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GAUSSIAN JN_p SPACES

JORGE J. BETANCOR[1], ESTEFANÍA DALMASSO[2], AND PABLO QUIJANO[2]

ABSTRACT. In this paper we introduce the John-Nirenberg's type spaces JN_p associated with the Gaussian measure $d\gamma(x) = \pi^{-d/2}e^{-|x|^2}dx$ in $\mathbb{R}^d$ where $1 < p < \infty$. We prove a John-Nirenberg inequality for $\mathrm{JN}_p(\mathbb{R}^d, \gamma)$. We also characterize the predual of $\mathrm{JN}_p(\mathbb{R}^d, \gamma)$ as a Hardy type space.

1. INTRODUCTION

In [12], John and Nirenberg introduced the well-known space $\mathrm{BMO}(\mathbb{R}^d)$ of functions with bounded mean oscillation. Also, they considered a variant of the BMO condition. This other condition is used to define the space of functions $\mathrm{JN}_p(\mathbb{R}^d)$, $1 < p < \infty$, as follows. Let Q_0 be a cube in $\mathbb{R}^d$ and $1 < p < \infty$. We always assume that the cubes have sides parallel to the coordinate axis and they are open. A function $f \in L^1(Q_0)$ is said to be in $\mathrm{JN}_p(Q_0)$ when

$$\|f\|_{\mathrm{JN}_p(Q_0)} \coloneqq \sup \left(\sum_i |Q_i| \left(\frac{1}{|Q_i|} \int_{Q_i} |f - f_{Q_i}| dx \right)^p \right)^{1/p} < \infty,$$

where the supremum is taken over all the countable families $\{Q_i\}_{i=1}^\infty$ of pairwise disjoint cubes in Q_0 and f_{Q_i} stands for the average of f over the cube Q_i. Similarly, a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is in $\mathrm{JN}_p(\mathbb{R}^d)$ when $\|f\|_{\mathrm{JN}_p(\mathbb{R}^d)} < \infty$, where $\|\cdot\|_{\mathrm{JN}_p(\mathbb{R}^d)}$ is defined analogously.

JN_p spaces were considered in the context of interpolation by Campanato [3] and Stampacchia [19]. More recently, in the last decade a number of papers have investigated about JN_p spaces ([1], [2], [6], [9], [15] and [17], for instance). Related with the JN_p spaces are the dyadic JN_p spaces ([13]), the John-Nirenberg-Campanato spaces ([25]), localized versions of JN_p spaces ([23]) and the sparse JN_p spaces ([6]), among others.

Other definitions of JN_p spaces appear when the cubes are replaced by other classes of sets in more general measure metric spaces. Depending on the overlapping properties of the chosen sets we can obtain different spaces.

In [11], John studied BMO spaces using medians instead of integral averages. From the results in [21] and [22] it can be deduced that BMO spaces defined by using medians and averages coincide. Recently, median-type John-Nirenberg spaces in metric measure spaces have been studied in [18].

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It is not hard to see that $L^p \subset \mathrm{JN}_p$. Also we have that $\mathrm{JN}_p \subset L^{p,\infty}$. Further, both of these inclusions are strict. An example of a function $f \in \mathrm{JN}_p(I) \setminus L^p(I)$ where I is an interval in $\mathbb{R}$ was defined in [5]. Previously, some results related to the nonequality $L^p \neq \mathrm{JN}_p$ were contained in [15]. In [1], it was proved that $\mathrm{JN}_p \neq L^{p,\infty}$. Other examples of functions in JN_p spaces have been constructed in [24].

Our objective in this paper is to introduce and to study the JN_p spaces associated with the Gaussian measure $d\gamma(x) = \pi^{-d/2} e^{-|x|^2} dx$ on $\mathbb{R}^d$ that we name $\mathrm{JN}_p(\mathbb{R}^d, \gamma)$ with $1 < p < \infty$.

We consider the function m defined on $\mathbb{R}^d$ by

$$m(x) = \begin{cases} 1 & \text{if } x = 0, \\ \min\left\{1, \frac{1}{|x|}\right\} & \text{if } x \neq 0. \end{cases}$$

If B is a ball in $\mathbb{R}^d$ we denote by c_B and r_B the center and the radius of B, respectively. Let $a > 0$. By $\mathcal{B}_a$ we represent the family of balls B in $\mathbb{R}^d$ satisfying $r_B \leq a m(c_B)$. It is usual to name the balls in $\mathcal{B}_a$ **admissible balls** with parameter a. The Gaussian measure has not the doubling property. However, the Gaussian measure is doubling on $\mathcal{B}_a$ but the doubling constant depends on a ([16, Proposition 2.1]).

The bounded mean oscillation function space associated with γ in $\mathbb{R}^d$, in short $\mathrm{BMO}(\mathbb{R}^d, \gamma)$, was introduced in [16]. A function $f \in L^1(\mathbb{R}^d, \gamma)$ is said to be in $\mathrm{BMO}(\mathbb{R}^d, \gamma)$ when

$$\|f\|_{\star,\mathcal{B}_1} = \sup_{B \in \mathcal{B}_1} \frac{1}{\gamma(B)} \int_B |f - f_B| d\gamma < \infty,$$

where, for a function f and a ball B, $f_B = \frac{1}{\gamma(B)} \int_B f d\gamma$. The space $\mathrm{BMO}(\mathbb{R}^d, \gamma)$ is endowed with the norm

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^d,\gamma)} = \|f\|_{L^1(\mathbb{R}^d,\gamma)} + \|f\|_{\star,\mathcal{B}_1}, \quad f \in \mathrm{BMO}(\mathbb{R}^d, \gamma).$$

Thus, $(\mathrm{BMO}(\mathbb{R}^d, \gamma), \|\cdot\|_{\mathrm{BMO}(\mathbb{R}^d,\gamma)})$ is a Banach space.

In [16, Proposition 2.4] it was proved that if we define the space $\mathrm{BMO}(\mathbb{R}^d, \gamma)$ associated to the family $\mathcal{B}_a$ with $a \neq 1$ instead of $\mathcal{B}_1$ we obtain again the same space and the corresponding norms are equivalent.

If Q is a cube in $\mathbb{R}^d$ we denote by c_Q and ℓ_Q the center and the side length of Q respectively. The family $\mathcal{Q}_a$ consists of all those cubes $Q \subset \mathbb{R}^d$ such that $\ell_Q \leq a m(c_Q)$. If we consider the family $\mathcal{Q}_a$ instead of $\mathcal{B}_1$ to define $\mathrm{BMO}(\mathbb{R}^d, \gamma)$ the new space coincides with that defined using $\mathcal{B}_1$ and the corresponding norms are equivalent.

The main properties of the space $\mathrm{BMO}(\mathbb{R}^d, \gamma)$ were established in [16] (see also [4], [14] and [26]).

Let $a > 0$ and $1 < p < \infty$. A function $f \in L^1(\mathbb{R}^d, \gamma)$ is said to be in $\mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d, \gamma)$ when

$$K_p^{\mathcal{Q}_a}(f) = \sup \left(\sum_i \gamma(Q_i) \left(\frac{1}{\gamma(Q_i)} \int_{Q_i} |f - f_{Q_i}| d\gamma \right)^p \right)^{1/p} < \infty,$$

where the supremum is taken over all the countable collections $\{Q_i\}_{i\in\mathbb{N}}$ of pairwise disjoint cubes in $\mathcal{Q}_a$. The space $\mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d, \gamma)$ is endowed with the norm

$$\|f\|_{\mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)} = \|f\|_{L^1(\mathbb{R}^d,\gamma)} + K_p^{\mathcal{Q}_a}(f), \quad f \in \mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d, \gamma).$$

We will prove in Proposition 2.1 that $\mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d, \gamma)$ actually does not depend on $a > 0$. Then we will write in the sequel $\mathrm{JN}_p(\mathbb{R}^d, \gamma)$ to name $\mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d, \gamma)$, $a > 0$. We

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also prove that $\mathrm{BMO}(\mathbb{R}^d,\gamma)$ is contained in $\mathrm{JN}_p(\mathbb{R}^d,\gamma)$, $1<p<\infty$, and appears when $p\to\infty$ in $\mathrm{JN}_p(\mathbb{R}^d,\gamma)$ (see Proposition 2.2).

The following property is a John-Nirenberg type inequality for $\mathrm{JN}_p(\mathbb{R}^d,\gamma)$.

Theorem 1.1. *Let $a>0$ and $1<p<\infty$. There exists $C>0$ such that, for every $Q\in\mathcal{Q}_a$, $\sigma>0$ and $f\in \mathrm{JN}_p(\mathbb{R}^d,\gamma)$,*

$$\gamma\left(\{x\in Q:|f-f_Q|>\sigma\}\right)\leq C\left(\frac{K_p^{\mathcal{Q}_a}(f)}{\sigma}\right)^p.$$

It is a celebrated result due to Fefferman and Stein ([7]) that the Hardy space $H^1(\mathbb{R}^d)$ is the predual of $\mathrm{BMO}(\mathbb{R}^d)$. In the Gaussian setting, the predual of the space $\mathrm{BMO}(\mathbb{R}^d,\gamma)$ was characterized in [16, Theorem 5.2] as a Hardy type space $H^1(\mathbb{R}^d,\gamma)$ defined by using atoms whose support is contained in admissible balls. In [5, §6] it was defined a Hardy type space $H^{p'}(Q)$ whose dual coincide with $\mathrm{JN}_p(Q)$, where $p'=\frac{p}{p-1}$ and $1<p<\infty$. The ideas in [5] inspired the duality properties for John-Nirenberg-Campanato spaces ([23]).

Our main result characterizes a new Hardy type space as the dual of $\mathrm{JN}_p(\mathbb{R}^d,\gamma)$, $1<p<\infty$.

For every $1\leq s\leq\infty$ and every cube in $\mathbb{R}^d$ we denote by $L_0^s(Q,\gamma)$ the space consisting of all those $f\in L^s(Q,\gamma)$ such that $\int_Q f d\gamma=0$. For $1<q\leq\infty$ and $a>0$ we say that a function $b\in\mathcal{A}(q,a,Q)$ if b is supported on a cube $Q\in\mathcal{Q}_a$ and $b\in L_0^q(Q,\gamma)$.

Let $a>0$ and $1<p<q\leq\infty$. We consider a measurable function g on $\mathbb{R}^d$ defined by $g=\sum_{j=1}^\infty b_j$, where, for every $j\in\mathbb{N}$, $b_j\in\mathcal{A}(q,a,Q_j)$ being $Q_j\in\mathcal{Q}_a$ and the sequence $\{Q_j\}_{j=1}^\infty$ is pairwise disjoint. We say that g is a (p,q,a)**-polymer** when

$$\sum_{j=1}^\infty \gamma(Q_j)\left(\frac{1}{\gamma(Q_j)}\int_{Q_j}|b_j|^q d\gamma\right)^{p/q}<\infty. \tag{1.1}$$

Note that the series defining g is pointwise convergent because $\{Q_j\}_{j=1}^\infty$ is pairwise disjoint. We also define

$$\|g\|_{(p,q,a)}=\inf\left(\sum_{j=1}^\infty \gamma(Q_j)\left(\frac{1}{\gamma(Q_j)}\int_{Q_j}|b_j|^q d\gamma\right)^{p/q}\right)^{1/p}$$

where the infimum is taken over all the sequences $\{b_j\}_{j=1}^\infty$ as above such that $g=\sum_{j=1}^\infty b_j$ and (1.1) holds. By using Jensen inequality we can see that if g is a (p,q,a)-polymer defined as above, then $g\in L^p(\mathbb{R}^d,\gamma)$ and

$$\|g\|_{L^p(\mathbb{R}^d,\gamma)}\leq\left(\sum_{j=1}^\infty \gamma(Q_j)\left(\frac{1}{\gamma(Q_j)}\int_{Q_j}|b_j|^q d\gamma\right)^{p/q}\right)^{1/p}. \tag{1.2}$$

Observe that $g=0$ a.e. provided that $\|g\|_{(p,q,a)}=0$. The above estimate implies that if $\{g_i\}_{i\in\mathbb{N}}$ is a sequence of (p,q,a)-polymers such that $\sum_{i=1}^\infty\|g_i\|_{(p,q,q)}<\infty$, then the series $\sum_{i=1}^\infty g_i$ converges in $L^p(\mathbb{R}^d,\gamma)$.

