ISSN 2451-7100

Publicaciones del IMAL

IMAL Preprints
#2024–0075

Sparse approach for the two-norm inequality of local fractional maximal and integral operators

Mauricio Ramseyer, Oscar Salinas, Juan Sotto Ríos, Marisa Toschi

Publication Date: December 27, 2024

2024

Publisher: Instituto de Matemática Aplicada del Litoral IMAL "Dra. Eleonor Harboure" (CCT CONICET Santa Fe – UNL)

https://imal.conicet.gov.ar
https://imal.conicet.gov.ar/preprints-del-imal

Publishing Director: Dra. Estefanía Dalmasso

edalmasso@santafe-conicet.gov.ar

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

SPARSE APPROACH FOR THE TWO-NORM INEQUALITY OF LOCAL FRACTIONAL MAXIMAL AND INTEGRAL OPERATORS.

MAURICIO RAMSEYER, OSCAR SALINAS, JUAN SOTTO RÍOS, AND MARISA TOSCHI.

ABSTRACT. Let Ω be a proper open subset of $\mathbb{R}^n$ We give sufficient conditions about local weights for the two-weight norm inequality for the local fractional maximal and the local fractional integral operator acting on weighted Lebesgue spaces. By using the technique of sparse operators we obtained improved results taking into account the proposed hypotheses. As applications we obtain a priori estimate for solutions of $\Delta^m u = f$ in Ω, acting in weighted Sobolev spaces involving the distance to the boundary and different local weights. In the context of Schödinger type operators we prove as another application the boundedness of the Riesz potential I^α_μ, for $0 < \alpha < 2$, $n \geq 3$ and μ be a Radon measure on $\mathbb{R}^n$. Some illuminating examples are set out at the end of the article.

1. INTRODUCTION

Let Ω be a proper open and non empty subset of $\mathbb{R}^n$. The notation $Q = Q(x_Q, l_Q)$ means a cube Q with sides parallel to the coordinate axes, where x_Q and l_Q are the center and the length of half of its side respectively. Here, when we mention the metric d, we mean the usual d_∞. As usual we will denote by λQ the cube with same center and side length λ-times of Q.

For $0 < \beta < 1$, we consider the family of cubes well-inside of Ω defined by

$$\mathcal{F}_\beta = \{Q = Q(x_Q, l_Q) : \; x_Q \in \Omega, \; l_Q < \beta \, \mathrm{d}(x_Q, \Omega^c)\} .$$

Associated to this families we give the definitions of the operators with which we will work.

Definition 1.1. *Let $0 \leq \gamma < 1$, $0 < \beta < 1$ and $f \in L^1_{loc}(\Omega)$. For $x \in \Omega$ we define the **local fractional maximal operator** as*

$$M^\gamma_\beta f(x) = \sup_{Q \in \mathcal{F}_\beta} \left(|Q|^{\gamma - 1} \int_Q |f| \right) \chi_Q(x) .$$

Definition 1.2. *Let $0 < \gamma < 1$, $0 < \beta < 1$ and $f \in L^1_{loc}(\Omega)$. For $x \in \Omega$ we define the **local fractional integral operator** as*

$$I^\gamma_\beta f(x) = \int_{Q(x, \beta d(x, \Omega^c))} \frac{f(y)}{|x - y|^{n(1-\gamma)}} dy .$$

2020 *Mathematics Subject Classification.* Primary 42B35.

Key words and phrases. local singular operator, local fractional operator, class of weights.

This research is partially supported by grants from Agencia Nacional de Promoción Científica y Tecnológica (ANPCyT), Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) and Universidad Nacional del Litoral (UNL), Argentina.

