ISSN 2451-7100

Publicaciones del IMAL
IMAL Preprints
#2024–0074

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Publication Date: December 6, 2024

Publisher: Instituto de Matemática Aplicada del Litoral IMAL "Dra. Eleonor Harboure" (CCT CONICET Santa Fe – UNL)

https://imal.conicet.gov.ar
https://imal.conicet.gov.ar/preprints-del-imal

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2024

IMAL PREPRINT # 2024-0074

ISSN 2451-7100
Publication date: December 6, 2024

WEIGHTED ESTIMATES FOR SCHRÖDINGER–CALDERÓN–ZYGMUND OPERATORS WITH EXPONENTIAL DECAY

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Abstract. In this work we obtain weighted boundedness results for singular integral operators with kernels exhibiting exponential decay. We also show that the classes of weights are characterized by a suitable maximal operator. Additionally, we study the boundedness of various operators associated with the generalized Schrödinger operator $-\Delta+\mu$, where μ is a nonnegative Radon measure in $\mathbb{R}^d$, for $d \geq 3$.

1. Introduction and main result

In [3], J. Bailey studied about harmonic analysis related to the Schrödinger operator $\mathcal{L}=-\Delta+V$, for V a locally integrable function in $\mathbb{R}^d$ with $d \geq 3$. Asking the question whether it is possible to construct a Muckenhoupt-type class of weights adapted to the underlying differential operator and whether this class can be characterized in terms of the corresponding operators, he defined new classes of weights. These classes, named $S^V_{p,c}$ and $H^{V,m}_{p,c}$, are comparable under certain dependence on the parameters m and c. The second one looks at the weights in the Euclidean distance, while the first one is adapted to another distance depending on the potential V. Both classes of weights are strictly larger that the class $A^{V,\infty}_p$ given in [7] (see §2.4).

Actually, since the classes can be described through a critical radius function associated to the operator $\mathcal{L}$, we will name the classes $S^\rho_{p,c}$, $H^{\rho,m}_{p,c}$ and A^ρ_p, respectively. This idea together with the main tool used throughout the work, the decay of the fundamental solution of the operator $\mathcal{L}$, leads us to think about leaving particular examples aside, for the moment, and address a general theory to prove the boundedness of singular integral operators having kernels with exponential decay on weighted L^p spaces. Regarding the weights, we will deal with the class $H^\rho_{p,c}$ since one can have a better understanding of the behavior of the weights with respect to the Euclidean distance. An analysis of the classes of weights is given in §3.1, including a simple example that shows the difference between them.

In the spirit of [4, Theorem 5.1], we give an extrapolation theorem (see Theorem 4.1) on weighted Lebesgue spaces for weights associated to the maximal operator defined in [3], and prove boundedness results for certain collection of operators that we shall introduce in §2.3. In order to do so, we first complete the work done by J. Bailey and characterize the classes $H^\rho_{p,c}$ as those that ensure the boundedness of the mentioned maximal function, which is an important result by itself (see §3.2).

The article will conclude with the presentation of certain examples of operators associated with the differential operator $\mathcal{L}_\mu=-\Delta+\mu$, along with their corresponding boundedness properties. Here, μ is a nonnegative Radon measure on $\mathbb{R}^d$, as considered in [11] and [3]. These operators include the imaginary powers $(-\Delta+\mu)^{i\gamma}$, the Riesz transforms $R_\mu=\nabla(-\Delta+\mu)^{-\frac{1}{2}}$ and its adjoint $R^*_\mu=(-\Delta+\mu)^{-\frac{1}{2}}\nabla$, as well as operators of the form $(-\Delta+V)^{-j/2}V^{j/2}$ for $j=1,2$ with V a nonnegative function belonging to the classical reverse Hölder class RH_q for $q>\frac{d}{2}$.

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Date: December 5, 2024.

Key words and phrases. Extrapolation, Schrödinger operators, Riesz transforms, imaginary powers, weights.

IMAL PREPRINT # 2024-0074
ISSN 2451-7100
Publication date: December 6, 2024

Throughout this paper C and c will always denote positive constants than may change in each occurrence. Also, by $a \lesssim b$ we mean there exists a positive constant c such that $a \leq cb$, and $a \gtrsim b$ is the same as $b \lesssim a$. When both are valid, we simply write $a \sim b$.

2. Preliminaries

2.1. Critical radius function and Agmon distance.

We introduce the notion of critical radius function as any function satisfying the definition given below.

Definition 2.1. A function $\rho : \mathbb{R}^d \to [0,\infty)$ will be called a **critical radius function** if there exist constants $k_0, C_0 \geq 1$ such that

$$C_0^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C_0\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}} \tag{2.1}$$

for $x, y \in \mathbb{R}^d$.

Remark 2.2. It follows from the definition that $\rho(x) \sim \rho(y)$ if $|x-y| \lesssim \rho(x)$.

A ball of the form $B(x,\rho(x))$, with $x \in \mathbb{R}^d$, is called **critical** and a ball $B(x,r)$ with $r \leq \rho(x)$ will be called **sub-critical**. We denote by $\mathcal{B}_\rho$ the family of all sub-critical balls, that is,

$$\mathcal{B}_\rho = \left\{B(x,r) : x \in \mathbb{R}^d, r \leq \rho(x)\right\}.$$

It is well-known (see, for instance, [1] and [9]) that many phenomena involving the classical Schrödinger operator $L_V = -\Delta + V$, with a potential V, can be better described in terms of the Agmon distance, which is comparable to the Euclidean distance locally, but is a Riemannian-like distance obtained by deforming the Euclidean one with respect to a quadratic form involving the potential.

In our setting, a notion of Agmon distance can also be given as:

$$d_\rho(x,y) := \inf_\gamma \int_0^1 \rho(\gamma(t))^{-1}|\gamma'(t)|dt,$$

where the infimum is taken over every absolutely continuous function $\gamma : [0,1] \to \mathbb{R}^d$ with $\gamma(0) = x$ and $\gamma(1) = y$.

The following lemma establishes the aforementioned local comparison between the Agmon distance and the Euclidean distance.

Lemma 2.3 ([3, Lemma 2.2])**.** *Let $\rho : \mathbb{R} \to [0,\infty)$ be a critical radius function. If $|x-y| \leq 2\rho(x)$, there exists $D_0 > 1$ such that*

$$\frac{1}{D_0}\frac{|x-y|}{\rho(x)} \leq d_\rho(x,y) \leq D_0\frac{|x-y|}{\rho(x)}. \tag{2.2}$$

The Agmon distance can also be compared with the quantity $1 + \frac{|x-y|}{\rho(x)}$ as stated below.

Lemma 2.4 ([3, Lemma 2.3], [11, (3.19)])**.** *Let $\rho : \mathbb{R} \to [0,\infty)$ be a critical radius function. Then,*

$$d_\rho(x,y) \leq C_0\left(1+\frac{|x-y|}{\rho(x)}\right)^{k_0+1} \tag{2.3}$$

for every $x, y \in \mathbb{R}^d$.

Moreover, when $|x-y| \geq \rho(x)$, there exists a constant $D_1 > 1$ such that

$$d_\rho(x,y) \geq \frac{1}{D_1}\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{1}{k_0+1}}. \tag{2.4}$$

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ISSN 2451-7100
Publication date: December 6, 2024

2.2. **Spaces of functions.** Given a weight w, i.e., a nonnegative and locally integrable function defined on $\mathbb{R}^d$, for $1 \le p < \infty$ we say that a measurable function f on $\mathbb{R}^d$ belongs to $L^p(w)$ iff $\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^d} |f(x)|^p w(x)dx\right)^{1/p} < \infty$. For $w \equiv 1$ we simply write $L^p(w) = L^p$.

When $p = \infty$, $f \in L^\infty(w)$ iff $fw \in L^\infty$ and we write $\|f\|_{L^\infty(w)} = \|fw\|_\infty$ where this quantity denotes, as usual, the essential supremum of fw on $\mathbb{R}^d$.

Given a critical radius function ρ and a weight w, we say that a locally integrable function f belongs to $\mathrm{BMO}_\rho(w)$ if it satisfies

$$\frac{\|\chi_B w\|_\infty}{|B|}\int_B |f - f_B| \le C, \quad \text{for every ball } B = B(x,r) \text{ with } r < \rho(x), \tag{2.5}$$

and

$$\frac{\|\chi_B w\|_\infty}{|B|}\int_B |f| \le C, \qquad \text{for every ball } B = B(x,r) \text{ with } r = \rho(x), \tag{2.6}$$

where f_B stands for the average of f over the ball B. As usual, the norm $\|f\|_{\mathrm{BMO}_\rho(w)}$ is defined as the maximum of the infima constants appearing in (2.5) and (2.6).

It is a classical fact that

$$\sup_B \frac{1}{|B|}\int_B |f - f_B| \sim \sup_B \inf_{a\in\mathbb{R}} \frac{1}{|B|}\int_B |f - a|.$$

This allows us to verify (2.5) for some constant $a \in \mathbb{R}$, not necessarily the average f_B.

Moreover, as it was proved in [4], the space $\mathrm{BMO}_\rho(w)$ can be characterized in terms of a suitable sharp maximal function.