When $q=\infty$ the above expressions are understood in the usual way.

We now introduce a Hardy type space as follows. A measurable function g is in $H_{p,q,a}(\mathbb{R}^d,\gamma)$ when $g=c_0+\sum_{i=1}^\infty g_i$, where $c_0\in\mathbb{C}$, g_i is a (p,q,a)-polymer for every $i\in\mathbb{N}$ and $\sum_{i=1}^\infty\|g_i\|_{(p,q,a)}<\infty$. The convergence of the series is understood in $L^p(\mathbb{R}^d,\gamma)$. Note that if $g\in H_{p,q,a}(\mathbb{R}^d,\gamma)$ then $g\in L^p(\mathbb{R}^d,\gamma)$. Observe that c_0 is actually unique, since each polymer g_i can be written in terms of functions

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$b_{ij} \in \mathcal{A}(q, a, Q_{ij})$ and all of them have zero integral with respect to the Gaussian measure.

We define the following quantity

$$\|g\|_{H_{p,q,a}(\mathbb{R}^d,\gamma)} = |c_0| + \inf \sum_{i=1}^{\infty} \|g_i\|_{(p,q,a)},$$

where the infimum is taken over all the sequences $\{g_i\}_{i=1}^{\infty}$ of (p, q, a)-polymers such that $g = c_0 + \sum_{i=1}^{\infty} g_i$ with $c_0 = \int_{\mathbb{R}^d} g d\gamma$ and $\sum_{i=1}^{\infty} \|g_i\|_{(p,q,a)} < \infty$. The functional $\|\cdot\|_{H_{p,q,a}(\mathbb{R}^d,\gamma)}$ is a norm for $H_{p,q,a}(\mathbb{R}^d, \gamma)$.

Given $f \in \mathrm{JN}_p(\mathbb{R}^d, \gamma)$, we define the functional Λ_f by

$$\Lambda_f g := \lim_{N\to\infty} \int_{\mathbb{R}^d} f_N g d\gamma, \tag{1.3}$$

where for every $N \in \mathbb{N}$,

$$f_N(x) = \begin{cases} f(x), & \text{if } |f(x)| \leq N \\ N \operatorname{sgn}(f(x)), & \text{if } |f(x)| > N. \end{cases}$$

The functional Λ_f is well-defined, as we shall see in the proof of Theorem 1.2(a).

Theorem 1.2. *Let $1 < q < p < \infty$ and $a > 0$.*

(a) Let $f \in \mathrm{JN}_p(\mathbb{R}^d, \gamma)$. Then $\Lambda_f \in (H_{p',q',a}(\mathbb{R}^d, \gamma))'$ and

$$\|\Lambda_f\|_{(H_{p',q',a}(\mathbb{R}^d,\gamma))'} \leq C\|f\|_{\mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)},$$

where $C > 0$ does not depend on f.

(b) If $\Lambda \in (H_{p',q',a}(\mathbb{R}^d, \gamma))'$ there exists a unique $f \in \mathrm{JN}_p(\mathbb{R}^d, \gamma)$ such that $\Lambda = \Lambda_f$, defined as in (1.3), and

$$\|f\|_{\mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)} \leq C\|\Lambda\|_{(H_{p',q',a}(\mathbb{R}^d,\gamma))'}$$

where $C > 0$ does not depend on Λ.

Note that from Theorem 1.2 and [10, Lemma 4.14] we can deduce that $\mathrm{JN}_p(\mathbb{R}^d, \gamma)$ is a Banach space for every $1 < p < \infty$. Moreover, as we shall see in Proposition 4.3, $H_{p,q,a_1}(\mathbb{R}^d, \gamma) = H_{p,q,a_2}(\mathbb{R}^d, \gamma)$ whenever $a_1, a_2 > 0$ and $1 < p < q < \infty$.

2. Some properties of the $\mathrm{JN}_p(\mathbb{R}^d, \gamma)$ spaces

We first prove that the space $\mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d, \gamma)$ does not depend on $a > 0$.

Proposition 2.1. *Let $1 < p < \infty$ and $a_1, a_2 > 0$. We have that*

$$\mathrm{JN}_p^{\mathcal{Q}_{a_1}}(\mathbb{R}^d, \gamma) = \mathrm{JN}_p^{\mathcal{Q}_{a_2}}(\mathbb{R}^d, \gamma)$$

algebraically and topologically.

Proof. Without loss of generality, we may assume that $0 < a_2 < a_1$. It is clear that $\mathrm{JN}_p^{\mathcal{Q}_{a_1}}(\mathbb{R}^d, \gamma) \subseteq \mathrm{JN}_p^{\mathcal{Q}_{a_2}}(\mathbb{R}^d, \gamma)$ since $K_p^{\mathcal{Q}_{a_2}}(f) \leq K_p^{\mathcal{Q}_{a_1}}(f)$ for every $f \in \mathrm{JN}_p^{\mathcal{Q}_{a_1}}(\mathbb{R}^d, \gamma)$ and, therefore,

$$\|f\|_{\mathrm{JN}_p^{\mathcal{Q}_{a_2}}(\mathbb{R}^d,\gamma)} \leq \|f\|_{\mathrm{JN}_p^{\mathcal{Q}_{a_1}}(\mathbb{R}^d,\gamma)}, \quad f \in \mathrm{JN}_p^{\mathcal{Q}_{a_1}}(\mathbb{R}^d, \gamma).$$

We are going to see the other inclusion. Let $f \in L^1(\mathbb{R}^d, \gamma)$ and $Q \in \mathcal{Q}_{a_1}$. As in the proof of [16, Proposition 2.3], there exist N cubes $Q_1, \dots, Q_N \in \mathcal{Q}_{a_2}$ contained in Q and a positive constant C such that $\gamma(Q_j) \leq \gamma(Q) \leq C\gamma(Q_j)$ for every $j = 1, \dots, N$, and

$$\frac{1}{\gamma(Q)} \int_Q |f - f_Q| d\gamma \leq C \sum_{j=1}^{N} \frac{1}{\gamma(Q_j)} \int_{Q_j} |f - f_{Q_j}| d\gamma.$$

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Here, $C > 0$ and $N \in \mathbb{N}$ do not depend on the cube Q.

Consider now a family $\{Q_i\}_{i\in\mathbb{N}}$ of cubes in $\mathcal{Q}_{a_1}$ such that $Q_i \cap Q_j = \emptyset$ for every $i, j \in \mathbb{N}$, $i \neq j$. For a fixed $i \in \mathbb{N}$, we consider the collection of cubes $\{Q_{i,1}, \dots, Q_{i,N}\}$ in $\mathcal{Q}_{a_2}$ associated with Q_i as above. Hence, there exists $C > 0$ for which

$$\begin{aligned}
&\sum_{i=1}^{\infty}\gamma(Q_i)\left(\frac{1}{\gamma(Q_i)}\int_{Q_i}|f - f_{Q_i}|d\gamma\right)^p \\
&\quad\le C\sum_{i=1}^{\infty}\gamma(Q_i)\left(\sum_{j=1}^{N}\frac{1}{\gamma(Q_{i,j})}\int_{Q_{i,j}}|f - f_{Q_{i,j}}|d\gamma\right)^p \\
&\quad\le C\sum_{i=1}^{\infty}\gamma(Q_i)\sum_{j=1}^{N}\left(\frac{1}{\gamma(Q_{i,j})}\int_{Q_{i,j}}|f - f_{Q_{i,j}}|d\gamma\right)^p \\
&\quad\le C\sum_{j=1}^{N}\sum_{i=1}^{\infty}\gamma(Q_{i,j})\left(\frac{1}{\gamma(Q_{i,j})}\int_{Q_{i,j}}|f - f_{Q_{i,j}}|d\gamma\right)^p \\
&\quad\le CN\left(K_p^{\mathcal{Q}_{a_2}}(f)\right)^p.
\end{aligned}$$

Taking the supremum on the pairwise disjoint families $\{Q_i\}_{i\in\mathbb{N}}$ in $\mathcal{Q}_{a_1}$, we get that

$$K_p^{\mathcal{Q}_{a_1}}(f) \le C_p N^{1/p} K_p^{\mathcal{Q}_{a_2}}(f),$$

which gives $\mathrm{JN}_p^{\mathcal{Q}_{a_2}}(\mathbb{R}^d,\gamma) \subseteq \mathrm{JN}_p^{\mathcal{Q}_{a_1}}(\mathbb{R}^d,\gamma)$ and the inclusion is also continuous. □

The following proposition establishes some relations between $\mathrm{BMO}(\mathbb{R}^d,\gamma)$ and $\mathrm{JN}_p(\mathbb{R}^d,\gamma)$.

Proposition 2.2.

(a) $\mathrm{BMO}(\mathbb{R}^d,\gamma)$ *is continuously contained in* $\mathrm{JN}_p(\mathbb{R}^d,\gamma)$ *for every* $1 < p < \infty$.

(b) For every $a > 0$ *and* $f \in \mathrm{BMO}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)$,

$$\lim_{p\to\infty}\|f\|_{\mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)} = \|f\|_{\mathrm{BMO}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)}.$$

Here, $\|f\|_{\mathrm{BMO}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)} = \sup_{Q\in\mathcal{Q}_a}\frac{1}{\gamma(Q)}\int_Q|f - f_Q|d\gamma + \|f\|_{L^1(\mathbb{R}^d,\gamma)}$, *for* $f \in \mathrm{BMO}(\mathbb{R}^d,\gamma)$.

Proof.

(a) Let $f \in \mathrm{BMO}(\mathbb{R}^d,\gamma)$, and suppose $\{Q_i\}_{i\in\mathbb{N}}$ is a pairwise disjoint family of cubes in $\mathcal{Q}_1$. Thus, for every $1 < p < \infty$,

$$\begin{aligned}
\sum_{i=1}^{\infty}\gamma(Q_i)\left(\frac{1}{\gamma(Q_i)}\int_{Q_i}|f - f_{Q_i}|d\gamma\right)^p &\le \|f\|^p_{\mathrm{BMO}(\mathbb{R}^d,\gamma)}\sum_{i=1}^{\infty}\gamma(Q_i) \\
&\le \|f\|^p_{\mathrm{BMO}(\mathbb{R}^d,\gamma)}.
\end{aligned}$$

Then,

$$\|f\|_{\mathrm{JN}_p^{\mathcal{Q}_1}(\mathbb{R}^d,\gamma)} \le \|f\|_{\mathrm{BMO}(\mathbb{R}^d,\gamma)}, \quad 1 < p < \infty.$$

(b) We adapt an idea given in the proof of [25, Proposition 2.6]. Let $a > 0$, $1 < p < \infty$, $f \in \mathrm{BMO}(\mathbb{R}^d,\gamma)$, and consider $Q \in \mathcal{Q}_a$. We have that

$$\|f\|_{\mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)} \ge \|f\|_{L^1(\mathbb{R}^d,\gamma)} + \gamma(Q)^{1/p}\frac{1}{\gamma(Q)}\int_Q|f - f_Q|d\gamma.$$

Thus,

$$\liminf_{p\to\infty}\|f\|_{\mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)} \ge \|f\|_{L^1(\mathbb{R}^d,\gamma)} + \frac{1}{\gamma(Q)}\int_Q|f - f_Q|d\gamma,$$