IMAL PREPRINT # 2024-0075
ISSN 2451-7100
Publication date: December 27, 2024

Next, we present the class of weights involves in this work. But first we introduce some notation. For a locally integrable function f and a cube Q in $\mathbb{R}^n$, with $0 < |Q| < \infty$, we define the localized L^p norm, $1 \le p < \infty$, by

$$\|f\|_{p,Q} = \left(\frac{1}{|Q|} \int_Q |f(x)|^p \, dx \right)^{\frac{1}{p}} .$$

A function $\varphi : [0,\infty) \to [0,\infty)$ is a Young function if φ is continuous, convex and strictly increasing, $\varphi(0) = 0$ and $\varphi(t)/t \to \infty$ as $t \to \infty$. There exists an associate function to φ namely $\overline{\varphi}$ defined by $\overline{\varphi}(t) = \sup_{s>0} \big(st - \varphi(t)\big)$. This function is important because it allows us to recover tools widely used in theory, among them the Hölder inequality. Finally, if φ is a Young function, the normalized Luxembourg norm of f in Q is

$$\|f\|_{\varphi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \varphi\left(\frac{|f(x)|}{\lambda} \right) dx \le 1 \right\} .$$

In each of the following three definitions the parameters are $0 < \beta < 1$, $0 \le \gamma < 1$ and $1 < p \le q < \infty$.

Definition 1.3. *We say that the pair of weights (u, v) belongs to the class $\boldsymbol{A^{\beta,\gamma}_{p,q}}$ if and only if*

$$[u, v]_{A^{\beta,\gamma}_{p,q}} = \sup_{Q\in\mathcal{F}_\beta} |Q|^{\gamma+\frac{1}{q}-\frac{1}{p}} \left\| u^{\frac{1}{q}} \right\|_{q,Q} \left\| v^{-\frac{1}{p}} \right\|_{p',Q} < \infty.$$

Definition 1.4. *Let ψ be a Young function. We say that the pair of weights (u, v) belongs to the class $\boldsymbol{A^{\beta,\gamma}_{p,q,\psi}}$ if and only if*

$$[u, v]_{A^{\beta,\gamma}_{p,q,\psi}} = \sup_{Q\in\mathcal{F}_\beta} |Q|^{\gamma+\frac{1}{q}-\frac{1}{p}} \left\| u^{\frac{1}{q}} \right\|_{q,Q} \left\| v^{-\frac{1}{p}} \right\|_{\psi,Q} < \infty .$$

Definition 1.5. *Lets ϕ and ψ Young functions. We say that the pair of weights (u, v) belongs to the class $\boldsymbol{A^{\beta,\gamma}_{p,q,\phi,\psi}}$ if and only if*

$$[u, v]_{A^{\beta,\gamma}_{p,q,\phi,\psi}} = \sup_{Q\in\mathcal{F}_\beta} |Q|^{\gamma+\frac{1}{q}-\frac{1}{p}} \left\| u^{\frac{1}{q}} \right\|_{\phi,Q} \left\| v^{-\frac{1}{p}} \right\|_{\psi,Q} < \infty .$$

For a Young function φ we say that it belongs to the class B_p, with $1 < p < \infty$, if for some $c > 0$

$$\int_c^\infty \frac{\varphi(t)}{t^p} \frac{dt}{t} < \infty .$$

This class of functions has an associated maximal operator defined as

$$M_\varphi f(x) = \sup_{Q:\, x\in Q} \|f\|_{\varphi,Q} .$$

In $\mathbb{R}^n$ we know that this operator is bounded on L^p (see [8], [9]).

Finally, if φ is doubling, that is, $\varphi(2t) \le C\varphi(t)$, and $\overline{\varphi}$ belongs to B_p, then $\overline{\varphi}(t^{1/p})$ is a concave function. Therefore, its inverse $(\overline{\varphi}^{-1}(t))^p$ is a convex function.