Lemma 2.5 ([4, Lemma 2]). *For any weight w and any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $\|f\|_{\mathrm{BMO}_\rho(w)} \sim \|M^\sharp_{\mathrm{loc}}(f)w\|_\infty$ where*

$$M^\sharp_{\mathrm{loc}}(f)(x) := \sup_{x\in B\in\mathcal{B}_\rho} \frac{1}{|B|}\int_B |f - f_B| + \sup_{x\in B = B(y,\rho(y))} \frac{1}{|B|}\int_B |f|.$$

2.3. **Schrödinger–Calderón–Zygmund operators with exponential decay.** In the following results, we consider an operator T with a kernel K, understood in the principal value sense, that satisfies certain Calderón–Zygmund or Hörmander-type conditions, having exponential decay. The terminology used here follows [6] (see also [10]), although their operators, characterized by polynomial decay, were previously considered in [4] without being explicitly named.

Definition 2.6. For $1 < s < \infty$ and $0 < \delta \le 1$ we say that T is an exponential Schrödinger–Calderón–Zygmund operator of (s, δ) type if

(i) T is bounded from $L^{s'}(\mathbb{R}^d)$ to $L^{s',\infty}(\mathbb{R}^d)$;
(ii) T has an associated kernel K verifying the following conditions:
 (a) there exist constants $c, C > 0$ and $m > 0$ such that

$$\left(\frac{1}{R^d}\int_{R<|x_0-y|\le 2R} |K(x,y)|^s dy\right)^{1/s} \le \frac{C}{R^d}\exp\left(-c\left(1+\frac{R}{\rho(x)}\right)^m\right), \tag{2.7}$$

 whenever $|x - x_0| < R/2$;
 (b) there exist a constant $C > 0$ such that

$$\left(\frac{1}{R^d}\int_{R<|x_0-y|\le 2R} |K(x,y) - K(x_0,y)|^s dy\right)^{1/s} \le \frac{C}{R^d}\left(\frac{r}{R}\right)^\delta \tag{2.8}$$

 for every $|x - x_0| < r \le \rho(x_0)$ and $r < R/2$.

For the case $s = \infty$, we consider pointwise estimates instead, for which we introduce the following definition.

Definition 2.7. Given $0 < \delta \le 1$ we say that T is a an exponential Schrödinger–Calderón–Zygmund operator of (∞, δ) type if

(i) T is bounded on L^p for every $p > 1$;
(ii) T has an associated kernel K verifying the following conditions:

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ISSN 2451-7100
Publication date: December 6, 2024

(a) there exist constants $c, C > 0$ and $m > 0$ such that
$$|K(x,y)| \le \frac{C}{|x-y|^d} \exp\left(-c\left(1+\frac{|x-y|}{\rho(x)}\right)^m\right), \qquad \text{for every } x \neq y; \tag{2.9}$$
(b) there exist a constant $C > 0$ such that
$$|K(x,y) - K(x_0,y)| \le \frac{C}{|x-y|^d}\left(\frac{|x-x_0|}{|x-y|}\right)^\delta, \tag{2.10}$$
for every $|x-y| > 2|x-x_0|$.

2.4. **Weights associated to a critical radius function.** The class A_p^ρ introduced in [7] properly contain the classical Muckenhoupt weights. For $1 < p < \infty$, they are denoted by $A_p^\rho = \bigcup_{\theta \ge 0} A_p^{\rho,\theta}$, where $w \in A_p^{\rho,\theta}$ means that
$$\left(\int_B w\right)^{\frac{1}{p}} \left(\int_B w^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}} \le C|B|\left(1+\frac{r}{\rho(x)}\right)^\theta$$
for every ball $B = B(x,r)$ with center $x \in \mathbb{R}^d$ and radius $r > 0$. Similarly, when $p = 1$, we denote by $A_1^\rho = \bigcup_{\theta \ge 0} A_1^{\rho,\theta}$, where $A_1^{\rho,\theta}$ collects weights w such that
$$\int_B w \le C|B|\left(1+\frac{r}{\rho(x)}\right)^\theta \inf_B w,$$
for every ball $B = B(x,r)$.

Let us also consider $A_p^{\rho,\mathrm{loc}}$ as defined in [5]. That is, a weight w belongs to the class $A_p^{\rho,\mathrm{loc}}$ for $1 < p < \infty$ if there exists a constant $C > 0$ such that
$$\left(\int_B w\right)^{\frac{1}{p}} \left(\int_B w^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}} \le C|B|$$
for balls $B \in \mathcal{B}_\rho$, and for $p = 1$ if
$$\int_B w \le C|B| \inf_B w,$$
holds for every $B \in \mathcal{B}_\rho$.

The weights involved in our main result where given by J. Bailey in [3]. For $1 < p < \infty$ and $c \ge 0$ we denote by $H_{p,c}^\rho := \bigcup_{m \ge 0} H_{p,c}^{\rho,m}$, where $H_{p,c}^{\rho,m}$ is the class of weights w for which there exists a constant C such that
$$\left(\int_B w\right)^{\frac{1}{p}} \left(\int_B w^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}} \le C|B| \exp\left(c\left(1+\frac{r}{\rho(x)}\right)^m\right),$$
for each ball $B = B(x,r)$ with center $x \in \mathbb{R}^d$ and radius $r > 0$.

When $p = 1$, for $c \ge 0$ we denote by $H_{1,c}^\rho := \bigcup_{m \ge 0} H_{1,c}^{\rho,m}$, where $w \in H_{1,c}^{\rho,m}$ means that
$$\int_B w \le C|B| \exp\left(c\left(1+\frac{r}{\rho(x)}\right)^m\right) \inf_B w,$$
for each ball $B = B(x,r)$ with center $x \in \mathbb{R}^d$ and radius $r > 0$.

Clearly, for any $1 \le p < \infty$, $H_{p,c_1}^{\rho,m_1} \subseteq H_{p,c_2}^{\rho,m_2}$ whenever $c_1 \le c_2$ and $m_1 \le m_2$.

Notice that when $c = 0$, we recover the Muckenhoupt A_p classes. Moreover, it is easy to see that the classes $H_{p,c}^\rho$ satisfy similar properties to those of the A_p weights, as we establish in Lemma 3.2.

It is easy to see that the weights in $H_{p,c}^{\rho,m}$ satisfy the following doubling condition.

Definition 2.8. Let $\kappa \ge 1$ and $c > 0$. We denote by $D_{\kappa,c}^\rho := \bigcup_{m \ge 0} D_{\kappa,c}^{\rho,m}$, where $D_{\kappa,c}^{\rho,m}$ is the class of weights w such that there exists a constant C for which
$$w(B(x,R)) \le C\left(\frac{R}{r}\right)^{d\kappa} \exp\left(c\left(1+\frac{R}{\rho(x)}\right)^m\right) w(B(x,r))$$
for all $x \in \mathbb{R}^d$ and $r < R$.

Remark 2.9. If $w \in H_{p,c}^{\rho,m}$, by Hölder's inequality it easy to check that $w \in D_{p,cp}^{\rho,m}$.

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ISSN 2451-7100
Publication date: December 6, 2024

We will also deal with the reverse Hölder classes of weights. In [7, Lemma 5] the authors prove that a weight $w \in A_p^\rho$, $1 \le p < \infty$, verifies the following reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B w^\eta\right)^{\frac{1}{\eta}} \le C\left(\frac{1}{|B|}\int_B w\right)\left(1+\frac{r}{\rho(x)}\right)^\beta \tag{2.11}$$

for every ball $B = B(x, r)$ and some constants $\beta, C > 0$ and $\eta > 1$.

Reverse Hölder classes adapted to the context of weights with exponential growth will also be needed.

Definition 2.10. Let $\eta > 1$ and $c > 0$. We denote by $RH^\rho_{\eta,c} := \bigcup_{m\ge 0} RH^{\rho,m}_{\eta,c}$, where $RH^{\rho,m}_{\eta,c}$ is defined as those weights w such that

$$\left(\frac{1}{|B|}\int_B w^\eta\right)^{\frac{1}{\eta}} \le C\left(\frac{1}{|B|}\int_B w\right)\exp\left(c\left(1+\frac{r}{\rho(x)}\right)^m\right) \tag{2.12}$$

for every ball $B = B(x, r)$.

Clearly, (2.11) implies (2.12).

Remark 2.11. Note that when taking $m = 0$, $RH^{\rho,0}_{p,c}$ and $D^{\rho,0}_{\kappa,c}$ are the classical reverse Hölder classes and doubling classes, respectively.

3. Auxiliary results

3.1. **Some properties for $H^\rho_{p,c}$.** The importance of the following result lies in the fact that the class of weights involved in Theorem 4.1 generalizes those presented in [7]. Additionally, the second inclusion enables the application of established tools for $A_p^{\rho,\mathrm{loc}}$. While this inclusion was originally obtained in [3], we provide a direct proof that avoids the need for intermediate weight classes and eliminates certain parameter restrictions.

Proposition 3.1. *Let $1 < p < \infty$. For any $c > 0$ we have*

$$A_p^\rho \subsetneq H^\rho_{p,c} \subseteq A_p^{\rho,\mathrm{loc}}. \tag{3.1}$$

Proof. The first inclusion can be found in [3, Proposition 3.2]. For the second one, we consider $w \in H^{\rho,m}_{p,c}$ for some $m \ge 0$ and a ball $B \in \mathcal{B}_\rho$. We get

$$\left(\int_B w\right)^{\frac{1}{p}}\left(\int_B w^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}} \le C|B|\exp\left(c\left(1+\frac{r}{\rho(x)}\right)^m\right) \le C|B|\exp\left(c2^m\right) \le C|B|,$$

where the constant is independent of B. Hence, $w \in A_p^{\rho,\mathrm{loc}}$.

