 2013.

[2] B. Bongioanni, E. Harboure, and O. Salinas. Weighted inequalities for negative powers of Schrödinger operators. *J. Math. Anal. Appl.*, 348(1):12–27, 2008.

[3] B. Bongioanni, E. Harboure, and O. Salinas. Weigthed inequalities for negative powers of Schrödinger operators. *J. Math. Anal. Appl.*, 348:12–27, 2008.

[4] B. Bongioanni, E. Harboure, and O. Salinas. Classes of weights related to Schrödinger operators. *J. Math. Anal. Appl.*, 373(2):563–579, 2011.

ISSN 2451-7100
Publication date: October 23, 2023

[5] Enrique Adrián Cabral. Análisis en el semigrupo generado por el operador de Schrödinger. Tesis doctoral, Instituto de Matemática Aplicada del Litoral (IMAL), 2009.

[6] Jun Cao, Yu Liu, and Dachun Yang. Hardy spaces $H^1_{\mathcal{L}}(\mathbb{R}^n)$ associated to Schrödinger type operators $(-\Delta)^2 + V^2$. *Houston J. Math.*, 36(4):1067–1095, 2010.

[7] Wei Chen and Chao Zhang. Regularity properties and lipschitz spaces adapted to high-order schrödinger operators. 10(2600), 2022.

[8] E. B. Davies. L^p spectral theory of higher-order elliptic differential operators. *Bull. London Math. Soc.*, 29(5):513–546, 1997.

[9] J. Dziubański, G. Garrigós, T. Martínez, J. L. Torrea, and J. Zienkiewicz. *BMO* spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality. *Math. Z.*, 249(2):329–356, 2005.

[10] Jacek Dziubański and Jacek Zienkiewicz. H^p spaces for Schrödinger operators. In *Fourier analysis and related topics (Bedlewo, 2000)*, volume 56 of *Banach Center Publ.*, pages 45–53. Polish Acad. Sci. Inst. Math., Warsaw, 2002.

[11] Jacek Dziubański and Jacek Zienkiewicz. H^p spaces associated with Schrödinger operators with potentials from reverse Hölder classes. *Colloq. Math.*, 98(1):5–38, 2003.

[12] Eleonor Harboure, Oscar Salinas, and Beatriz Viviani. Boundedness of the fractional integral on weighted Lebesgue and Lipschitz spaces. *Trans. Amer. Math. Soc.*, 349(1):235–255, 1997.

[13] Herbert Koch and Tobias Lamm. Geometric flows with rough initial data. *Asian J. Math.*, 16(2):209–235, 2012.

[14] Tao Ma, Pablo Raúl Stinga, José L. Torrea, and Chao Zhang. Regularity estimates in Hölder spaces for Schrödinger operators via a $T1$ theorem. *Ann. Mat. Pura Appl. (4)*, 193(2):561–589, 2014.

[15] Z. Shen. L^p estimates for Schrödinger operators with certain potentials. *Ann. Inst. Fourier (Grenoble)*, 45(2):513–546, 1995.