Lemma 1.6. *Let β, γ, p, q and ψ as above. If $\overline{\psi}$ belongs to B_p then*

$$A^{\beta,\gamma}_{p,q,\psi} \subset A^{\beta,\gamma}_{p,q} .$$

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

Proof. By the definition of the classes it is sufficient to show that

$$\left\| v^{-\frac{1}{p}} \right\|_{p',Q} \leq \left\| v^{-\frac{1}{p}} \right\|_{\psi,Q} .$$

For a given function g, by using Hölder's inequality with $\varphi = \psi(t^{1/p'})$ we have

$$\|g\|_{p',Q}^{p'} = \frac{1}{|Q|}\int_Q |g|^{p'} \leq 2 \left\| g^{p'} \right\|_{\varphi,Q} \|1\|_{\overline{\varphi},Q} \leq C \left\| g^{p'} \right\|_{\varphi,Q} .$$

Furthermore, we have that

$$\begin{aligned} \left\| g^{p'} \right\|_{\varphi,Q} &= \inf\left\{ \lambda > 0 : \frac{1}{|Q|}\int_Q \varphi\left(\frac{|g|^{p'}}{\lambda}\right) \leq 1 \right\} \\ &= \inf\left\{ \lambda > 0 : \frac{1}{|Q|}\int_Q \psi\left(\frac{|g|}{\lambda^{1/p'}}\right) \leq 1 \right\} \\ &= \inf\left\{ \tilde{\lambda}^{p'} > 0 : \frac{1}{|Q|}\int_Q \psi\left(\frac{|g|}{\tilde{\lambda}}\right) \leq 1 \right\} = \|g\|_{\psi,Q}^{p'} . \end{aligned}$$

as we wanted to prove. □

Lemma 1.7. *Let β, γ, p, q, ϕ and ψ as above. If $\overline{\phi}$ belongs to $B_{q'}$ and $\overline{\psi}$ belongs to B_p then*

$$A_{p,q,\phi,\psi}^{\beta,\gamma} \subset A_{p,q}^{\beta,\gamma} .$$

Proof. The proof is analogous to the previous lemma. □

Now, we introduce our main results.

Theorem 1.8. *Let $0 \leq \gamma < 1$, $1 < p \leq q < \infty$ and u, v be two weights such that $(u, v) \in A_{p,q,\psi}^{\tau,\gamma}$, where $0 < \tau < 1$ and ψ is a Young function such that $\overline{\psi} \in B_p$. Then for each $\beta < \tau$ holds*

$$M_\beta^\gamma : L^p(\Omega, v) \to L^q(\Omega, u).$$

Theorem 1.9. *Let $0 < \gamma < 1$, $1 < p \leq q < \infty$ and u, v be two weights such that $(u, v) \in A_{p,q,\phi,\psi}^{\tau,\gamma}$, where $0 < \tau < 1$ and ϕ and ψ are two Young function such that $\overline{\phi}$ and $\overline{\psi}$ belong to $B_{q'}$ and B_p respectively. Then for each $\beta < \tau$ holds*

$$I_\beta^\gamma : L^p(\Omega, v) \to L^q(\Omega, u).$$

The paper is organized as follows. In section 1 we give the tools in the local context and introduce the local dyadic grids. In section 2 and 3 we prove the principal theorems related to the boundedness of the local fractional maximal operator and the local fractional integral operator respectively. In the section we give some applications related to Sobolev embedding and the principal application is the boundedness of the Riesz potential. In the last section we expose some examples that justify the hypotheses used in the results. Throughout this paper, unless otherwise indicated, we will use C to denote constants, which are not necessarily the same at each occurrence,

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

2. Local geometry context

We will work in this paper with the notion of "cloud" of a given subset E in Ω. More precisely, for each $\beta \in (0,1)$ we consider the set

$$\mathcal{N}_\beta(E) = \bigcup_{\substack{R\cap E\neq\emptyset \\ R\in\mathcal{F}_\beta}} R\,, \tag{2.1}$$

as the cloud of E. For balls in a general metric spaces, this notion was introduced in [5]. The following two lemmas can be found in [**?**] in the euclidean version.