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and we obtain that

$$\liminf_{p\to\infty} \|f\|_{\mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)} \geq \|f\|_{\mathrm{BMO}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)}. \tag{2.1}$$

On the other hand, by proceeding as in (a), for every $1 < p < \infty$,

$$\|f\|_{\mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)} \leq \|f\|_{\mathrm{BMO}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)}. \tag{2.2}$$

From (2.1) and (2.2), it follows that

$$\lim_{p\to\infty} \|f\|_{\mathrm{JN}_p^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)} = \|f\|_{\mathrm{BMO}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)}$$

as desired. □

3. A John-Nirenberg inequality for $\mathrm{JN}_p(\mathbb{R}^d,\gamma)$

We now prove Theorem 1.1. Let $f \in \mathrm{JN}_p(\mathbb{R}^d,\gamma)$ and $Q \in \mathcal{Q}_a$. We denote by λ the Lebesgue measure in $\mathbb{R}^d$. Proceeding as in the proof of [12, Lemma 3] we can see that

$$\lambda\left(\{x \in Q : |f(x) - f_{Q,\lambda}| > \sigma\}\right) \leq C\left(\frac{\mathbb{K}_p^{\mathcal{Q}_a}(f)}{\sigma}\right)^p, \tag{3.1}$$

for $\sigma > 0$ where

$$\mathbb{K}_p^{\mathcal{Q}_a}(f) = \sup\left(\sum_{i=1}^\infty \lambda(Q_i)\left(\frac{1}{\lambda(Q_i)}\int_{Q_i}|f - f_{Q_i,\lambda}|d\lambda\right)^p\right)^{1/p}$$

and the supremum is taken over all the pairwise disjoint sequences $\{Q_i\}_{i=1}^\infty$ of cubes in $\mathcal{Q}_a$. Here, $f_{H,\lambda} = \frac{1}{\lambda(H)}\int_H f d\lambda$ for every measurable set H in $\mathbb{R}^d$.

To see this, suppose that H is a cube contained in Q. We have that

$$|c_H| \leq |c_H - c_Q| + |c_Q| \leq \sqrt{d}\ell_Q + |c_Q| \leq a\sqrt{d}m(c_Q) + |c_Q|.$$

Then,

$$\begin{aligned} m(c_H)^{-1} &= \max\{1,|c_H|\} \leq \max\{1,|c_Q|\} + a\sqrt{d}m(c_Q) \\ &\leq m(c_Q)^{-1} + a\sqrt{d}m(c_Q) \leq m(c_Q)^{-1} + a\sqrt{d}. \end{aligned}$$

Therefore

$$m(c_Q) \leq m(c_H)(1 + a\sqrt{d}m(c_Q)) \leq m(c_H)(1 + a\sqrt{d})$$

and it follows that

$$\ell_H \leq \ell_Q \leq am(c_Q) \leq a(1 + a\sqrt{d})m(c_H).$$

According to [12, Lemma 3] we deduce that

$$\lambda\left(\{x \in Q : |f(x) - f_{Q,\lambda}| > \sigma\}\right) \leq C\left(\frac{\mathbb{K}_p^{\mathcal{Q}_{a(1+a\sqrt{d})}}(f)}{\sigma}\right)^p,$$

for $\sigma > 0$ and proceeding as in the proof of Proposition 2.1 we obtain that

$$\mathbb{K}_p^{\mathcal{Q}_{a(1+a\sqrt{d})}}(f) \leq C\mathbb{K}_p^{\mathcal{Q}_a}(f)$$

and (3.1) is proved.

If $H \in \mathcal{Q}_a$, by using [16, Proposition 2.1(i)] we get

$$\frac{1}{\lambda(H)}\int_H |f - f_{H,\lambda}|d\lambda \leq \frac{2}{\lambda(H)}\int_H |f - f_H|d\lambda \leq \frac{C}{\gamma(H)}\int_H |f - f_H|d\gamma.$$

Also, [16, Proposition 2.1(i)] implies that if $b > 0$ there exists $C > 0$ such that for every measurable set $B \subset D$ with $D \in \mathcal{Q}_b$

$$C^{-1}\gamma(B) \leq e^{-|c_D|^2}\lambda(B) \leq C\gamma(B).$$

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It follows that, for $\sigma > 0$,

$$\gamma\left(\{x \in Q : |f(x) - f_{Q,\lambda}| > \sigma\}\right) \leq C \left(\frac{K_p^{\mathcal{Q}_a}(f)}{\sigma}\right)^p. \tag{3.2}$$

Let $\sigma > 0$. We have that

$$\begin{aligned}\gamma\left(\{x \in Q : |f(x) - f_Q| > \sigma\}\right) \leq & \gamma\left(\{x \in Q : |f(x) - f_{Q,\lambda}| > \sigma/2\}\right) \\ & + \gamma\left(\{x \in Q : |f_{Q,\lambda} - f_Q| > \sigma/2\}\right).\end{aligned}$$

As above, we can write

$$|f_{Q,\lambda} - f_Q| \leq \frac{1}{\lambda(Q)} \int_H |f - f_Q| d\lambda \leq \frac{C_0}{\gamma(Q)} \int_H |f - f_Q| d\gamma \leq C_0 \gamma(Q)^{-1/p} K_p^{\mathcal{Q}_a}(f),$$

for certain $C_0 > 0$.

Then

$$\gamma\left(\{x \in Q : |f_{Q,\lambda} - f_Q| > \sigma/2\}\right) \leq \begin{cases} \gamma(Q) & \text{if } 0 < \sigma \leq 2C_0\gamma(Q)^{-1/p} K_p^{\mathcal{Q}_a}(f), \\ 0 & \text{if } \sigma < 2C_0\gamma(Q)^{-1/p} K_p^{\mathcal{Q}_a}(f). \end{cases}$$

We obtain

$$\gamma\left(\{x \in Q : |f_{Q,\lambda} - f_Q| > \sigma/2\}\right) \leq C\gamma(Q) \left(\frac{\gamma(Q)^{-1/p} K_p^{\mathcal{Q}_a}(f)}{\sigma}\right)^p \leq C \left(\frac{K_p^{\mathcal{Q}_a}(f)}{\sigma}\right)^p$$

Using this and estimate (3.2) we conclude that

$$\gamma\left(\{x \in Q : |f(x) - f_Q| > \sigma\}\right) \leq C \left(\frac{K_p^{\mathcal{Q}_a}(f)}{\sigma}\right)^p$$

and the proof of Theorem 1.1 is finished.

4. Duality

In this section we prove Theorem 1.2. In order to do so, we will establish some preliminary results related to the spaces involved and a covering lemma of admissible cubes.

4.1. **Properties of function spaces.** First we consider the space $\mathrm{JN}_{p,q}(\mathbb{R}^d, \gamma)$ as follows. Let $1 < p < \infty$, $1 \leq q < \infty$ and $a > 0$. A function $f \in L^1(\mathbb{R}^d, \gamma)$ is said to be in $\mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d, \gamma)$ when

$$K_{p,q}^{\mathcal{Q}_a}(f) := \sup \left(\sum_{i=1}^{\infty} \gamma(Q_i) \left(\frac{1}{\gamma(Q_i)} \int_{Q_i} |f - f_{Q_i}|^q d\gamma\right)^{p/q}\right)^{1/p} < \infty,$$

where the supremum is taken over all the pairwise disjoint sequences $\{Q_i\}_{i \in \mathbb{N}}$ of cubes in $\mathcal{Q}_a$. The space $\mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d, \gamma)$ is equipped with the norm $\|\cdot\|_{\mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)}$ defined by

$$\|f\|_{\mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)} := \|f\|_{L^1(\mathbb{R}^d,\gamma)} + K_{p,q}^{\mathcal{Q}_a}(f), \quad f \in \mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d, \gamma).$$

When $a > 0$ and $1 \leq q < p$, $\mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d, \gamma)$ actually does not depend on a and q, as shown below.

Proposition 4.1. *Let $a > 0$ and $1 \leq q < p$. Then, $\mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d, \gamma) = \mathrm{JN}_p(\mathbb{R}^d, \gamma)$ algebraically and topologically.*

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Proof. By using Hölder inequality with q and q' we easily get $\mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma) \subseteq \mathrm{JN}_{p,1}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma) = \mathrm{JN}_p(\mathbb{R}^d,\gamma)$ for any $a > 0$, and the inclusion is continuous. Here, we have used Proposition 2.1.

We will now prove the other inclusion. Let $Q \in \mathcal{Q}_a$. Since $1 \le q < p$, $L^{p,\infty}(Q,\gamma)$ is continuously contained in $L^q(Q,\gamma)$, and there exists a constant $C > 0$ independent of Q such that

$$\|g\|_{L^q(Q,\gamma)} \le C\gamma(Q)^{1/q-1/p}\|g\|_{L^{p,\infty}(Q,\gamma)}, \quad g \in L^{p,\infty}(Q,\gamma).$$

Indeed, given $g \in L^{p,\infty}(Q,\gamma)$ we can write, for $t = \|g\|_{L^{p,\infty}(Q,\gamma)}\gamma(Q)^{-1/p}$,

$$\begin{aligned}\|g\|_{L^q(Q,\gamma)}^q &= q\int_0^\infty \sigma^{q-1}\gamma(\{x\in Q : |g(x)| > \sigma\})d\sigma \\ &\le q\left(\int_0^t \sigma^{q-1}\gamma(Q)d\gamma + \int_t^\infty \sigma^{q-1}\gamma(\{x \in Q : |g(x)| > \sigma\})\right) \\ &\le \gamma(Q)t^q + q\int_t^\infty \sigma^{q-1-p}\|g\|_{L^{p,\infty}(Q,\gamma)}^p d\sigma \\ &= \gamma(Q)t^q + q\|g\|_{L^{p,\infty}(Q,\gamma)}^p\frac{t^{q-p}}{p-q} \\ &= \gamma(Q)^{1-q/p}\|g\|_{L^{p,\infty}(Q,\gamma)}^q + \frac{q}{p-q}\gamma(Q)^{1-q/p}\|g\|_{L^{p,\infty}(Q,\gamma)}^q \\ &= \frac{p}{p-q}\gamma(Q)^{1-q/p}\|g\|_{L^{p,\infty}(Q,\gamma)}^q.\end{aligned}$$

Hence,

$$\|g\|_{L^q(Q,\gamma)} \le C_{p,q}\gamma(Q)^{1/q-1/p}\|g\|_{L^{p,\infty}(Q,\gamma)}^q.$$

Now, let $f \in \mathrm{JN}_p(Q,\gamma)$. By proceeding as in the proof of Theorem 1.1 we can deduce that

$$\|f - f_Q\|_{L^{p,\infty}(Q,\gamma)} \le CK_{p,Q}^{\mathcal{Q}_{a(1+\sqrt{d}a)}}(f) \le CK_{p,Q}^{\mathcal{Q}_a}(f),$$

where

$$K_{p,Q}^{\mathcal{Q}_b}(f) := \sup\left(\sum_{i=1}^\infty \gamma(Q_i)\left(\frac{1}{\gamma(Q_i)}\int_{Q_i}|f - f_{Q_i}|d\gamma\right)^p\right)^{1/p} < \infty,$$

and the supremum is taken over all the pairwise disjoint sequences $\{Q_i\}_{i\in\mathbb{N}}$ of cubes in $\mathcal{Q}_a$ contained in Q.