For the critical radius function $\rho(x) = \min\left\{1, \frac{1}{|x|}\right\}$ (which arises when dealing with the harmonic oscillator whose potential is $V(x) = |x|^2$), we shall see that $w(x) = e^{|x|}$ belongs to $H^\rho_{p,c}$ for some $c > 0$, but $w \notin A_p^\rho$.

Let $B = B(y, r)$ be a ball for $y \in \mathbb{R}^d$ and $r > 0$. Then,

$$\left(\int_B e^{|x|}dx\right)^{\frac{1}{p}} \lesssim |B|^{\frac{1}{p}} e^{\frac{|y|+r}{p}}$$

and

$$\left(\int_B e^{-\frac{|x|}{p-1}}dx\right)^{\frac{p-1}{p}} \lesssim |B|^{1-\frac{1}{p}} e^{-\frac{|y|-r}{p}},$$

which give

$$\left(\int_B e^{|x|}dx\right)^{\frac{1}{p}}\left(\int_B e^{-\frac{|x|}{p-1}}dx\right)^{\frac{p-1}{p}} \lesssim |B| e^{\frac{2}{p}r}.$$

Now, notice that $r \le \frac{r}{\rho(y)}$ for every $y \in \mathbb{R}^d$ and $r > 0$, since $r = \frac{r}{\rho(y)}$ when $|y| \le 1$ and $r \le r|y| = \frac{r}{\rho(y)}$ when $|y| > 1$. Thus,

$$\left(\int_B e^{|x|}dx\right)^{\frac{1}{p}}\left(\int_B e^{-\frac{|x|}{p-1}}dx\right)^{\frac{p-1}{p}} \lesssim |B|\exp\left(\frac{2}{p}\left(1+\frac{r}{\rho(y)}\right)\right)$$

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ISSN 2451-7100
Publication date: December 6, 2024

meaning that $w \in H^{\rho}_{p,\frac{2}{p}}$.

In order to see that $w \notin A^{\rho}_p$, we will follow the ideas given in [2, p. 367]. Let us consider the balls $B_\ell = B(0,\ell)$ for $\ell > 1$. On one hand, we can obtain

$$\left(\int_{B_{2\ell}} e^{|x|}dx\right)^{\frac{1}{p}} \geq \left(\int_{B_{2\ell}\setminus B_\ell} e^{|x|}dx\right)^{\frac{1}{p}} = \left(\int_\ell^{2\ell} e^t t^{d-1}dt\right)^{\frac{1}{p}} \geq \left(e^{2\ell}-e^{\ell}\right)^{\frac{1}{p}}.$$

On the other hand,

$$\left(\int_{B_{2\ell}} e^{-\frac{|x|}{p-1}}dx\right)^{\frac{p-1}{p}} \geq \left(\int_{B_{2\ell}\setminus B_\ell} e^{-\frac{|x|}{p-1}}dx\right)^{\frac{p-1}{p}} \gtrsim \left(e^{-\frac{\ell}{p-1}}-e^{-\frac{2\ell}{p-1}}\right)^{\frac{p-1}{p}} \gtrsim e^{-\frac{\ell}{p}}.$$

These yields

$$\left(\int_{B_{2\ell}} e^{|x|}dx\right)^{\frac{1}{p}}\left(\int_{B_{2\ell}} e^{-\frac{|x|}{p-1}}dx\right)^{\frac{p-1}{p}} \gtrsim (e^\ell - 1)^{\frac{1}{p}},$$

and since this estimate holds for any $\ell > 1$, the left-hand-side cannot be bounded by any polynomial on ℓ. That is, $w \notin A^{\rho,\theta}_p$ for any $\theta \geq 0$, so $w \notin A^{\rho}_p$ as we claimed. □

As it was observed in [5], the A^{ρ}_p classes preserve the favorable properties of the A_p weights. Similarly, as the reader may verify, the $H^{\rho}_{p,c}$ classes also exhibit these properties.

Lemma 3.2. *The following properties hold:*

(i) $H^{\rho,m}_{p,c} \subset H^{\rho,m}_{q,c}$ *for every* $1 \leq p \leq q < \infty$ *and* $m, c \geq 0$.

(ii) If $w \in H^{\rho,m}_{p,c}$ *for some* $1 < p < \infty$ *and* $m \geq 0$*, then* $\sigma := w^{1-p'} \in H^{\rho,m}_{p',c}$.

(iii) If $w_1 \in H^{\rho,m_1}_{1,c_1}$ *and* $w_2 \in H^{\rho,m_2}_{1,c_2}$ *for some* $m_1, m_2, c_1, c_2 \geq 0$*, then for every* $1 \leq p < \infty$ *there exist* $m, c \geq 0$ *such that* $w_1 w_2^{1-p} \in H^{\rho,m}_{p,c}$.

Next lemma states that $H^{\rho}_{p,c}$ weights satisfy the reverse Hölder condition given above. The proof follows the lines of [7, Lemma 5], with some modifications that we will provide.

Lemma 3.3. *Given* $1 \leq p < \infty$ *and* $c > 0$*, for any* $w \in H^{\rho}_{p,c}$*, there exist constants* $\eta > 1$ *and* $c^* > 0$ *such that* $w \in RH^{\rho}_{\eta,c^*}$.

Proof. Since $w \in H^{\rho,m}_{p,c}$ for some $m \geq 0$, by Proposition 3.1 we get $w \in A^{\rho,\mathrm{loc}}_p$. We follow the proof given in [7], where an important tool is Proposition 2. It guarantees the existence of a sequence of points $\{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^d$ that satisfies the following two properties:

- $\mathbb{R}^d = \bigcup_{j\in\mathbb{N}} B(x_j, \rho(x_j))$;
- there exist constants $C, N_1 > 0$ such that for every $\sigma \geq 1$, $\sum_{j\in\mathbb{N}} \chi_{B(x_j,\sigma\rho(x_j))} \leq C\sigma^{N_1}$.

When estimating the integral of w^η on B, for $\eta > 1$, the same arguments can be applied to get that

$$\left(\int_B w^\eta\right)^{\frac{1}{\eta}} \lesssim w(c(x,r)B)\rho(x)^{-\frac{d(\eta-1)}{\eta}}\left(1+\frac{r}{\rho(x)}\right)^{\frac{dk_0(\eta-1)}{\eta}},$$

where $c(x,r) = 4C_0^2\left(1+\frac{r}{\rho(x)}\right)^{k_0/(k_0+1)}$.

In the context of our class of weights, we use Remark 2.9 to obtain

$$w(c(x,r)B) \lesssim \exp\left(\tilde{c}\left(1+\frac{r}{\rho(x)}\right)^{\tilde{m}}\right)w(B),$$

where $\tilde{c} = 4cC_0^2 p$ and $\tilde{m} = m\frac{k_0}{k_0+1} + m$. Therefore we have

$$\left(\frac{1}{|B|}\int_B w^\eta\right)^{\frac{1}{\eta}} \lesssim \left(\frac{1}{|B|}\int_B w\right)\exp\left(c^*\left(1+\frac{r}{\rho(x)}\right)^{m^*}\right)$$

where $c^* = \tilde{c}+1$ and $m^* = \max\left\{\tilde{m}, \frac{d(k_0+1)(\eta-1)}{\eta}\right\}$. □

ISSN 2451-7100
Publication date: December 6, 2024

We also have the following property for $RH^{\rho}_{\eta,c}$ families of weights, which gives us an openness property. It is important to clarify that in the hypotheses of the following lemma the parameters c and c_1 are not necessarily the same, which broadens the result.

Lemma 3.4. *Let $w \in H^{\rho}_{p,c} \cap RH^{\rho}_{\eta,c_1}$ for some $\eta > 1$, $c, c_1 > 0$, and $p > 1$. There exist $\beta > 1$ and $\widetilde{c} > 0$ such that $w \in RH^{\rho}_{\eta\beta,\widetilde{c}}$.*

Proof. Let $w \in H^{\rho}_{p,c} \cap RH^{\rho}_{\eta,c_1}$. Then, there exists $m, m_1 > 0$ such that $w \in H^{\rho,m}_{p,c} \cap RH^{\rho,m_1}_{\eta,c_1}$. Using both conditions, we get $w^{\eta} \in H^{\rho}_{q,c_2}$ with $c_2 > 0$ and $q = \eta(p-1)+1$. Indeed,

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B w^{\eta}\right)^{\frac{1}{\eta(p-1)+1}}\left(\frac{1}{|B|}\int_B w^{-\frac{\eta}{\eta(p-1)}}\right)^{\frac{\eta(p-1)}{\eta(p-1)+1}}\\
&\lesssim \left(\exp\left(c_1\left(1+\frac{r}{\rho(x)}\right)^{m_1}\right)\frac{1}{|B|}\int_B w\right)^{\frac{\eta}{\eta(p-1)+1}}\left(\frac{1}{|B|}\int_B w^{-\frac{1}{(p-1)}}\right)^{\frac{\eta(p-1)}{\eta(p-1)+1}}\\
&\lesssim \exp\left(c_1\frac{\eta}{\eta(p-1)+1}\left(1+\frac{r}{\rho(x)}\right)^{m_1}\right)\left(\left(\frac{1}{|B|}\int_B w\right)^{\frac{1}{p}}\left(\frac{1}{|B|}\int_B w^{-\frac{1}{(p-1)}}\right)^{\frac{p-1}{p}}\right)^{\frac{\eta p}{\eta(p-1)+1}}\\
&\lesssim \exp\left(c_1\frac{\eta}{\eta(p-1)+1}\left(1+\frac{r}{\rho(x)}\right)^{m_1}\right)\exp\left(c\frac{\eta p}{\eta(p-1)+1}\left(1+\frac{r}{\rho(x)}\right)^{m}\right)\\
&\lesssim \exp\left(c_2\left(1+\frac{r}{\rho(x)}\right)^{m_2}\right),
\end{aligned}$$

with $c_2 = \frac{(c_1+cp)\eta}{\eta(p-1)+1} > 0$ and $m_2 = \max\{m, m_1\}$.