Then, $f - f_Q \in L^q(Q,\gamma)$ and

$$\begin{aligned}\left(\frac{1}{\gamma(Q)}\int_Q |f - f_Q|^q d\gamma\right)^{1/q} &\le C_{p,q}\gamma(Q)^{-1/p}\|f - f_Q\|_{L^{p,\infty}(Q,\gamma)} \\ &\le C\gamma(Q)^{-1/p}K_{p,Q}^{\mathcal{Q}_a}(f).\end{aligned}$$

Suppose that $\{Q_i\}_{i\in\mathbb{N}}$ is a pointwise sequence of cubes in $\mathcal{Q}_a$. From the above inequality we have

$$\sum_{i=1}^\infty \gamma(Q_i)\left(\frac{1}{\gamma(Q_i)}\int_{Q_i}|f - f_{Q_i}|^q d\gamma\right)^{p/q} \le C\sum_{i=1}^\infty \left(K_{p,Q_i}^{\mathcal{Q}_a}(f)\right)^p.$$

Let $\epsilon > 0$. For every $i \in \mathbb{N}$, we choose a pairwise disjoint sequence $\{Q_{i,j}\}_{j\in\mathbb{N}}$ of cubes in $\mathcal{Q}_a$ for which

$$\left(K_{p,Q_i}^{\mathcal{Q}_a}(f)\right)^p \le \sum_{j=1}^\infty \gamma(Q_{i,j})\left(\frac{1}{\gamma(Q_{i,j})}\int_{Q_{i,j}}|f - f_{Q_{i,j}}|d\gamma\right)^p + \frac{\epsilon}{2^i}.$$

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We get

$$\begin{aligned}\sum_{i=1}^{\infty}\gamma(Q_i)\left(\frac{1}{\gamma(Q_i)}\int_{Q_i}|f-f_{Q_i}|^q d\gamma\right)^{p/q}\\ &\leq C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\gamma(Q_{i,j})\left(\frac{1}{\gamma(Q_{i,j})}\int_{Q_{i,j}}|f-f_{Q_{i,j}}|d\gamma\right)^p+\sum_{i=1}^{\infty}\frac{\epsilon}{2^i}\\ &\leq C\left(K_p^{\mathcal{Q}_a}(f)\right)^p+\epsilon.\end{aligned}$$

The arbitrariness of $\epsilon > 0$ allows us to obtain

$$\left(\sum_{i=1}^{\infty}\gamma(Q_i)\left(\frac{1}{\gamma(Q_i)}\int_{Q_i}|f-f_{Q_i}|^q d\gamma\right)^{p/q}\right)^{1/p} \leq CK_p^{\mathcal{Q}_a}(f).$$

This implies that $\mathrm{JN}_p(\mathbb{R}^d,\gamma)$ is continuously contained in $\mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)$ so the proof is now finished. □

Proposition 4.2. *Let $1 < r < s < \infty$ and $a > 0$. The linear space*

$$A_{a,s} := \operatorname{span}\left\{\left(\bigcup_{Q\in\mathcal{Q}_a} L_0^s(Q,\gamma)\right)\cup\{c\chi_{\mathbb{R}^d} : c\in\mathbb{C}\}\right\}$$

is dense in $H_{r,s,a}(\mathbb{R}^d,\gamma)$.

Proof. Suppose first that g is an (r,s,a)-polymer. We can write $g=\sum_{j=1}^{\infty}b_j$ where, for every $j\in\mathbb{N}$, $\operatorname{supp} b_j\subset Q_j\in\mathcal{Q}_a$ and $b_j\in L_0^s(Q_j,\gamma)$, being $\{Q_j\}_{j\in\mathbb{N}}$ a sequence of pairwise disjoint cubes. Also, we have that (1.1) holds.

We can write

$$\left\|g-\sum_{j=1}^{k}b_j\right\|_{(r,s,a)} \leq \left(\sum_{j=k+1}^{\infty}\gamma(Q_j)\left(\frac{1}{\gamma(Q_j)}\int_{Q_j}|b_j|^s d\gamma\right)^{r/s}\right)^{1/r}.$$

Due to the convergence of the series, for every $\epsilon > 0$, there exists $j_0\in\mathbb{N}$ such that

$$\left\|g-\sum_{j=1}^{j_0}b_j\right\|_{(r,s,a)} < \epsilon.$$

Given now $g\in H_{r,s,a}(\mathbb{R}^d,\gamma)$, where $g=c_0+\sum_{i=1}^{\infty}g_i$ with $c_0\in\mathbb{C}$, g_i is an (r,s,a)-polymer for every $i\in\mathbb{N}$, and $\sum_{i=1}^{\infty}\|g_i\|_{(r,s,a)}<\infty$, for every $\epsilon>0$, there exists $i_0\in\mathbb{N}$ such that

$$\left\|g-c_0-\sum_{i=1}^{i_0}g_i\right\|_{H_{r,s,a}(\mathbb{R}^d,\gamma)} \leq \sum_{i=i_0+1}^{\infty}\|g_i\|_{(r,s,a)} < \frac{\epsilon}{2}.$$

For each of these (r,s,a)-polymers g_i, we have that $g_i=\sum_{j=1}^{\infty}b_{ij}$ as above. Therefore, for every $i\in\mathbb{N}$, there exists $l_i\in\mathbb{N}$ such that

$$\left\|g_i-\sum_{j=1}^{l}b_{ij}\right\|_{(r,s,a)} < \frac{\epsilon}{2i_0}, \quad \text{for every } l\geq l_i.$$

Then,

$$\left\|g-c_0-\sum_{i=1}^{i_0}\sum_{j=1}^{l_i}b_{ij}\right\|_{H_{r,s,a}(\mathbb{R}^d,\gamma)}$$

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$$\leq \left\| g - c_0 - \sum_{i=1}^{i_0} g_i \right\|_{H_{r,s,a}(\mathbb{R}^d,\gamma)} + \left\| \sum_{i=1}^{i_0} \left(g_i - \sum_{j=1}^{l_i} b_{ij} \right) \right\|_{H_{r,s,a}(\mathbb{R}^d,\gamma)}$$
$$\leq \frac{\epsilon}{2} + \sum_{i=1}^{i_0} \left\| g_i - \sum_{j=1}^{l_i} b_{ij} \right\|_{(r,s,a)} < \epsilon.$$

The proof is now concluded. □

Proposition 4.3. *Let $a_1, a_2 > 0$ and $1 < p < q < \infty$. Then*

$$H_{p,q,a_1}(\mathbb{R}^d,\gamma) = H_{p,q,a_2}(\mathbb{R}^d,\gamma)$$

algebraically and topologically.

Proof. Without loss of generality, we may assume $0 < a_2 < a_1$. Since $\mathcal{Q}_{a_2} \subset \mathcal{Q}_{a_1}$ it is immediate that $H_{p,q,a_2}(\mathbb{R}^d,\gamma)$ is continuously contained in $H_{p,q,a_1}(\mathbb{R}^d,\gamma)$.

We now prove the converse inclusion.

Let us fix $Q \in \mathcal{Q}_{a_1}$ and $v \in L^q_0(Q,\gamma)$. We consider, as in [16, Lemma 2.3], the family of 2^d cubes P_i contained in Q with sides parallel to the axes, each having sidelength $\ell_{P_i} = \frac{2}{3}\ell_Q$ and a vertex in common with the cube Q. They verify that

$$\bigcap_{i=1}^{2^d} P_i = P_0,$$

where P_0 is a cube with $c_{P_0} = c_Q$ and $\ell_{P_0} = \frac{1}{3}\ell_Q$. As in the proof of [16, Lemma 2.3], it can be obtained that $P_i \in \mathcal{Q}_{\frac{2}{3}a_1(1+\sqrt{d}\frac{a_1}{2})}$ for each $i = 0, \dots, 2^d$. Consequently,

$$\gamma(P_i) \leq C_{d,a_1}\gamma(P_0), \quad i = 1, \dots, 2^d. \tag{4.1}$$

We define the functions and scalars given also in the aforementioned proof. For every $i = 1, \dots, 2^d$,

$$\psi_i = \frac{\chi_{P_i}}{\sum_{k=1}^{2^d} \chi_{P_k}}, \quad \lambda_i = \frac{1}{\gamma(P_0)} \int_{\mathbb{R}^d} v\psi_i d\gamma,$$

where χ_E denotes the characteristic function of the measurable set $E \subset \mathbb{R}^d$, and set

$$v_i = v\psi_i - \lambda_i \chi_{P_0}, \quad v_0 = v - \sum_{i=1}^{2^d} v_i.$$

For every $i = 0, \dots, 2^d$ it is clear that $\operatorname{supp} v_i \subset P_i$ and $\int_{P_i} v_i d\gamma = 0$. Moreover, for $i = 1, \dots, 2^d$, by Hölder inequality and (4.1) we get

$$\begin{aligned}\|v_i\|_{L^q_0(P_i,\gamma)} &\leq \|v\|_{L^q(P_i,\gamma)} + \left(\frac{1}{\gamma(P_0)} \int_{P_i} |v||\psi_i| d\gamma \right) \gamma(P_0)^{1/q} \\ &\leq \|v\|_{L^q(P_i,\gamma)} + \|v\|_{L^q(P_i,\gamma)} \frac{\gamma(P_i)}{\gamma(P_0)} \\ &\leq C\|v\|_{L^q(P_i,\gamma)},\end{aligned}$$

and

$$\|v_0\|_{L^q_0(P_0,\gamma)} \leq \|v\|_{L^q(P_0,\gamma)} + \sum_{i=1}^{2^d} \|v_i\|_{L^q(P_0,\gamma)} \leq (1 + C2^d)\|v\|_{L^q(P_0,\gamma)}.$$

If $\frac{2}{3}a_1(1+\sqrt{d}\frac{a_1}{2}) \leq a_2$, we have $P_i \in \mathcal{Q}_{a_2}$ for every $i = 0, \dots, 2^d$, so we are done. If, otherwise, there exists some P_i not in $\mathcal{Q}_{a_2}$, we repeat the previous construction for each of these cubes not belonging to $\mathcal{Q}_{a_2}$. If necessary, we iterate the argument.

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Notice that it will suffice to repeat this construction at most n times, where

$$n := \min\left\{k \in \mathbb{N} : \left(\tfrac{2}{3}\right)^k a_1 \left(1 + \sqrt{d}\tfrac{a_1}{2}\right) \le a_2\right\}.$$

This process produce a decomposition of Q into a family of cubes $\{P_i\}_{i=1}^{i_0}$ in $\mathcal{Q}_{a_2}$, a decomposition of v into a family of functions $\{v_i\}_{i=1}^{i_0}$ for some $i_0 \in \mathbb{N}$ with $i_0 \le (1+2^d)^n$ such that, for every $i = 1, \dots, i_0$, $\operatorname{supp} v_i \subset P_i$, $v_i \in L_0^q(P_i, \gamma)$ and $\|v_i\|_{L_0^q(P_i,\gamma)} \le (1+C2^d)^n \|v\|_{L_0^q(Q,\gamma)}$. According to [16, Proposition 2.1(i)], for every $i = 0, \dots, i_0$, $\gamma(P_i) \sim \gamma(Q)$, where the equivalence does not depend on Q.

Let g be a (p, q, a_1)-polymer, that is, $g = \sum_{j=1}^{\infty} v_j$ where, for each $j \in \mathbb{N}$, $\operatorname{supp} v_j \subset Q_j \in \mathcal{Q}_{a_1}$ and $v_j \in L_0^q(Q_j, \gamma)$, and the sequence of cubes $\{Q_j\}_{j\in\mathbb{N}}$ is pairwise disjoint with

$$\sum_{j=1}^{\infty} \gamma(Q_j) \left(\frac{1}{\gamma(Q_j)} \int_{Q_j} |v_j|^q d\gamma\right)^{p/q} < \infty. \tag{4.2}$$

Let $j \in \mathbb{N}$. We work as before with v_j and Q_j. Hence, we obtain a family of cubes $\{P_{ij}\}_{i=1}^{i_0}$ in $\mathcal{Q}_{a_2}$ and a collection $\{v_{ij}\}_{i=1}^{i_0}$ of functions satisfying that

(a) $Q_j = \left(\bigcup_{i=1}^{i_0} P_{ij}\right) \cup E_j$, where $|E_j| = 0$;
(b) $\operatorname{supp} v_{ij} \subset P_{ij}$, $v_{ij} \in L_0^q(P_{ij}, \gamma)$ and $\|v_{ij}\|_{L_0^q(P_{ij},\gamma)} \le (1 + C2^d)^n \|v_j\|_{L_0^q(Q_j,\gamma)}$ for every $i = 1, \dots, i_0$;
(c) there exists $C = C(a_1, a_2, d) > 0$ such that

$$\frac{1}{C}\gamma(P_{ij}) \le \gamma(Q_j) \le C\gamma(P_{ij}), \quad i = 1, \dots, i_0;$$

(d) $v_j = \sum_{i=1}^{i_0} v_{ij}$.