Then, from Lemma 3.3 there exists $\beta > 1$ and $c' > 0$ such that $w^{\eta} \in RH^{\rho}_{\beta,c'}$. Using again that $w \in RH^{\rho}_{\eta,c_1}$, we get $w \in RH^{\rho}_{\eta\beta,\widetilde{c}}$ with $\widetilde{c} = c_1 + \frac{c}{\eta} > 0$. □

3.2. **Maximal operators associated with a critical radius function.** For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, we recall the definition of the local maximal operator over sub-critical balls, studied in [7] and [5]:

$$M^{\text{loc}}_{\rho} f(x) = \sup_{x\in B\in\mathcal{B}_{\rho}} \frac{1}{|B|}\int_B |f(y)|dy \tag{3.2}$$

As it was proved in [7, Theorem 1], for any $1 < p < \infty$, $w \in A^{\rho,\text{loc}}_p$ if and only if M^{loc}_{ρ} is bounded on $L^p(w)$.

In a similar way, the classes of weights $H^{\rho}_{p,c}$ are related with the boundedness of certain maximal functions that capture the exponential growth. This operators were already considered by J. Bailey in [3], but we define them below for the sake of completeness.

Definition 3.5. Let $c, m \geq 0$ and let ρ be a critical radius function. For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we consider the maximal operator given by

$$\widetilde{\mathcal{M}}^m_{\rho,c} f(x) := \sup_{B(x',r)\ni x} \frac{1}{\exp\left(c\left(1+\frac{r}{\rho(x')}\right)^m\right)} \fint_{B(x',r)} |f(y)|dy, \tag{3.3}$$

and its centered version

$$\widetilde{M}^m_{\rho,c} f(x) := \sup_{r>0} \frac{1}{\exp\left(c\left(1+\frac{r}{\rho(x)}\right)^m\right)} \fint_{B(x,r)} |f(y)|dy. \tag{3.4}$$

For any $x \in \mathbb{R}^d$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, the inequality $\widetilde{M}^m_{\rho,c} f \lesssim \widetilde{\mathcal{M}}^m_{\rho,c} f$ trivially holds. The following proposition states that the reverse inequality also holds under some restrictions on the parameters.

Proposition 3.6 ([3, Proposition 3.5])**.** *Let ρ be a critical radius function with constants C_0 and k_0 as in Definition 2.1. Given $c_1, c_2, m_1, m_2 > 0$ with $m_1 > (k_0+1)m_2$ and $c_2 \geq c_1(2C_0)^{m_2}$ we have*

$$\widetilde{\mathcal{M}}^{m_1}_{\rho,c_1} f \lesssim \widetilde{M}^{m_2}_{\rho,c_2} f \tag{3.5}$$

for every $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

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ISSN 2451-7100
Publication date: December 6, 2024

The following theorem is significant in its own right, since it allows us to characterize the weight classes $H^\rho_{p,c}$ as those that ensure the boundedness of the maximal operator $\widetilde{M}^m_{\rho,c}$. This result will be useful to give an extrapolation tool in Section 4.

Theorem 3.7. *Let $1 < p < \infty$. Then, $w \in H^\rho_{p,c}$ if and only if there exists $m \geq 0$ such that $\widetilde{M}^m_{\rho,c}$ is bounded on $L^p(w)$.*

Proof. The necessity of the class $H^\rho_{p,c}$ was already stated in [3, Proposition 3.6] using Proposition 3.6. We shall prove the sufficiency.

Let $w \in H^\rho_{p,c}$. Then, $w \in H^{\rho,m_1}_{p,c}$ for some $m_1 \geq 0$, i.e., there exists $C > 0$ such that

$$\left(\int_B w\right)^{1/p}\left(\int_B w^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}} \leq C|B|\exp\left(c\left(1+\frac{r}{\rho(x)}\right)^{m_1}\right) \tag{3.6}$$

holds for every ball $B = B(x,r)$.

Let $m \geq 0$ to be chosen later. We can split the maximal operator as follows,

$$\widetilde{M}^m_{\rho,c}f(x) \leq \widetilde{M}^{m,(1)}_{\rho,c}f(x) + \widetilde{M}^{m,(2)}_{\rho,c}f(x) \tag{3.7}$$

where

$$\widetilde{M}^{m,(1)}_{\rho,c}f(x) = \sup_{r\leq\rho(x)} \frac{1}{\exp\left(c\left(1+\frac{r}{\rho(x)}\right)^m\right)} \fint_{B(x,r)} |f(y)|dy$$

and

$$\widetilde{M}^{m,(2)}_{\rho,c}f(x) = \sup_{r>\rho(x)} \frac{1}{\exp\left(c\left(1+\frac{r}{\rho(x)}\right)^m\right)} \fint_{B(x,r)} |f(y)|dy$$

By Proposition 3.1, $H^\rho_{p,c} \subset A^{\rho,\mathrm{loc}}_p$, and since $\widetilde{M}^{m,(1)}_{\rho,c}f(x) \leq M^{\mathrm{loc}}_\rho f(x)$, we deduce that $\widetilde{M}^{m,(1)}_{\rho,c}$ is bounded on $L^p(w)$ for any $m \geq 0$ and $w \in H^\rho_{p,c}$.

We will now deal with $\widetilde{M}^{m,(2)}_{\rho,c}f$. Let $\{Q_k\}_{k\geq 1}$ the covering given in [7, Proposition 2] of critical balls $Q_k = B(x_k, \rho(x_k))$. We fix $x \in \mathbb{R}^d$ and set $R_j = \{r : 2^{j-1}\rho(x) < r \leq 2^j\rho(x)\}$. Then, for any $Q_k \ni x$,

$$\begin{aligned}
\widetilde{M}^{m,(2)}_{\rho,c}f(x) &= \sup_{r>\rho(x)} \frac{1}{\exp\left(c\left(1+\frac{r}{\rho(x)}\right)^m\right)} \fint_{B(x,r)} |f(y)|dy \\
&= \sup_{j\geq 1}\sup_{r\in R_j} \frac{1}{\exp\left(c\left(1+\frac{r}{\rho(x)}\right)^m\right) c_d r^d} \int_{B(x,r)} |f(y)|dy \\
&\lesssim \sup_{j\geq 1} \frac{1}{\exp\left(c\left(1+2^{j-1}\right)^m\right)} \frac{1}{(2^{j-1}\rho(x))^d} \int_{B(x,2^j\rho(x))} |f(y)|dy \\
&\lesssim \sup_{j\geq 1} \frac{2^{-jd}}{\exp\left(c\left(1+2^{j-1}\right)^m\right)\rho(x_k)^d} \int_{c_jQ_k} |f(y)|dy
\end{aligned}$$

where $c_j = 2^j\left(C_0 2^{k_0}+1\right)$ can be obtained from the definition of ρ, and we have used that $\rho(x_k) \sim \rho(x)$ as $x \in Q_k$.