Consequently, for any representation of $g = \sum_{j=1}^{\infty} v_j$ as above, we have another representation of $g = \sum_{ij} v_{ij} = \sum_{i=1}^{i_0} V_i$ satisfying the properties (a)-(d). Moreover, for each $i = 1, \dots, i_0$, V_i is a (p, q, a_2)-polymer since

$$\begin{aligned}\|V_i\|_{(p,q,a_2)}^p &\le \sum_{j=1}^{\infty} \gamma(P_{ij}) \left(\frac{1}{\gamma(P_{ij})} \int_{P_{ij}} |v_{ij}|^q\right)^{p/q} \\ &\le C \sum_{j=1}^{\infty} \gamma(Q_j) \left(\frac{1}{\gamma(Q_j)} \int_{Q_j} |v_j|^q\right)^{p/q} < \infty,\end{aligned}$$

and thus

$$\|g\|_{H_{p,q,a_2}(\mathbb{R}^d,\gamma)} \le \sum_{i=1}^{i_0} \|V_i\|_{(p,q,a_2)} \le C\ i_0 \|g\|_{(p,q,a_1)}$$

being $g \in H_{p,q,a_2}(\mathbb{R}^d, \gamma)$. Notice that the constant does not depend on g. Then,

$$\|g\|_{H_{p,q,a_2}(\mathbb{R}^d,\gamma)} \le C\|g\|_{(p,q,a_1)}.$$

Suppose now that $g = c_0 + \sum_{j=1}^{\infty} g_j$ where $c_0 \in \mathbb{C}$ and g_j is a (p, q, a_1)-polymer for each $j \in \mathbb{N}$ with $\sum_{j=1}^{\infty} \|g_j\|_{(p,q,a_1)} < \infty$. We can represent each $g_j = \sum_{i=1}^{i_0} V_{ij}$ where V_{ij} is a (p, q, a_2)-polymer. Then, as before we can estimate

$$\sum_{i=1}^{i_0} \sum_{j=1}^{\infty} \|V_{ij}\|_{(p,q,a_2)} \le C \sum_{j=1}^{\infty} \|g_j\|_{(p,q,a_1)} < \infty.$$

We conclude that $g \in H_{p,q,a_2}(\mathbb{R}^d, \gamma)$ with

$$\|g\|_{H_{p,q,a_2}(\mathbb{R}^d,\gamma)} \le C\|g\|_{H_{p,q,a_1}(\mathbb{R}^d,\gamma)}$$

and the constant does not depend on g. □

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4.2. **A covering lemma.** We now introduce a covering by cubes in $\mathbb{R}^n$ that will be very useful in the proof of Theorem 1.2.

Lemma 4.4. *There exists a sequence of cubes $\{Q_n\}_{n\in\mathbb{N}}$ such that*

(a) $\mathbb{R}^d = \cup_{n\in N} Q_n \cup E$ with E having null Lebesgue (equivalently Gaussian) measure.
(b) For every $n\in\mathbb{N}$, $Q_n \in \mathcal{Q}_{A_d}$, where $A_d = 2\sqrt{d}$.
(c) There exists C_d depending only on the dimension d such that

$$\#\{j\in\mathbb{N} : Q_n\cap Q_j \neq \emptyset\} \le C_d, \quad n\in\mathbb{N}.$$

(d) There exists, for each $k\in\mathbb{N}$, $k\ge 2$, a subfamily of cubes in the covering, named the k-th layer L_k, such that $\#L_k \le Ck^{d-1}$ where C only depends on d, and if $Q\in L_k$ then $m(c_Q)\le \ell_Q \le 2\sqrt{d}m(c_Q)$ and there exists a constant M, independent of k, such that $\frac{1}{M}\sqrt{k}\le |c_Q| \le Mk^{d/2}$.

Proof. We consider the interval $I=(\alpha,\beta)$ with $-\infty<\alpha<\beta<\infty$, and let $0<\delta<\beta-\alpha$. We divide the interval I in subintervals with length $(\beta-\alpha)/\delta$ as follows:

(a) If $\beta = \alpha + \ell\,\frac{\beta-\alpha}{\delta}$, for some $\ell\in\mathbb{N}$, we write I_j as above for $j=1,\dots,\ell$,

$$I_j = \left(\alpha + (j-1)\frac{\beta-\alpha}{\delta}, \alpha + j\,\frac{\beta-\alpha}{\delta}\right)$$

(b) If $\alpha + (\ell-1)\,\frac{\beta-\alpha}{\delta} < \beta < \alpha + \ell\,\frac{\beta-\alpha}{\delta}$, for some $\ell\in\mathbb{N}$, we write I_j as above for $j=1,\dots,\ell-1$ and

$$I_\ell = \left(\beta - \frac{\beta-\alpha}{\delta}, \beta\right)$$

When we divide the interval I as above we say that I is divided by δ finishing in β.

Consider the sequence defined by

$$a_1 = 1 \qquad \text{and} \qquad a_{k+1} = a_k + \frac{1}{a_k},\ k\ge 1,$$

which is increasing and $a_k\to\infty$ as $k\to\infty$. By proceeding as in [16, p. 298] we can see that $a_k \sim \sqrt{k}$. In fact,

$$\sqrt{2k} \le a_{k+1} \le \sqrt{3k},$$

for $k\in\mathbb{N}$.

Let $k\in\mathbb{N}$, $k\ge 1$. We define P_k the cube of center $c_{P_k}=0$ and side $\ell_{P_k} = 2a_k$, and $R_k = P_{k+1}\setminus P_k$.

We have that

$$R_k = \bigcup_{j=1}^{d} (R^+_{k,j}\cup R^-_{k,j})\cup E,$$

where E has null Lebesgue measure and, for every $j=1,\dots,d$,

$$R^+_{k,j} = \{(x_1,\dots,x_d)\in\mathbb{R}^d : a_k < x_j < a_{k+1},\ |x_i| < a_{k+1}, i=1,\dots,d, i\neq j\}$$

and

$$R^-_{k,j} = \{(x_1,\dots,x_d)\in\mathbb{R}^d : -a_{k+1} < x_j < -a_k,\ |x_i| < a_{k+1}, i=1,\dots,d, i\neq j\}.$$

Let $j=1,\dots,d$. We denote by $\{I^k_s\}_{s=1}^{\ell_{k+1}}$ the division of $(-a_{k+1},a_{k+1})$ by $1/a_k$ finishing in a_{k+1}. We name $I_0^{k,+} = (a_k, a_{k+1})$. We define

$$H^+_{j,s_1,\dots,s_{j-1},s_{j+1},\dots,s_d} = \left(\prod_{i=1}^{j-1} I^k_{s_i}\right)\times I_0^{k,+}\times\left(\prod_{i=j+1}^{d} I^k_{s_i}\right),$$

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where $s_i \in \{1, \dots, \ell_{k+1}\}$, $i = 1, \dots, d$, $i \neq j$.

We have that

$$R^+_{k,j} = \cup H^+_{j,s_1,\dots,s_{j-1},s_0,s_{j+1},\dots,s_k}.$$

In a similar way we can write

$$R^-_{k,j} = \cup H^-_{j,s_1,\dots,s_{j-1},s_{j+1},\dots,s_k},$$

where H^- is defined as H^+ by replacing $I_0^{k,+}$ with $I_0^{k,-} = (-a_{k+1}, -a_k)$.

Thus we obtain a covering (modulo a set with null Lebesgue measure) of R_k. The cubes in this covering will be called the cubes in the k-th layer.

Note that if Q and Q' are cubes in different layers then $Q \cap Q' = \emptyset$. Also, there exists $C = C(d)$ such that, for every $k \in \mathbb{N}$ and every cube Q in the k-th layer

$$\#\{Q' : Q' \cap Q \neq \emptyset,\ Q' \text{ is a cube in the } k\text{-th layer}\} \leq C.$$

On the other hand, since $a_k \sim \sqrt{k}$ we have that

$$\# L_k \leq C k^{d-1},$$

where L_k collects the cubes of the covering in the k-th layer, and there exists $M > 0$ such that for every $Q \in L_k$, $\frac{1}{M}\sqrt{k} \leq |c_Q| \leq M k^{d/2}$. We define $\{Q_n\}_{n\in\mathbb{N}} = \bigcup_{k\in\mathbb{N}} L_k$. □

Let $A > 1$ and $Q \in \mathcal{Q}_A$ such that $m(c_Q) \leq \ell_Q$ and $|c_Q| > 1/2$. Let B be the ball inscribed in Q, that is, $c_B = c_Q$ and $r_B = \ell_Q/2$. We can construct a ball $M(B)$ such that $c_{M(B)}$ is in the segment joining 0 and c_B, c_B is in the boundary of $M(B)$ and $r_{M(B)} = m(c_{M(B)})/2$.

To show this construction can be made suppose first that $|c_B| \geq 3/2$. In this case we can set $c_{M(B)}$ in the segment joining 0 and c_B such that

$$|c_{M(B)}| = \frac{|c_B| + \sqrt{|c_B|^2 - 2}}{2}.$$

Therefore $|c_{M(B)}| \geq 1$ and setting $r_{M(B)} = 1/(2|c_{M(B)}|)$ we obtain

$$r_{M(B)} = \frac{m(c_{M(B)})}{2} \qquad \text{and} \qquad |c_{M(B)}| + r_{M(B)} = |c_B|$$

showing that c_B lies in the boundary of $M(B) = B(c_{M(B)}, r_{M(B)})$.

On the other hand, if $1/2 < |c_B| < 3/2$ we can choose $c_{M(B)}$ in the segment joining 0 and c_B such that $|c_{M(B)}| = |c_B| - 1/2$. Then, choosing $r_{M(B)} = 1/2$ we again that, since $|c_B| < 1$,

$$r_{M(B)} = \frac{m(c_{M(B)})}{2} \qquad \text{and} \qquad |c_{M(B)}| + r_{M(B)} = |c_B|$$

showing that c_B lies in the boundary of $M(B) = B(c_{M(B)}, r_{M(B)})$.

Also, if we name $M(Q)$ the cube circumscribed around $M(B)$ we have that $c_{M(B)} = c_{M(Q)}$ and $\ell_{M(Q)} = 2r_{M(B)} = m(c_{M(Q)})$. Therefore, it is possible to iterate this procedure obtaining $M^2(Q) = M(M(Q))$ as long as $|c_{M(Q)}| > 1/2$. Given a cube $Q \in \mathcal{Q}_A$ such that $m(c_Q) \leq \ell_Q$ and $|c_Q| > 1/2$ we will denote K_Q the integer such that $|c_{M^{K_Q}(Q)}| \leq 1/2$ and $|c_{M^{K_Q-1}(Q)}| > 1/2$.

Lemma 4.5. *Let $A > 1$ and $Q \in \mathcal{Q}_A$ such that $m(c_Q) \leq \ell_Q$ and $|c_Q| > 1/2$. Let B be the ball inscribed in Q, $M(B)$ and $M(Q)$ the ball and the cube obtained in the construction above, B' the larger ball contained in $M(B) \cap B$ and Q' the cube circumscribed around B'. Then there exists $K > 0$ independent of Q such that*

$$\frac{\gamma(M(Q))}{\gamma(Q')} \leq K.$$

Also if $|c_Q| \leq D\sqrt{d}$ for some constant D, then $K_Q \leq D^2 d$.