Finally, from Hölder's inequality and (3.6) we get

$$\begin{aligned}
\int_{\mathbb{R}^d} |\widetilde{M}^{m,(2)}_{\rho,c}f|^p w &\leq \sum_{k\geq 1}\int_{Q_k} |\widetilde{M}^{m,(2)}_{\rho,c}f|^p w \\
&\leq \sum_{k\geq 1}\sup_{j\geq 1} \frac{2^{-jdp}}{\exp\left(cp\left(1+2^{j-1}\right)^m\right)\rho(x_k)^{dp}} \left(\int_{c_jQ_k}|f|\right)^p\left(\int_{Q_k} w\right) \\
&\leq \sum_{k\geq 1}\sup_{j\geq 1} \frac{2^{-jdp}}{\exp\left(c\left(1+2^{j-1}\right)^m\right)\rho(x_k)^{dp}} \left(\int_{c_jQ_k}|f|^p w\right)\left(\int_{c_jQ_k} w^{\frac{-1}{p-1}}\right)^{p/p'}\int_{c_jQ_k} w \\
&\leq \sum_{k\geq 1}\sup_{j\geq 1} \frac{2^{-jdp}\left(c_j\rho(x_k)\right)^{dp}\exp\left(cp\left(1+\frac{c_j\rho(x_k)}{\rho(x_k)}\right)^{m_1}\right)}{\exp\left(cp\left(1+2^{j-1}\right)^m\right)\rho(x_k)^d} \int_{c_jQ_k} |f|^p w
\end{aligned}$$

IMAL PREPRINT # 2024-0074
ISSN 2451-7100
Publication date: December 6, 2024

$$\leq \left(C_0 2^{k_0}+1\right)^{dp} \sum_{k\geq 1} \sup_{j\geq 1} \frac{\exp\left(cp\left(1+2^j(C_0 2^{k_0}+1)\right)^{m_1}\right)}{\exp\left(cp\left(1+2^{j-1}\right)^m\right)} \int_{c_j Q_k} |f|^p w$$
$$\lesssim \sum_{j\geq 1} \exp\left(\tilde{c}\left(1+2^j\right)^{m_1-m}\right) \sum_{k\geq 1} \int_{c_j Q_k} |f|^p w.$$

Since $\sum_k \chi_{c_j Q_k} \leq C c_j^{N_1} \sim C 2^{jN_1}$,

$$\int_{\mathbb{R}^d} |\widetilde{M}^{m,(2)}_{\rho,c} f|^p w \lesssim \int_{\mathbb{R}^d} |f|^p w \sum_{j\geq 1} \exp\left(\tilde{c}\left(1+2^j\right)^{m_1-m}\right) 2^{jN_1} \lesssim \int_{\mathbb{R}^d} |f|^p w,$$

provided that $m > m_1 + N_1$ is chosen. □

4. Boundedness results for exponential Schrödinger–Calderón–Zygmund operators

We first present useful tools for proving our main theorem. While many of them follow the approach outlined in [5], additional considerations may be necessary in this new context. For this reason, we will provide details whenever clarification is required.

In order to get the boundedness on $L^p(w)$ for exponential Schrödinger–Calderón–Zygmund operators, we state an extrapolation result from the extreme point $p_0 = \infty$. Since many of the operators arising on the Schrödinger context are not bounded on L^∞, the following result will allow us to include them by establishing their continuity from $L^\infty(w)$ to $\mathrm{BMO}_\rho(w)$, for certain classes of weights.

Theorem 4.1. *Let $q > 0$ and let T be a bounded operator from $L^\infty(w)$ into $BMO_\rho(w)$ for any weight w such that $w^{-q} \in H^\rho_{1,c}$, with constant depending on w through the constant of the condition $w^{-q} \in H^{\rho,\theta}_{1,c}$. Then T is bounded on $L^p(w)$ for $q < p < \infty$ and every $w \in H^\rho_{p/q,c}$.*

Proof. First we observe that from Lemma 2.5 and the hypothesis on T, for every $f \in L^\infty(w)$ we have

$$\|M^\sharp_{\mathrm{loc}}(Tf)w\|_\infty \leq \|Tf\|_{\mathrm{BMO}_\rho(w)} \leq C\|f\|_{L^\infty(w)} = C\|fw\|_\infty \tag{4.1}$$

whenever w is a weight such that $w^{-q} \in H^\rho_{1,c}$.

Now, we need to state that $H^\rho_{p,c}$ is in the context of [4, Corollary 4], that is, it is a class of weights associated to the boundedness of a certain family of operators and they verify similar properties to those of the Muckenhoupt weights. Indeed, both conditions hold by Lemma 3.2 and Theorem 3.7 with the maximal operators $\widetilde{M}^m_{\rho,c}$. Hence, we can obtain the boundedness of $M^\sharp_{\mathrm{loc}} T$ in $L^p(w)$, whenever $q < p < \infty$ and $w \in H^\rho_{p/q,c}$. Finally, Proposition 3.1 allows us to apply [4, Corollary 5] to finish the proof. □

The next two propositions show that, for certain parameters, the exponential Schrödinger–Calderón–Zygmund operators of (s,δ) type for $1 < s \leq \infty$ and $0 < \delta \leq 1$, fall under the hypothesis of the theorem above.

Proposition 4.2. *Let T be an exponential Schrödinger–Calderón–Zygmund operator of (∞,δ) type as given in Definition 2.7 with constants c_1 and m_1 in (2.9). Then, T is bounded from $L^\infty(w)$ into $BMO_\rho(w)$ for every weight w such that $w^{-1} \in H^{\rho,m_1}_{1,c}$ with $c < c_1$.*

Proof. Let $x_0 \in \mathbb{R}^d$, $r \leq \rho(x_0)$ and $B = B(x_0, r)$. We consider

$$f = f\chi_{2B} + f\chi_{B(x_0,2\rho(x_0))\setminus 2B} + f\chi_{B(x_0,2\rho(x_0))^c} =: f_1 + f_2 + f_3.$$

Here, we need to observe that as $w^{-1} \in H^{\rho,m_1}_{1,c}$, from Lemma 3.3 there exist $\eta > 1$ and $c^* > 0$ such that $w \in RH^\rho_{\eta,c^*}$. Therefore there exist a constant C depending on m_1 and c^* such that

$$\left(\frac{1}{|B|}\int_B w^{-\gamma}\right)^{1/\gamma} \leq C\frac{1}{|B|}\int_B w^{-1}$$

since $r \leq \rho(x_0)$.

IMAL PREPRINT # 2024-0074
ISSN 2451-7100
Publication date: December 6, 2024

Using that T is bounded on $L^\gamma(\mathbb{R}^d)$, and that $w^{-1}\in A_1^{\rho,\mathrm{loc}}$ (by Proposition 3.1), we have

$$\begin{aligned}\frac{1}{|B|}\int_B |Tf_1(x)|dx &\le \left(\frac{1}{|B|}\int_B |Tf_1(x)|^\gamma dx\right)^{1/\gamma}\\ &\le C\left(\frac{1}{|B|}\int_{2B} |f(x)|^\gamma dx\right)^{1/\gamma}\\ &\le C\|fw\|_\infty\left(\frac{1}{|2B|}\int_{2B} w^{-\gamma}(x)dx\right)^{1/\gamma}\\ &\le C\|fw\|_\infty\left(\frac{1}{|B|}\int_{B} w^{-1}(x)dx\right)\\ &\le C\|fw\|_\infty \inf_B w^{-1}\\ &= C\frac{\|fw\|_\infty}{\|w\chi_B\|_\infty}.\end{aligned}$$

To estimate $|Tf_3(x)|$ for $x\in B$, we denote by $B_\rho=B(x_0,\rho(x_0))$ and $B_\rho^k=2^kB_\rho$ for each $k\in\mathbb{N}$. Thus

$$\int_{B_\rho^{k+1}}|f|\le \|fw\|_\infty\int_{B_\rho^{k+1}} w^{-1}\le C\exp\left(c\left(1+2^{k+1}\right)^{m_1}\right)|B_\rho^{k+1}|\frac{\|fw\|_\infty}{\|w\chi_{B_\rho^{k+1}}\|_\infty} \tag{4.2}$$

for each $k\in\mathbb{N}$. Then, by the size condition (2.9) on the kernel K with c_1 and m_1, applying the above inequality and Remark 2.2, we get

$$\begin{aligned}|Tf_3(x)| &\le \sum_{k\ge1}\int_{B_\rho^{k+1}\setminus B_\rho^k}|K(x,z)||f(z)|dz\\ &\le C\sum_{k\ge1}\int_{B_\rho^{k+1}\setminus B_\rho^k}\frac{1}{|B_\rho^{k-1}|}\exp\left(-c_1\left(1+\frac{2^{k-1}\rho(x_0)}{\rho(x)}\right)^{m_1}\right)|f(z)|dz\\ &\le C\sum_{k\ge1}\exp\left(-c_1\left(1+2^kC_02^{k_0}\right)^{m_1}\right)\frac{1}{|B_\rho^{k+1}|}\int_{B_\rho^{k+1}}|f(z)|dz\\ &\le C\sum_{k\ge1}\exp\left(-(c_1-c)\left(1+2^{k+1}\right)^{m_1}\right)\frac{\|fw\|_\infty}{\|w\chi_{B_\rho^{k+1}}\|_\infty}\\ &\le C\frac{\|fw\|_\infty}{\|w\chi_B\|_\infty}\sum_{k\ge1}\exp\left(-(c_1-c)\left(1+2^{k+1}\right)^{m_1}\right)\end{aligned}$$

where the series converges since $c<c_1$.

The bound for f_2 follows as in the proof of [4, Proposition 5], since only the smoothness condition is required, and the operator T under consideration satisfies [4, (41)], which is precisely (2.10). ☐

Proposition 4.3. *Let T be an exponential Schrödinger–Calderón–Zygmund operator of (s,δ) type as given in Definition 2.6 with constants c_1 and m_1 in (2.7). Then, T is bounded from $L^\infty(w)$ into* $\mathrm{BMO}_\rho(w)$ *for every weight w such that $w^{-s'}\in H_{1,c}^{\rho,m_1}$ with $c<s'c_1$.*

Proof. Let $x_0\in\mathbb{R}^d$, $r\le\rho(x_0)$ and $B=B(x_0,r)$. We consider

$$f=f\chi_{2B}+f\chi_{(B(x_0,2\rho(x_0))\setminus 2B)}+f\chi_{B(x_0,2\rho(x_0))^c}=f_1+f_2+f_3.$$

Here, we need to observe that as $w^{-s'}\in H_{1,c}^{\rho}$ and $r\le\rho(x_0)$, the estimates of the average $\frac{1}{|B|}\int_B|Tf_1|dx$ follows using Kolmogorov's inequality and the hypothesis on w as in [5]. Then we have

$$\frac{1}{|B|}\int_B|Tf_1(x)|dx\le C\frac{\|fw\|_\infty}{\|w\chi_B\|_\infty}.$$

To estimate $|Tf_3(x)|$ for $x\in B$, we denote, as in the previous proof, $B_\rho=B(x_0,\rho(x_0))$ and $B_\rho^k=2^kB_\rho$ for $k\in\mathbb{N}$. From the condition on the weight we can obtain

$$\left(\int_{B_\rho^{k+1}}|f|^{s'}\right)^{1/s'}\le\|fw\|_\infty\left(\int_{B_\rho^{k+1}}w^{-s'}\right)^{1/s'}\le C\exp\left(\frac{c}{s'}\left(1+2^{k+1}\right)^{m_1}\right)\left|B_\rho^{k+1}\right|^{1/s'}\frac{\|fw\|_\infty}{\|w\chi_{B_\rho^{k+1}}\|_\infty}.$$

IMAL PREPRINT # 2024-0074
ISSN 2451-7100
Publication date: December 6, 2024

From this inequality and (2.7) with $c_1, m_1 > 0$, we estimate

$$\begin{aligned}|Tf_3(x)| &\leq \sum_{k\geq 1} \int_{B_\rho^{k+1}\setminus B_\rho^k} |K(x,z)||f(z)|dz \\ &\leq \sum_{k\geq 1} \left(\int_{B_\rho^{k+1}\setminus B_\rho^k} |K(x,z)|^s dz\right)^{1/s} \left(\int_{B_\rho^{k+1}} |f(z)|^{s'} dz\right)^{1/s'} \\ &\leq C \sum_{k\geq 1} \rho(x_0)^{-d/s'} \exp\left(-c_1\left(1+2^{k+1}\right)^{m_1}\right) \exp\left(\frac{c}{s'}\left(1+2^{k+1}\right)^{m_1}\right) \left|B_\rho^{k+1}\right|^{1/s'} \frac{\|fw\|_\infty}{\|w\chi_{B_\rho^{k+1}}\|_\infty} \\ &\leq C \frac{\|fw\|_\infty}{\|w\chi_{B_\rho}\|_\infty} \sum_{k\geq 1} \exp\left(-\left(c_1 - \frac{c}{s'}\right)\left(1+2^{k+1}\right)^{m_1}\right)\end{aligned}$$

where the series converges as $c < s'c_1$.

The bound for f_2 can be obtained as in the proof of [4, Proposition 6] since (2.8) implies [4, (45)]. □

5. Applications for operators associated to $-\Delta + \mu$

We consider a Schrödinger operator with measure μ in $\mathbb{R}^d$, with $d \geq 3$,

$$\mathcal{L}_\mu = -\Delta + \mu,$$

where μ is a nonnegative Radon measure on $\mathbb{R}^d$ that satisfies the following conditions: there exist constants $\delta_\mu, C_\mu > 0$ and $D_\mu \geq 1$ such that

$$\mu(B(x,r)) \leq C_\mu \left(\frac{r}{R}\right)^{d-2+\delta_\mu} \mu(B(x,R)) \tag{5.1}$$

and

$$\mu(B(x,2r)) \leq D_\mu \left(\mu(B(x,r)) + r^{d-2}\right) \tag{5.2}$$

for all $x \in \mathbb{R}^d$ and $0 < r < R$, where $B(x,r)$ denotes the open ball centered at x with radius r.

From (5.1) and (5.2) it can be proved that (see [11, Remark 0.13])

$$\int_{B(x,R)} \frac{d\mu(y)}{|y-x|^{d-2}} \leq C \frac{\mu(B(x,R))}{R^{d-2}}, \tag{5.3}$$

and if $\delta_\mu > 1$ then we also have

$$\int_{B(x,R)} \frac{d\mu(y)}{|y-x|^{d-1}} \leq C \frac{\mu(B(x,R))}{R^{d-1}}, \tag{5.4}$$

for all $x \in \mathbb{R}^d$ and $R > 0$.

For any nonnegative Radon measure μ on $\mathbb{R}^d$ verifying conditions (5.1) and (5.2), the function given by

$$\rho_\mu(x) := \sup\left\{r > 0 : \frac{\mu(B(x,r))}{r^{d-2}} \leq 1\right\}, \quad x \in \mathbb{R}^d, \tag{5.5}$$

is a critical radius function (see Definition 2.1).

It is immediate from the definition of ρ_μ that for every $x \in \mathbb{R}^d$

$$\frac{\mu(B(x,\rho_\mu(x)))}{\rho_\mu(x)^{d-2}} \sim 1. \tag{5.6}$$

The following technical lemma for the Agmon distance will be an important tool for the examples below.

Lemma 5.1. *Let $\rho : \mathbb{R} \to [0,\infty)$ be a critical radius function. Then, there exists a constant $C > 0$, depending on D_0, D_1 and k_0, such that*

$$d_\rho(x,y) \geq D_1^{-1}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{1}{k_0+1}} + D_0^{-1}\left(1 + \frac{\rho(x)}{|x-y|}\right)^{-1} - C. \tag{5.7}$$

IMAL PREPRINT # 2024-0074
ISSN 2451-7100
Publication date: December 6, 2024

Proof. We will consider two cases. If $|x-y| \leq 2\rho(x)$, by Lemma 2.3 we have

$$d_\rho(x,y) \geq D_0^{-1} \frac{|x-y|}{\rho(x)}.$$

Since

$$\frac{|x-y|}{\rho(x)} \geq \frac{|x-y|}{\rho(x)+|x-y|} = \left(1+\frac{\rho(x)}{|x-y|}\right)^{-1},$$

we get

$$d_\rho(x,y) \geq D_0^{-1} \left(1+\frac{\rho(x)}{|x-y|}\right)^{-1}.$$

As $1+\frac{|x-y|}{\rho(x)} \leq 3$, we can take $C \geq D_1^{-1} 3^{\frac{1}{k_0+1}}$ to get the desired inequality in this case.

On the other hand, if $|x-y| > 2\rho(x)$ we can apply Lemma 2.4 to get

$$d_\rho(x,y) \geq D_1^{-1} \left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{1}{k_0+1}}$$

Trivially, $1+\frac{\rho(x)}{|x-y|} \geq 1$, so choosing $C \geq D_0^{-1}$ gives (5.7) for $|x-y| > 2\rho(x)$. Finally, taking $C = \max\left\{D_1^{-1} 3^{\frac{1}{k_0+1}}, D_0^{-1}\right\}$, the proof is finished. □

Now, we shall apply Propositions 4.2 and 4.3 to some operators that were considered by Z. Shen in [11].

Remark 5.2. It is worth noticing that, as a particular case of a measure μ we will be able to consider $d\mu = V(x)dx$, with $V \geq 0$ in the reverse Hölder class $RH_{\frac{d}{2}}$, that is,

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)^{\frac{d}{2}} dy\right)^{\frac{2}{d}} \leq C \frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)dy,$$

for every $x \in \mathbb{R}^d$ and $r > 0$. Under this condition it is easy to see that μ satisfy conditions (5.3) and (5.4) for some $\delta_\mu, C_\mu > 0$ and $D_\mu \geq 1$. In this context, we recover the operators studied in [12].

Hereafter, we denote by Γ_μ the fundamental solution of $\mathcal{L}_\mu = -\Delta + \mu$ and by $\Gamma_{\mu+\lambda}$ the fundamental solution of $\mathcal{L}_{\mu+\lambda} = -\Delta + (\mu+\lambda)$ for $\lambda \geq 0$. As observed in [11, p. 554], the measure $\mu+\lambda$ verifies (5.1) and (5.2) for any $\lambda \geq 0$. Also, we may write d_μ and $d_{\mu+\lambda}$ to indicate the Agmon distances associated with the critical radius functions ρ_μ and $\rho_{\mu+\lambda}$ defined as in (5.5) for the measures μ and $\mu+\lambda$, respectively.

We refer to [11, Theorems 0.8 and 0.17] for the following results. Actually, although the estimate (5.9) given below is not explicitly stated in [11], it is essentially contained within the proof of [11, Theorem 0.19] (see also [3, Theorem 4.2]).