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Proof. Notice that if $M(B) \subset B$, then $M(Q) = Q'$ and

$$\frac{\gamma(M(Q))}{\gamma(Q')} = 1.$$

If $M(B) \subsetneq B$ we distinguish 3 cases. First, suppose that $|c_{M(B)}| \geq 1$. Then

$$\begin{aligned}\ell_{M(Q)} - \ell_Q &= 2(r_{M(B)} - r_B) \leq m(c_{M(B)}) - m(c_B)\\ &= \frac{1}{|c_{M(B)}|} - \frac{1}{|c_B|} = \frac{|c_B| - |c_{M(B)}|}{|c_B||c_{M(B)}|}\\ &= \frac{r_{M(B)}}{|c_B||c_{M(B)}|} \leq 4r_{M(B)}^2 r_B = \frac{\ell_{M(Q)}^2 \ell_Q}{2}.\end{aligned}$$

Thus

$$\ell_{M(Q)} \leq \ell_Q \left(1 + \frac{\ell_{M(Q)}^2}{2}\right).$$

Now,

$$\ell_Q \leq A\ell_{M(Q)} \leq A\ell_Q \left(1 + \frac{\ell_{M(Q)}^2}{2}\right) \leq A\ell_Q \left(1 + \frac{m^2(c_{M(Q)})}{2}\right) \leq \frac{3A}{2}\ell_Q.$$

Therefore

$$\frac{2}{3}\ell_{M(Q)} \leq \ell_Q \leq A\ell_{M(Q)}.$$

Since $2\ell_{Q'} = \ell_Q$ the conclusion follows.

Second, suppose that $M(B) \subsetneq B$ and $|c_{M(B)}| \leq |c_B| \leq 1$. In this case, $r_B = 2r_{B'} \geq r_{M(B)}$ and the conclusion follows immediately.

Suppose now that $|c_{M(B)}| \leq 1 \leq |c_B|$. Then, proceeding similarly to the first case

$$\ell_{M(Q)} - \ell_Q \leq 2r_{M(B)} r_B = \frac{\ell_{M(Q)}\ell_Q}{2}.$$

Thus

$$\ell_{M(Q)} \leq \ell_Q \left(1 + \frac{\ell_{M(Q)}}{2}\right).$$

Now,

$$\ell_Q \leq A\ell_{M(Q)} \leq A\ell_Q \left(1 + \frac{\ell_{M(Q)}}{2}\right) \leq A\ell_Q \left(1 + \frac{m(c_{M(Q)})}{2}\right) \leq \frac{3A}{2}\ell_Q.$$

Therefore

$$\frac{2}{3}\ell_{M(Q)} \leq \ell_Q \leq A\ell_{M(Q)}.$$

Since $2\ell_{Q'} = \ell_Q$ the conclusion follows.

Finally, assume that $|c_Q| \leq D\sqrt{d}$, then

$$\ell_{M^j(Q)} = m(c_{M^j(Q)}) \geq m(c_Q) \geq \frac{1}{2|c_Q|} \geq \frac{1}{2D\sqrt{d}}.$$

Thus,

$$K_Q \leq 4D\sqrt{d}|c_Q| \leq D^2 d.$$ □

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4.3. **Proof of Theorem 1.2.** We will prove (a).

We first establish a result that will help us to better understand the proof of (a).

Lemma 4.6. *Let $1<q<p$, $a>0$, $f\in \mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)$, and $C_1, C_2>0$. Assume that for every $i\in\mathbb{N}$, g_i is a (p',q',a)-polymer with $\sum_{i=1}^\infty \|g\|_{(p',q',a)}<\infty$. For every $i\in\mathbb{N}$, we have that $g_i=\sum_{j=1}^\infty b_{ij}$ where, for every $j\in\mathbb{N}$, $b_{ij}\in L_0^{q'}(Q_j,\gamma)$ and $\operatorname{supp} b_{ij}\subset Q_{ij}$, being $\{Q_{ij}\}_{j\in\mathbb{N}}$ a family of pairwise disjoint cubes in $\mathcal{Q}_a$, and suppose that*

$$\left(\sum_{j=1}^\infty \gamma(Q_{ij})\left(\frac{1}{\gamma(Q_{ij})}\int_{Q_{ij}}|b_{ij}|^{q'}d\gamma\right)^{p'/q'}\right)^{1/p'}\leq C_1\|g_i\|_{(p',q',a)},\quad i\in\mathbb{N}.$$

Then,

$$\sum_{i=1}^\infty\sum_{j=1}^\infty\left|\int_{Q_{ij}} f b_{ij}d\gamma\right|\leq C_1\|f\|_{\mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)}\sum_{i=1}^\infty\|g_i\|_{(p',q',a)}. \tag{4.3}$$

If, in addition, $g=c_0+\sum_{i=1}^\infty g_i$ for some $c_0\in\mathbb{C}$, and

$$|c_0|+\sum_{i=1}^\infty\|g_i\|_{(p',q',a)}\leq C_2\|g\|_{H_{p',q',a}(\mathbb{R}^d,\gamma)},$$

then

$$\left|c_0\int_{\mathbb{R}^d} f d\gamma+\sum_{i=1}^\infty\sum_{j=1}^\infty\int_{Q_{ij}} f b_{ij}d\gamma\right|\leq (1+C_1)C_2\|f\|_{\mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)}\|g\|_{H_{p',q',a}(\mathbb{R}^d,\gamma)}. \tag{4.4}$$

Proof. First, let us show that, for every $i\in\mathbb{N}$, (4.3) holds, where f, g_i, b_{ij} and the cubes are as in the hypotheses. Indeed, for $i\in\mathbb{N}$, since $\int_{Q_{ij}} b_{ij}d\gamma=0$ for each $j\in\mathbb{N}$, by Hölder inequality we get

$$\begin{aligned}
\sum_{j=1}^\infty\left|\int_{Q_{ij}} f b_{ij}d\gamma\right| &= \sum_{j=1}^\infty \gamma(Q_{ij})\frac{1}{\gamma(Q_{ij})}\left|\int_{Q_{ij}}(f-f_{Q_{ij}})b_{ij}d\gamma\right|\\
&\leq \sum_{j=1}^\infty \gamma(Q_{ij})\left(\frac{1}{\gamma(Q_{ij})}\int_{Q_{ij}}|f-f_{Q_{ij}}|^q d\gamma\right)^{1/q}\left(\frac{1}{\gamma(Q_{ij})}\int_{Q_{ij}}|b_{ij}|^{q'}d\gamma\right)^{1/q'}\\
&\leq \left(\sum_{j=1}^\infty \gamma(Q_{ij})\left(\frac{1}{\gamma(Q_{ij})}\int_{Q_{ij}}|f-f_{Q_{ij}}|^q d\gamma\right)^{p/q}\right)^{1/p}\\
&\quad\times\left(\sum_{j=1}^\infty \gamma(Q_{ij})\left(\frac{1}{\gamma(Q_{ij})}\int_{Q_{ij}}|b_{ij}|^{q'}d\gamma\right)^{p'/q'}\right)^{1/p'}\\
&\leq \|f\|_{\mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)}\left(\sum_{j=1}^\infty \gamma(Q_{ij})\left(\frac{1}{\gamma(Q_{ij})}\int_{Q_{ij}}|b_{ij}|^{q'}d\gamma\right)^{p'/q'}\right)^{1/p'}\\
&\leq C_1\|f\|_{\mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)}\|g_i\|_{(p',q',a)}.
\end{aligned} \tag{4.5}$$

On the other hand, if $g=c_0+\sum_{i=1}^\infty g_i$ for some $c_0\in\mathbb{C}$ and g_i as before for every $i\in\mathbb{N}$, with $|c_0|+\sum_{i=1}^\infty\|g_i\|_{(p',q',a)}\leq C_2\|g\|_{H_{p',q',a}(\mathbb{R}^d,\gamma)}$, we can write

$$\left|c_0\int_{\mathbb{R}^d} f d\gamma+\sum_{i=1}^\infty\sum_{j=1}^\infty\int_{Q_{ij}} f b_{ij}d\gamma\right|$$

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$$\begin{aligned}&\leq |c_0|\|f\|_{\mathrm{JN}^{\mathcal{Q}_a}_{p,q}(\mathbb{R}^d,\gamma)} + C_1\|f\|_{\mathrm{JN}^{\mathcal{Q}_a}_{p,q}(\mathbb{R}^d,\gamma)}\sum_{i=1}^\infty \|g_i\|_{(p',q',a)}\\ &\leq (1+C_1)\|f\|_{\mathrm{JN}^{\mathcal{Q}_a}_{p,q}(\mathbb{R}^d,\gamma)}\left(|c_0|+\sum_{i=1}^\infty \|g_i\|_{(p',q',a)}\right)\\ &\leq (1+C_1)C_2\|f\|_{\mathrm{JN}^{\mathcal{Q}_a}_{p,q}(\mathbb{R}^d,\gamma)}\|g\|_{H_{p',q',a}(\mathbb{R}^d,\gamma)}. \qquad \square\end{aligned}$$

Let $f \in \mathrm{JN}_p(\mathbb{R}^d,\gamma)$. According to Proposition 4.1, given $a > 0$, $f \in \mathrm{JN}^{\mathcal{Q}_a}_{p,q}(\mathbb{R}^d,\gamma)$ for any $1 < q < p$ so the previous results apply to f.

We now adapt some of the ideas of [5, pp. 599-600]. We define, for every $N \in \mathbb{N}$,

$$f_N(x) = \begin{cases} f(x), & \text{if } |f(x)| \leq N \\ N\,\mathrm{sgn}(f(x)), & \text{if } |f(x)| > N. \end{cases}$$

Assume that $g \in H_{p',q',a}(\mathbb{R}^d,\gamma)$. Our next objective is to see that the limit

$$\lim_{N\to\infty}\int_{\mathbb{R}^d} f_N g\, d\gamma$$

exists, and also that

$$\Lambda_f g := \lim_{N\to\infty}\int_{\mathbb{R}^d} f_N g\, d\gamma$$

it satisfies

$$|\Lambda_f g| \leq C\|f\|_{\mathrm{JN}_p(\mathbb{R}^d,\gamma)}\|g\|_{H_{p',q',a}(\mathbb{R}^d,\gamma)}$$

for some constant $C > 0$ independent of f and g.

According to [20, Remark 1.1.3, p. 141] and [8, Exercise 3.1.4], for every cube Q in $\mathbb{R}^d$,

$$\int_Q |f_N - (f_N)_Q|^q d\gamma \leq C\int_Q |f - f_Q|^q d\gamma, \tag{4.6}$$

where the constant C does not depend on N and Q.

From (4.6) we deduce that

$$\|f_N\|_{\mathrm{JN}^{\mathcal{Q}_a}_{p,q}(\mathbb{R}^d,\gamma)} \leq \|f\|_{\mathrm{JN}^{\mathcal{Q}_a}_{p,q}(\mathbb{R}^d,\gamma)}, \quad N \in \mathbb{N}.$$

On the other hand, notice that $f_N \in L^\infty(\mathbb{R}^d,\gamma)$ for every $N \in \mathbb{N}$.

If $g \equiv c_0$ for some $c_0 \in \mathbb{C}$, our objective is clear. Otherwise, we can write $g = c_0 + \sum_{i=1}^\infty g_i$ for some $c_0 \in \mathbb{C}$ and $0 \not\equiv g_i$ being (p',q',a)-polymers for every $i \in \mathbb{N}$.