For a measure μ satisfying (5.1) and (5.2), there exist positive constants $C_1, C_2, C_3, \epsilon_1, \epsilon_2$ and ϵ_3 such that

$$C_1 \frac{e^{-\epsilon_1 d_\mu(x,y)}}{|x-y|^{d-2}} \leq \Gamma_\mu(x,y) \leq C_2 \frac{e^{-\epsilon_2 d_\mu(x,y)}}{|x-y|^{d-2}}, \tag{5.8}$$

and

$$|\nabla_1 \Gamma_\mu(x,y)| \leq C_3 \frac{e^{-\epsilon_3 d_\mu(x,y)}}{|x-y|^{d-2}} \left(\int_{B\left(x,\frac{|y-x|}{2}\right)} \frac{d\mu(z)}{|z-x|^{d-1}} + \frac{1}{|x-y|}\right) \tag{5.9}$$

for $x \neq y$. Also, for $\delta_\mu > 1$ we have that there exist $C, \epsilon > 0$ such that

$$|\nabla_1 \Gamma_\mu(x,y)| \leq C \frac{e^{-\epsilon d_\mu(x,y)}}{|x-y|^{d-1}}, \qquad \text{for } x \neq y. \tag{5.10}$$

IMAL PREPRINT # 2024-0074
ISSN 2451-7100
Publication date: December 6, 2024

5.1. **Imaginary powers of $\mathcal{L}_\mu$.** For $\gamma \in \mathbb{R}$, we consider $(-\Delta+\mu)^{i\gamma}$ to be the operator defined in [11] where, by functional calculus, we can write as

$$(-\Delta+\mu)^{i\gamma} f(x) = \int_{\mathbb{R}^n} K_\gamma(x,y) f(y) dy,$$

being

$$K_\gamma(x,y) = -\frac{\sin(\pi\gamma i)}{\pi} \int_0^\infty \lambda^{i\gamma} \Gamma_{\mu+\lambda}(x,y) d\lambda.$$

Proposition 5.3. *Let μ be as above with $\delta_\mu > 0$, and let $\epsilon_1 > 0$ be as in (5.8). Then, the operator $(-\Delta+\mu)^{i\gamma}$ is an exponential Schrödinger–Calderón–Zygmund operator of (∞,δ) type as given in Definition 2.7 with constants $c_1 = \frac{\epsilon_1}{2D_1}$ and $m_0 = \frac{1}{k_0+1}$.*

Proof. In [11, Theorem 0.19] it was proved that $(-\Delta+\mu)^{i\gamma}$ is a Calderón–Zygmund operator for $\delta_\mu > 0$, which gives (2.10) and implies its boundedness on $L^p(\mathbb{R}^d)$ for every $1 < p < \infty$. Moreover, from (5.8), by using that (see [3, (21)])

$$d_{\mu+\lambda}(x,y) \geq \frac{1}{2}\left(d_\mu(x,y) + d_\lambda(x,y)\right) \geq \frac{1}{2} d_\mu(x,y) + \frac{\sqrt{\lambda}}{2}|x-y|, \tag{5.11}$$

and Lemma 5.1, we get

$$|K_\gamma(x,y)| \leq C \frac{e^{-\frac{\epsilon_1}{2} d_\mu(x,y)}}{|x-y|^{d-2}} \leq C \frac{e^{-\frac{\epsilon_1}{2D_1}\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{1}{k_0+1}}}}{|x-y|^{d-2}},$$

for every $x \neq y$ and the condition it is satisfied with the desired parameters. □

As a consequence of Proposition 5.3, Proposition 4.2 and Theorem 4.1 we obtain the following result.

Theorem 5.4. *Let μ be as above with $\delta_\mu > 0$, and let $\epsilon_1 > 0$ be as in (5.8). Then for any $c < \frac{\epsilon_1}{2D_1}$, $m_0 = \frac{1}{k_0+1}$ and w such that $w^{-1} \in H_{1,c}^{\rho,m_0}$, we have that $(-\Delta+\mu)^{i\gamma}$ is bounded from $L^\infty(w)$ into $\mathrm{BMO}_\rho(w)$ and is also bounded on $L^p(w)$, for every $1 < p < \infty$.*

5.2. **Riesz transforms.** We consider the Riesz transform $R_\mu = \nabla(-\Delta+\mu)^{-\frac{1}{2}}$ and its adjoint $R_\mu^* = (-\Delta+\mu)^{-\frac{1}{2}}\nabla$ as in [11].

The Riesz transform R_μ can be expressed, for $f \in C_c^\infty(\mathbb{R}^d)$ and $x \notin \mathrm{supp}(f)$, as

$$\begin{aligned} R_\mu f(x) &= \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} \nabla(-\Delta+\mu+\lambda)^{-1} f(x) d\lambda \\ &= \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} \int_{\mathbb{R}^d} \nabla\Gamma_{\mu+\lambda}(x,y) f(y) dy\, d\lambda. \end{aligned}$$

By Fubini's theorem,

$$R_\mu f(x) = \int_{\mathbb{R}^d} \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} \nabla_1 \Gamma_{\mu+\lambda}(x,y) d\lambda f(y) dy = \int_{\mathbb{R}^d} K_\mu(x,y) f(y) dy,$$

where K_μ is the singular kernel of R_μ given by

$$K_\mu(x,y) = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} \nabla_1 \Gamma_{\mu+\lambda}(x,y) d\lambda. \tag{5.12}$$

The adjoint R_μ^* can be written as

$$R_\mu^* f(x) = \int_{\mathbb{R}^d} K_\mu^*(x,y) f(y) dy = \int_{\mathbb{R}^d} -K_\mu(y,x) f(y) dy.$$

By [11, Theorem 7.18] we know that R_μ is a Calderón–Zygmund operator when $\delta_\mu > 1$, so it is bounded on $L^p(\mathbb{R}^d)$ for every $1 < p < \infty$. By duality, R_μ^* is also bounded on $L^p(\mathbb{R}^d)$ for every $1 < p < \infty$. There also is proved condition (2.10). Moreover, we have that (see [11, Theorem 0.17])

$$|K_\mu(x,y)| \leq C \frac{e^{-\epsilon d_\mu(x,y)}}{|x-y|^d},$$

for some $\epsilon > 0$, and the same estimate hold for $|K_\mu^*(x,y)|$ (see also [3, Lemma 4.3]). Then, by Lemma 5.1 we obtain the following classification for the Riesz transform and its adjoint.

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ISSN 2451-7100
Publication date: December 6, 2024

Proposition 5.5. *Let μ be a measure verifying (5.1) and (5.2) with $\delta_\mu > 1$. Then, R_μ and R_μ^* are both exponential Schrödinger–Calderón–Zygmund operators of (∞, δ) type as given in Definition 2.7 with constants $c = \frac{\epsilon}{2D_1}$ and $m = \frac{1}{k_0+1}$.*

As a consequence of Proposition 5.5 above, Proposition 4.2 and Theorem 4.1 we obtain the following boundedness result for the case $\delta_\mu > 1$.

Theorem 5.6. *Let μ be a measure verifying (5.1) and (5.2) with $\delta_\mu > 1$. Then, for any $c < \frac{\epsilon_1}{2D_1}$, $m_0 = \frac{1}{k_0+1}$ and w such that $w^{-1} \in H_{1,c}^{\rho,m_0}$, we have that R_μ and R_μ^* are both bounded from $L^\infty(w)$ into* $\mathrm{BMO}_\rho(w)$ *and is also bounded on $L^p(w)$, for every $1 < p < \infty$.*

When $0 < \delta_\mu < 1$ we consider the adjoint operators R_μ^* and the estimates given in [3, Lemma 4.3], which establishes the existence of constants $C, \epsilon > 0$ such that

$$|K_\mu^*(x,y)| \leq C \frac{e^{-\epsilon d_\mu(x,y)}}{|x-y|^{d-1}} \left(\int_{B\left(y, \frac{|y-x|}{2}\right)} \frac{d\mu(z)}{|z-y|^{d-1}} + \frac{1}{|x-y|} \right), \tag{5.13}$$

for all $x, y \in \mathbb{R}^d$ with $x \neq y$.

Before determining the type of operator that R_μ^* is for $0 < \delta_\mu < 1$, we will need to make the following generalization of a known property for the case where $d\mu(x) = V(x)dx$, as given in [8, Lemma 1].

Lemma 5.7. *Let μ be a measure verifying (5.1) and (5.2) with $\delta_\mu > 0$. Then there exist a constant $C > 0$ such that for any $x_0 \in \mathbb{R}^d$ and $R > 0$*

$$\mu\left(B(x_0, R)\right) \leq C R^{d-2} \left(1 + \frac{R}{\rho(x_0)}\right)^N,$$

for all $N \geq \log_2 D_\mu$, with D_μ as in (5.2).

Proof. Let us first consider the case $R \leq \rho(x_0)$. By (5.1) and (5.6),

$$\mu\left(B(x_0, R)\right) \lesssim \left(\frac{R}{\rho(x_0)}\right)^{d-2+\delta_\mu} \mu\left(B(x_0, \rho(x_0))\right) \lesssim R^{d-2}.$$

Now, if $R > \rho(x_0)$, let $j_0 \in \mathbb{N}$ such that $2^{j_0-1}\rho(x_0) < R \leq 2^{j_0}\rho(x_0)$ and use (5.2) j_0 times to get

$$\mu(B(x_0, 2^{j_0} r)) \leq D_\mu^{j_0} \mu(B(x_0, r)) + D_\mu^{j_0} r^{d-2} 2^{(j_0-1)(d-2)}.$$

Then, again by (5.1) and (5.6)

$$\begin{aligned} \mu\left(B(x_0, R)\right) &\lesssim \left(\frac{R}{2^{j_0}\rho(x_0)}\right)^{d-2+\delta_\mu} \mu\left(B(x_0, 2^{j_0}\rho(x_0))\right) \\ &\lesssim \left(\frac{R}{2^{j_0}\rho(x_0)}\right)^{d-2+\delta_\mu} \left(D_\mu^{j_0} \mu\left(B(x_0, \rho(x_0))\right) + D_\mu^{j_0} \rho(x_0)^{d-2} 2^{(j_0-1)(d-2)}\right) \\ &\leq C R^{d-2} \left(\frac{R}{\rho(x_0)}\right)^{\delta_\mu} \left(\frac{D_\mu}{2^{\delta_\mu}}\right)^{j_0}. \end{aligned}$$

Finally, since $\log_2\left(\frac{R}{\rho(x_0)}\right) \leq j_0 \leq 1 + \log_2\left(\frac{R}{\rho(x_0)}\right)$,

$$\begin{aligned} D_\mu^{j_0} 2^{-j_0\delta_\mu} &\leq D_\mu^{1+\log_2\left(\frac{R}{\rho(x_0)}\right)} 2^{-\delta_\mu \log_2\left(\frac{R}{\rho(x_0)}\right)} \\ &\leq D_\mu D_\mu^{\log_2\left(1+\frac{R}{\rho(x_0)}\right)} \left(\frac{R}{\rho(x_0)}\right)^{-\delta_\mu} \\ &\lesssim \left(1 + \frac{R}{\rho(x_0)}\right)^{\log_2 D_\mu} \left(\frac{R}{\rho(x_0)}\right)^{-\delta_\mu}. \end{aligned}$$

Therefore, in both cases we obtain the desired estimate. □

Proposition 5.8. *Let μ be a measure verifying (5.1) and (5.2) with $0 < \delta_\mu < 1$. Then R_μ^* is an exponential Schrödinger–Calderón–Zygmund operator of (s, δ) type for some $0 < \delta < 1$ and every $1 < s < \frac{2-\delta_\mu}{1-\delta_\mu}$ as given in Definition 2.6 with constants $c = \frac{\epsilon}{4D_1}$ and $m = \frac{1}{k_0+1}$.*

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ISSN 2451-7100
Publication date: December 6, 2024

Proof. Condition (i) of Definition 2.6 follows from the proof of [11, Theorem 7.1] for every $s' > 2-\delta_\mu$, i.e., for every $1 < s < \frac{2-\delta_\mu}{1-\delta_\mu}$.

We now prove condition (2.7). Using [11, Lemma 7.9] with $1 < s < \frac{2-\delta_\mu}{1-\delta_\mu}$ Lemma 5.7, we have that, whenever $|x-x_0| < R/2$,

$$\begin{aligned}\left(\int_{R<|x_0-y|\le 2R}\left(\int_{B\left(y,\frac{|y-x|}{2}\right)}\frac{d\mu(z)}{|z-y|^{d-1}}\right)^s dy\right)^{1/s} &\le \left(\int_{B(x,4R)}\left(\int_{B(x,4R)}\frac{d\mu(z)}{|z-y|^{d-1}}\right)^s dy\right)^{1/s}\\ &\le C\frac{\mu\left(B(x,12R)\right)}{R^{d(1-(1/s))-1}}\\ &\le CR^{d/s-1}\left(1+\frac{R}{\rho(x)}\right)^{\log_2 D_\mu}. \end{aligned}\tag{5.14}$$

Then, by Lemma 5.1 and (5.13),

$$\left(\frac{1}{R^d}\int_{R<|x_0-y|\le 2R}|K_\mu^*(x,y)|^s dy\right)^{1/s} \lesssim \frac{1}{R^d}e^{-\frac{\epsilon}{2D_1}\left(1+\frac{R}{\rho(x)}\right)^{\frac{1}{k_0+1}}}\left(1+\frac{R}{\rho(x)}\right)^{\log_2 D_\mu} \lesssim \frac{1}{R^d}e^{-c\left(1+\frac{R}{\rho(x)}\right)^m}.$$

To prove condition (2.8) we follow the ideas given in the proof of [11, Theorem 7.18]. There exists $\delta_1 \in (0,1)$ such that for $x_0 \in B\left(x, \frac{|x-y|}{8}\right)$,

$$\begin{aligned}&|\nabla_1\Gamma_{\mu+\lambda}(y,x) - \nabla_1\Gamma_{\mu+\lambda}(y,x_0)|\\ &\quad\lesssim \left(\frac{|x-x_0|}{|x-y|}\right)^{\delta_1}\sup_{w\in B(x,|x-y|/2)}|\nabla_1\Gamma_{\mu+\lambda}(y,w)|\\ &\quad\lesssim \left(\frac{|x-x_0|}{|x-y|}\right)^{\delta_1}\sup_{w\in B(x,|x-y|/2)} e^{-\epsilon_3 d_{\mu+\lambda}(y,w)}\frac{1}{|y-w|^{d-2}}\left(\int_{B\left(y,\frac{|w-y|}{2}\right)}\frac{d\mu(z)}{|z-y|^{d-1}}+\frac{1}{|w-y|}\right)\\ &\quad\lesssim \left(\frac{|x-x_0|}{|x-y|}\right)^{\delta_1}\sup_{w\in B(x,|x-y|/2)} e^{-\frac{\epsilon_3}{2}|y-w|}\frac{1}{|y-w|^{d-2}}\left(\int_{B\left(y,\frac{|w-y|}{2}\right)}\frac{d\mu(z)}{|z-y|^{d-1}}+\frac{1}{|w-y|}\right),\end{aligned}$$

where we have used (5.9) and (5.11).

By taking the supremum on w we obtain

$$\begin{aligned}|K_\mu^*(x,y)-K_\mu^*(x_0,y)| &\lesssim \left(\frac{|x-x_0|}{|x-y|}\right)^{\delta_1}\frac{1}{|y-x|^{d-2}}\left(\int_{B(y,|y-x|)}\frac{d\mu(z)}{|z-y|^{d-1}}+\frac{1}{|y-x|}\right)\int_0^\infty \frac{e^{-\frac{\epsilon_3}{4}\lambda^{1/2}}}{\lambda^{1/2}}d\lambda\\ &\lesssim \left(\frac{|x-x_0|}{|x-y|}\right)^{\delta_1}\frac{1}{|y-x|^{d-2}}\left(\int_{B(y,|y-x|)}\frac{d\mu(z)}{|z-y|^{d-1}}+\frac{1}{|y-x|}\right).\end{aligned}$$

Finally, proceeding as in the estimate of the kernel size and using (5.14), we obtain (2.8), completing the proof of the proposition. □

5.3. **Operators associated to a potential.** In the particular case where the measure μ is given by $d\mu(x) = V(x)dx$, with V a nonnegative function belonging to the classical reverse Hölder class RH_q for $q > \frac{d}{2}$, we will be able to study the operators $T_j = (-\Delta+V)^{-j/2}V^{j/2}$ for $j = 1,2$. Actually, they can be found in [4], where weighted estimates were proved for weights in the class A_p^ρ.

Condition (i) of Definition 2.6 holds for T_j, $j = 1,2$ when $s \ge 2q$ and $s \ge q$, respectively (see [12, Theorems 5.10 and 3.1]) and condition (2.8) is contained in the proof of [8, Theorem 1] when $s = 2q$ and $s = q$ for T_1 and T_2, respectively, and some $\delta > 0$.

Now, from the results in Section 4, we shall obtain estimates for weights in $H_{p,c}^\rho$, for appropriated parameters, where we understand by ρ the critical radius function associated to the potential V. In order to do so, we will see that the size condition obtained in [4, Theorem 8] can be improved in order to have (2.7) for the kernels of T_j, $j = 1,2$.

ISSN 2451-7100
IMAL PREPRINT # 2024-0074
Publication date: December 6, 2024

For each $j=1,2$, we have

$$\begin{aligned} T_j f(x) &= (-\Delta+V)^{-j/2} V^{j/2} f(x) \\ &= \frac{1}{\pi}\int_0^\infty \lambda^{-j/2}\int_{\mathbb{R}^d} \Gamma_{V+\lambda}(x,y) V(y)^{j/2} f(y)\,dy\,d\lambda \\ &= \int_{\mathbb{R}^d} K_j(x,y) f(y)\,dy, \end{aligned}$$

and by the estimates for the fundamental solution of $-\Delta+V$ given in (5.9), (5.11) and Lemma 5.1, there exists $\epsilon_j>0$ such that

$$|K_j(x,y)| \le C\frac{\exp\left(-\frac{\epsilon_j}{2D_1}\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right)}{|x-y|^{d-2}} V(y)^{j/2}, \quad j=1,2. \tag{5.15}$$

Then, condition (2.7) follows in the same way as in the proof of [4, Theorem 8] considering $s=2q$ and $s=q$ for the cases $j=1$ and $j=2$, respectively.

Therefore, we get that both T_j are exponential Schrödinger–Calderón–Zygmund operators of (s,δ) type for s as above and some $\delta>0$.

As a consequence of Proposition 4.3 and Theorem 4.1 we obtain the following result for each T_j.

Theorem 5.9. *Let V be as above with $q>\frac{d}{2}$, and let $\epsilon_j>0$ be as in (5.15), $j=1,2$. Then, for any $c_j<\frac{\epsilon_j}{2D_1}$, $m_0=\frac{1}{k_0+1}$ and w such that $w^{-s'}\in H^{\rho,m_0}_{1,c_j}$, with $s=2q$ when $j=1$ and $s=q$ when $j=2$, we have that T_j is bounded from $L^\infty(w)$ into $\mathrm{BMO}_\rho(w)$ and is also bounded on $L^p(w)$, for every $s'<p<\infty$, for $j=1,2$.*

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