For every polymer g_i, since $\|g_i\|_{(p',q',a)} \neq 0$ we can write $g_i = \sum_{j=1}^\infty b_{ij}$ where, for every $j \in \mathbb{N}$, $b_{ij} \in L_0^{q'}(Q_{ij},\gamma)$ and $\operatorname{supp} b_{ij} \subset Q_{ij}$, being $\{Q_{ij}\}_{j\in\mathbb{N}}$ a family of pairwise disjoint cubes in $\mathcal{Q}_a$, with the property that

$$\left(\sum_{j=1}^\infty \gamma(Q_{ij})\left(\frac{1}{\gamma(Q_{ij})}\int_{Q_{ij}}|b_{ij}|^{q'}d\gamma\right)^{p'/q'}\right)^{1/p'} \leq 2\|g_i\|_{(p',q',a)}.$$

By using (1.2), the polymeric expansion converges in $L^{p'}(\mathbb{R}^d,\gamma)$ and, since γ is a probability measure, also converges in $L^1(\mathbb{R}^d,\gamma)$. We have that

$$\int_{\mathbb{R}^d} f_N g\, d\gamma = c_0\int_{\mathbb{R}^d} f\, d\gamma + \sum_{i=1}^\infty \int_{\mathbb{R}^d} f_N g_i\, d\gamma, \quad N \in \mathbb{N}.$$

Since $f \in L^1(\mathbb{R}^d,\gamma)$ and $|f_N| \leq |f|$ for every $N \in \mathbb{N}$, the Dominated Convergence Theorem leads to

$$\lim_{N\to\infty}\int_{\mathbb{R}^d} f_N\, d\gamma = \int_{\mathbb{R}^d} f\, d\gamma.$$

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Since, for every $i \in \mathbb{N}$, $\{Q_{ij}\}_{j\in\mathbb{N}}$ is pairwise disjoint and $\operatorname{supp}(b_{ij}) \subset Q_{ij}$, $j \in \mathbb{N}$, we can write

$$\sum_{i=1}^{\infty} \int_{\mathbb{R}^d} f_N g_i d\gamma = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{\mathbb{R}^d} f_N b_{ij} d\gamma, \quad N \in \mathbb{N}.$$

According to Proposition 4.1 and Theorem 1.1, there exists $C > 0$ such that for every $Q \in \mathcal{Q}_a$ and $\sigma > 0$,

$$\gamma\left(\{x \in Q : |f(x) - f_Q| > \sigma\}\right) \le C \left(\frac{K_p^{\mathcal{Q}_a}(f)}{\sigma}\right)^p.$$

Therefore, given $Q \in \mathcal{Q}_a$ and $\sigma > 0$, we can write

$$\begin{aligned}\gamma\left(\{x \in Q : |f(x)| > \sigma\}\right) &\le \gamma\left(\{x \in Q : |f(x) - f_Q| > \tfrac{\sigma}{2}\}\right) + \gamma\left(\{x \in Q : |f_Q| > \tfrac{\sigma}{2}\}\right)\\ &\le C \left(\frac{K_p^{\mathcal{Q}_a}(f)}{\sigma}\right)^p + \begin{cases} 0, & \text{if } |f_Q| \le \frac{\sigma}{2},\\ \gamma(Q), & \text{if } |f_Q| > \frac{\sigma}{2},\end{cases}\\ &\le C \left(\frac{K_p^{\mathcal{Q}_a}(f)}{\sigma}\right)^p + \gamma(Q) C \left(\frac{2|f_Q|}{\sigma}\right)^p,\end{aligned}$$

and we conclude that $f \in L^{p,\infty}(Q,\gamma)$. Since $1 < q < p$, $f \in L^q(Q,\gamma)$ (see the proof of Proposition 4.1).

Let $i, j \in \mathbb{N}$. We get $|f_N b_{ij}| \le |f||b_{ij}| \in L^1(Q_{ij},\gamma)$ for every $N \in \mathbb{N}$. Dominated Convergence Theorem leads to

$$\lim_{N\to\infty} \int_{Q_{ij}} f_N b_{ij} d\gamma = \int_{Q_{ij}} f b_{ij} d\gamma.$$

By proceeding as in (4.5) and using first (4.6), we get, for each $i, N \in \mathbb{N}$,

$$\begin{aligned}&\left|\int_{Q_{ij}} f_N b_{ij} d\gamma\right|\\ &\le \gamma(Q_{ij}) \left(\frac{1}{\gamma(Q_{ij})}\int_{Q_{ij}} |f_N - (f_N)_{Q_{ij}}|^q d\gamma\right)^{1/q} \left(\frac{1}{\gamma(Q_{ij})}\int_{Q_{ij}} |b_{ij}|^{q'} d\gamma\right)^{1/q'}\\ &\le C\gamma(Q_{ij}) \left(\frac{1}{\gamma(Q_{ij})}\int_{Q_{ij}} |f - f_{Q_{ij}}|^q d\gamma\right)^{1/q} \left(\frac{1}{\gamma(Q_{ij})}\int_{Q_{ij}} |b_{ij}|^{q'} d\gamma\right)^{1/q'}, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned}&\sum_{j=1}^{\infty} \gamma(Q_{ij}) \left(\frac{1}{\gamma(Q_{ij})}\int_{Q_{ij}} |f - f_{Q_{ij}}|^q d\gamma\right)^{1/q} \left(\frac{1}{\gamma(Q_{ij})}\int_{Q_{ij}} |b_{ij}|^{q'} d\gamma\right)^{1/q'}\\ &\quad \le 2\|f\|_{\mathrm{JN}_{p,q}^{\mathcal{Q}_a}(\mathbb{R}^d,\gamma)} \|g_i\|_{(p',q',a)}. \end{aligned} \tag{4.8}$$

By applying again the Dominated Convergence Theorem, we get, for every $i \in \mathbb{N}$,

$$\lim_{N\to\infty} \int_{\mathbb{R}^d} f_N g_i d\gamma = \lim_{N\to\infty} \sum_{j=1}^{\infty} \int_{Q_{ij}} f_N b_{ij} d\gamma = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{Q_{ij}} f b_{ij} d\gamma.$$

From (4.7) and (4.8), since $\sum_{i=1}^{\infty} \|g_i\|_{(p',q',a)} < \infty$, again Dominated Convergence Theorem leads to

$$\lim_{N\to\infty} \int_{\mathbb{R}^d} f_N (g - c_0) d\gamma = \lim_{N\to\infty} \sum_{i=1}^{\infty} \int_{\mathbb{R}^d} f_N g_i d\gamma = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{Q_{ij}} f b_{ij} d\gamma.$$

According to Lemma 4.6 and Proposition 4.1 we conclude that

$$|\Lambda_f g| \le C \|f\|_{\mathrm{JN}_p(\mathbb{R}^d,\gamma)} \|g\|_{H_{p',q',a}(\mathbb{R}^d,\gamma)},$$

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with C independent of f and g. Therefore, $\Lambda_f \in (H_{p',q'a}(\mathbb{R}^d,\gamma))'$ and

$$\|\Lambda_f\|_{(H_{p',q'a}(\mathbb{R}^d,\gamma))'} \leq C\|f\|_{\mathrm{JN}_p(\mathbb{R}^d,\gamma)}.$$

We will now see (b). Assume that $\Lambda \in (H_{p',q',a}(\mathbb{R}^d,\gamma))'$.

Note first that for any $1 < p' < q' < \infty$, $a > 0$ and $Q \in \mathcal{Q}_a$, $L_0^{q'}(Q,\gamma)$ is continuously contained in $H_{p',q',a}(\mathbb{R}^d,\gamma)$. Indeed, let $Q \in \mathcal{Q}_a$. Any $g \in L_0^{q'}(Q,\gamma)$ happens to be a (p',q',a)-polymer and

$$\|g\|_{H_{p',q',a}(\mathbb{R}^d,\gamma)} \leq \gamma(Q)^{1/p'-1/q'} \left(\int_Q |g|^{q'} d\gamma\right)^{1/q'} \leq \|g\|_{L_0^{q'}(Q,\gamma)}. \tag{4.9}$$

Then, $\Lambda|_{L_0^{q'}(Q,\gamma)} \in (L^{q'}(Q,\gamma))'$ and there exists $h_{q',Q} \in L^q(Q,\gamma)$ such that

$$\Lambda(g) = \int_Q g h_{q',Q} d\gamma, \quad g \in L_0^{q'}(Q,\gamma).$$

Replacing $h_{q',Q}$ by $h_{q',Q} - \gamma^{-1}(Q)\int_Q h_{q',Q} d\gamma$ shows that we can consider $h_{q',Q} \in L_0^q(Q,\gamma)$.

If H is a measurable set in $\mathbb{R}^d$ and h is a measurable function defined on H we say that h represents Λ on all the cubes in $\mathcal{Q}_a$ contained in H when for every cube $R \in \mathcal{Q}_a$ contained in H, $h|_R$ represents $\Lambda|_{L_0^{q'}(R,\gamma)}$. According to this definition we have that $h_{q',Q}$ represents Λ on all the cubes $R \in \mathcal{Q}_a$ contained in Q. Note that also, for every $c \in \mathbb{C}$, $h_{q',Q} + c$ represents Λ on all the cubes $R \in \mathcal{Q}_a$ contained in Q.

The arguments developed in [16, steps I, II and II, p. 300-301] allow us to define a function h^Λ on $\mathbb{R}^d$ which represents Λ on all the cubes in $\mathcal{Q}_a$.

Our objective is to prove that $h^\Lambda \in \mathrm{JN}_{p,q}(\mathbb{R}^d,\gamma)$. Then, by Proposition 4.1 we can conclude that $h^\Lambda \in \mathrm{JN}_p(\mathbb{R}^d,\gamma)$. To do this, we claim first that $h^\Lambda \in L^1(\mathbb{R}^d,\gamma)$.

By Proposition 4.3 it suffices to pick any value $a \in \mathbb{R}^d$. Let $a = A_d$ as in Lemma 4.4(b) and consider the function h^Λ that represents Λ on all the cubes in $\mathcal{Q}_{A_d}$. Given $Q \in \mathcal{Q}_{A_d}$, there exists $\alpha_Q \in \mathbb{C}$ such that

$$h^\Lambda\big|_Q = h_{q',Q} + \alpha_Q.$$

Using (4.9),

$$\begin{aligned}
\|h_{q',Q}\|_{L^1(Q,\gamma)} &\leq \gamma(Q)^{1/q'} \left(\int_Q |h_{q',Q}|^q d\gamma\right)^{1/q} \\
&\leq \gamma(Q)^{1/q'} \sup_{\|g\|_{L_0^{q'}(Q,\gamma)}\leq 1} \left|\int_Q h_{q',Q} g d\gamma\right| \\
&\leq \gamma(Q)^{1/q'} \|\Lambda\|_{(H_{p',q',a}(\mathbb{R}^d,\gamma))'} \sup_{\|g\|_{L_0^{q'}(Q,\gamma)}\leq 1} \|g\|_{H_{p',q',a}(\mathbb{R}^d,\gamma)} \\
&\leq \gamma(Q)^{1/q'} \|\Lambda\|_{(H_{p',q',a}(\mathbb{R}^d,\gamma))'}.
\end{aligned}$$

We get

$$\begin{aligned}
\|h^\Lambda\|_{L^1(Q,\gamma)} &\leq \|h_{q',Q}\|_{L^1(Q,\gamma)} + |\alpha_Q|\gamma(Q) \\
&\leq \gamma(Q)^{1/q'} \|\Lambda\|_{(H_{p',q',a}(\mathbb{R}^d,\gamma))'} + |\alpha_Q|\gamma(Q).
\end{aligned}$$

By proceeding as in [16, p. 302] we deduce that

$$|\alpha_Q| \leq C K_Q \|\Lambda\|_{(H_{p',q',a}(\mathbb{R}^d,\gamma))'}$$

Then,

$$\|h^\Lambda\|_{L^1(Q,\gamma)} \leq C\left(\gamma(Q)^{1/q'} + K_Q\gamma(Q)\right) \|\Lambda\|_{(H_{p',q',a}(\mathbb{R}^d,\gamma))'}.$$

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Now, we consider the covering $\{Q_n\}_{n\in\mathbb{N}}$ given in Lemma 4.4 and write

$$\begin{aligned}\|h^\Lambda\|_{L^1(\mathbb{R}^d,\gamma)} &\leq \sum_{n\in\mathbb{N}} \|h^\Lambda\|_{L^1(Q_n,\gamma)} \\ &\leq C\|\Lambda\|_{(H_{p',q',a}(\mathbb{R}^d,\gamma))'} \sum_{n\in\mathbb{N}} \left(\gamma(Q_n)^{1/q'} + K_{Q_n}\gamma(Q_n)\right).\end{aligned}$$

According to [16, Proposition 2.1] there exists $C > 1$ such that for every $Q \in \mathcal{Q}_{A_d}$ and every $x \in Q$

$$\frac{1}{C} \leq e^{|c_Q|^2-|x|^2} \leq C.$$

It follows that, for each $n \in \mathbb{N}$

$$\gamma(Q_n) \leq Ce^{-|c_{Q_n}|^2}\ell_{Q_n}^d$$

By Lemmas 4.4 (d) and 4.5, if Q is a cube in the k-th layer of $\{Q_n\}$ we get

$$\gamma(Q) \leq Ce^{-ck}k^{-d/2}, \quad \text{and} \quad K_Q \leq Ck, k \in \mathbb{N}.$$

Also by Lemma 4.4 (d), the number of cubes in the k-th layer of $\{Q_n\}$ is controlled by Ck^{d-1} for every $k \in \mathbb{N}$. Note that the constant C in the last three appearances only depend on the dimension d.

Therefore, we obtain

$$\sum_{n=1}^{\infty} \gamma(Q_n)^{1/q'} \leq C\sum_{k=1}^{\infty} \left(e^{-ck}k^{-d/2}k^{d-1}\right)^{1/q'} < \infty$$

and

$$\sum_{n=1}^{\infty} K_{Q_n}\gamma(Q_n) \leq C\sum_{k=1}^{\infty} e^{-ck}k^{-d/2}k^{d-1} < \infty.$$

Thus, we have proved that $h^\Lambda \in L^1(\mathbb{R}^d,\gamma)$.

Using the ideas in [5, p. 601-602], we are going to see now that

$$K_{p,q}^{\mathcal{Q}_{A_d}}(h^\Lambda) \leq C\|\Lambda\|_{(H_{p',q',a}(\mathbb{R}^d,\gamma))'},$$

where C does not depend on Λ.

We consider a finite pairwise disjoint family of cubes $\{Q_j\}_{j=1}^N \subset \mathcal{Q}_{A_d}$. We can write, for every $j = 1,\dots,N$,

$$\left(\frac{1}{\gamma(Q_j)}\int_{Q_j} |h^\Lambda - h^\Lambda_{Q_j}|^q d\gamma\right)^{1/q} = \sup \frac{1}{\gamma(Q_j)}\int_{Q_j} h^\Lambda b_j d\gamma,$$

where the supremum is taken over all the functions $b_j \in L_0^{q'}(Q_j)$ such that

$$\frac{1}{\gamma(Q_j)}\int_{Q_j} |b_j|^{q'} d\gamma = 1. \tag{4.10}$$

Let $\epsilon > 0$. We choose, for every $j = 1,\dots,N$, $b_j \in L_0^{q'}(Q_j,\gamma)$ such that (4.10) holds and

$$\left(\frac{1}{\gamma(Q_j)}\int_{Q_j} |h^\Lambda - h^\Lambda_{Q_j}|^q d\gamma\right)^{1/q} = \frac{1}{\gamma(Q_j)}\int_{Q_j} h^\Lambda b_j d\gamma + \frac{\epsilon}{\lambda_j N\gamma(Q_j)}.$$

Here $\{\lambda_j\}_{j=1}^N \subset (0,\infty)$ is such that $\sum_{j=1}^N \gamma(Q_j)\lambda_j^{p'} = 1$ and

$$\left(\sum_{j=1}^{N} \gamma(Q_j)\left(\frac{1}{\gamma(Q_j)}\int_{Q_j} |h^\Lambda - h^\Lambda_{Q_j}|^q d\gamma\right)^{p/q}\right)^{1/p}$$

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$$= \sum_{j=1}^N \gamma(Q_j)\lambda_j \left(\frac{1}{\gamma(Q_j)}\int_{Q_j} |h^\Lambda - h^\Lambda_{Q_j}|^q d\gamma\right)^{1/q}.$$

We define $g = \sum_{j=1}^N \lambda_j b_j$. It follows that

$$\begin{aligned}\left(\sum_{j=1}^N \gamma(Q_j)\left(\frac{1}{\gamma(Q_j)}\int_{Q_j}|h^\Lambda - h^\Lambda_{Q_j}|^q d\gamma\right)^{p/q}\right)^{1/p} &= \sum_{j=1}^N \lambda_j \int_{Q_j} h^\Lambda b_j d\gamma + \epsilon\\ &= \int_{\mathbb{R}^d} h^\Lambda \sum_{j=1}^N \lambda_j b_j d\gamma + \epsilon\\ &= \int_{\mathbb{R}^d} h^\Lambda g d\gamma + \epsilon\\ &= \Lambda(g) + \epsilon.\end{aligned}$$

The function g is a (p', q', A_d)-polymer and

$$\begin{aligned}\|g\|_{(p',q',A_d)} &\leq \left(\sum_{j=1}^N \gamma(Q_j)\left(\frac{1}{\gamma(Q_j)}\int_{Q_j}|\lambda_j b_j|^{q'} d\gamma\right)^{p'/q'}\right)^{1/p'}\\ &= \left(\sum_{j=1}^N \gamma(Q_j)\lambda_j^{p'}\right)^{1/p'} = 1.\end{aligned}$$

Then, $\|g\|_{H_{p',q',A_d}(\mathbb{R}^d,\gamma)} \leq 1$ and $|\Lambda(g)| \leq \|\Lambda\|_{(H_{p',q',A_d}(\mathbb{R}^d,\gamma))'}$. We obtain

$$\left(\sum_{j=1}^N \gamma(Q_j)\left(\frac{1}{\gamma(Q_j)}\int_{Q_j}|h^\Lambda - h^\Lambda_{Q_j}|^q d\gamma\right)^{p/q}\right)^{1/p} \leq \|\Lambda\|_{(H_{p',q',A_d}(\mathbb{R}^d,\gamma))'} + \epsilon.$$

It follows that

$$\inf \left(\sum_{j=1}^N \gamma(Q_j)\left(\frac{1}{\gamma(Q_j)}\int_{Q_j}|h^\Lambda - h^\Lambda_{Q_j}|^q d\gamma\right)^{p/q}\right)^{1/p} \leq \|\Lambda\|_{(H_{p',q',A_d}(\mathbb{R}^d,\gamma))'} + \epsilon,$$

where the infimum is taken over all the families $\{Q_j\}_{j=1}^N$ of pairwise disjoint cubes in $\mathcal{Q}_{A_d}$, with $N \in \mathbb{N}$. The arbitrariness of $\epsilon > 0$ allow us to conclude that

$$K_{p,q}^{\mathcal{Q}_{A_d}}(h^\Lambda) \leq C\|\Lambda\|_{(H_{p',q',a}(\mathbb{R}^d,\gamma))'},$$

and, by virtue of Proposition 4.1, that $h_\Lambda \in \mathrm{JN}_p(\mathbb{R}^d,\gamma)$.

We are going to see that there exists $\alpha \in \mathbb{C}$ such that $\Lambda = \Lambda_f$ where $f = h^\Lambda + \alpha$. Let $g \in H_{p',q',a}(\mathbb{R}^d,\gamma) \setminus \{0\}$.

Suppose that $c_0 \in \mathbb{C}$ and consider $g = c_0$. We have

$$\Lambda(g) = c_0\Lambda(1) = c_0\left(\Lambda(1) - \int_{\mathbb{R}^d} h^\Lambda d\gamma\right) + \int_{\mathbb{R}^d} g h^\Lambda d\gamma.$$

We define $\alpha = \Lambda(1) - \int_{\mathbb{R}^d} h^\Lambda d\gamma$, so it is clear that $f = h^\Lambda + \alpha \in \mathrm{JN}_p(\mathbb{R}^d,\gamma)$ and $\Lambda_f(g) = \Lambda(g)$.

Suppose now that g is a nonconstant function in $\mathrm{JN}_p(\mathbb{R}^d,\gamma)$. Therefore, $A := \inf \sum_{i=1}^\infty \|g_i\|_{(p',q',a)} > 0$, where the infimum is taken over all the sequences $\{g_i\}_{i\in\mathbb{N}}$ of (p',q',a)-polymers such that $\sum_{i=1}^\infty \|g_i\|_{(p',q',a)} < \infty$ and $g = c_0 + \sum_{i=1}^\infty g_i$, for some $c_0 \in \mathbb{C}$.

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Thus, there exists a sequence $\{g_i\}_{i\in\mathbb{N}}$ of (p',q',a)-polymers such that $g_i \not\equiv 0$ for every $i \in \mathbb{N}$, $g = c_0 + \sum_{i=1}^{\infty} g_i$, with

$$\sum_{i=1}^{\infty} \|g_i\|_{(p',q',a)} \leq 2A.$$

For every $i \in \mathbb{N}$, we write $g_i = \sum_{j=1}^{\infty} b_{ij}$, where for every $j \in \mathbb{N}$, $b_{ij} \in L_0^{q'}(Q_{ij},\gamma)$ and $\operatorname{supp} b_{ij} \subset Q_{ij}$, being $\{Q_{ij}\}_{j=1}^{\infty}$ is a family of pairwise disjoint cubes in $\mathcal{Q}_{A_d}$ such that

$$\left(\sum_{j=1}^{\infty} \gamma(Q_{ij}) \left(\frac{1}{Q_{ij}} \int_{Q_{ij}} |b_{ij}|^{q'} d\gamma\right)^{p'/q'}\right)^{1/q'} \leq 2\|g_i\|_{(p',q',a)}.$$

By proceeding as in the proof of Proposition 4.2 we can see that there exists two sequences $\{i_k\}_{k\in\mathbb{N}}$ and $\{j_k\}_{k\in\mathbb{N}}$ of nonnegative integers such that

$$c_0 + \sum_{i=1}^{i_k} \sum_{j=1}^{j_k} b_{ij} \longrightarrow g,$$

as $k \to \infty$, in $H_{p',q',a}(\mathbb{R}^d,\gamma)$. Since $\Lambda \in (H_{p',q',a}(\mathbb{R}^d,\gamma))'$ we get

$$\begin{aligned}\Lambda(g) &= c_0\Lambda(1) + \lim_{k\to\infty} \sum_{i=1}^{i_k} \sum_{j=1}^{j_k} \Lambda(b_{ij}) \\ &= c_0\Lambda(1) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{\mathbb{R}^d} h^{\Lambda} b_{ij}\, d\gamma,\end{aligned}$$

where the series is absolutely convergent (see part (a) of this proof). Then,

$$\Lambda(g) = c_0 \left(\Lambda(1) - \int_{\mathbb{R}^d} h^{\Lambda} d\gamma\right) + \Lambda_{h^{\Lambda}}(g).$$

We define $f = h^{\Lambda} - \int_{\mathbb{R}^d} h^{\Lambda} d\gamma + \Lambda(1) := h^{\Lambda} + \alpha$. It is clear that $f \in \mathrm{JN}_p(\mathbb{R}^d,\gamma)$ and we also have that

$$\begin{aligned}\Lambda_f(g) &= c_0 \int_{\mathbb{R}^d} f\, d\gamma + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{\mathbb{R}^d} f b_{ij}\, d\gamma \\ &= c_0\Lambda(1) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{\mathbb{R}^d} h^{\Lambda} b_{ij}\, d\gamma \\ &= \Lambda(g).\end{aligned}$$

The proof of Theorem 1.2 is now complete.

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