Lemma 2.2. *Let Ω be an open proper subset of $\mathbb{R}^n$. Given $0<\beta<1$, for each $t\in\mathbb{N}$ such that $2^{-t}\leq \beta/5$, there exists a covering $\mathcal{W}_t$ of Ω by disjoint dyadic cubes belonging to $\mathcal{F}_\beta$ and satisfying the following properties*

i) If $R=R(x_R,l_R)\in\mathcal{W}_t$, then $10\,R\in\mathcal{F}_\beta$ and

$$2^{-t-3}\,d(x_R,\Omega^c) \;\leq\; l_R \;\leq\; 2^{-t-1}\,d(x_R,\Omega^c)\,.$$

ii) There is a number M, only depending on β and t, such that for any cube $Q_0=Q(x_0,l_0)\in\mathcal{F}_\beta$ with $10Q_0\notin\mathcal{F}_\beta$, the cardinal of the set

$$\mathcal{W}_t(Q_0)=\{R\in\mathcal{W}_t\,:\, R\cap\mathcal{N}_\beta(Q_0)\neq\emptyset\}\,,$$

is at most M. We will call the union of this cubes as

$$\mathcal{W}_{t,Q_0} = \bigcup_{R\in\mathcal{W}_t(Q_0)} R\,.$$

The following lemma says that a certain element x can not belong to too many clouds of balls of the covering given by Lemma 2.2.

Lemma 2.3. *Let $\{Q_i\}$ be a pairwise disjoint collection of Whitney type cubes. Then their clouds have bounded overlapping. More precisely, there exists a natural number $N_0>0$ such that*

$$\sum_i \chi_{\mathcal{N}_\beta(Q_i)}(x)\leq N_0\,,$$

for every x in Ω.

For a cube $R=Q(x_R,l_R)\in\mathcal{F}_\beta$, we observe that the cloud $\mathcal{N}_\beta(R)$ can be written as

$$S_\beta(R) = \bigcup_{\substack{x\in\Omega \\ Q\cap R\neq\emptyset}} Q(x,\beta d(x,\Omega^c))\,.$$

By the definition we note that this set is the support of local fractional maximal operator acting on locally integrable functions f supported on R. Moreover, for small β values, is well-inside of Ω, in the following sense.

Lemma 2.4. *Let $\eta\in(0,1]$. If $\beta\in\left(0,\frac{\eta}{5}\right)$ then there exists $\tilde\beta\in(\beta,\eta)$ such that*

$$S_\beta(R)\subset\tilde R = Q(x_0,\tilde\beta\,d(x_0,\Omega^c))\,,$$

for every cube $R=Q(x_0,l)\in\mathcal{F}_\beta$.

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

Proof. Let us consider $R = Q(x_0, l) \in \mathcal{F}_\beta$ and $y \in S_\beta(R)$. Then $y \in Q = Q(x, \beta\, d(x, \Omega^c))$, with $R \cap Q \neq \emptyset$. Since for any $z \in R \cap Q$ we have

$$d(x, \Omega^c) \leq d(x_0, \Omega^c) + d(x_0, z) + d(z, x) \leq (1+\beta)\, d(x_0, \Omega^c) + \beta\, d(x, \Omega^c)\,.$$

Then

$$d(x, \Omega^c) \leq \frac{1+\beta}{1-\beta}\, d(x_0, \Omega^c)\,. \tag{2.5}$$

Now

$$\begin{aligned} d(y, x_0) &\leq d(y, z) + d(z, x_0) \\ &< 2\beta d(x, \Omega^c) + l \\ &< 2\beta \left(\frac{1+\beta}{1-\beta}\right) d(x_0, \Omega^c) + \beta d(x_0, \Omega^c) \\ &= \frac{\beta^2 + 3\beta}{1-\beta} d(x_0, \Omega^c) = \tilde{\beta}\, d(x_0, \Omega^c)\,. \end{aligned}$$

So, $y \in Q(x_0, \tilde{\beta}\, d(x_0, \Omega^c))$. Since $0 < \beta < \eta/5$ then $\tilde{\beta} = \frac{\beta^2+3\beta}{1-\beta} < \eta$ as we wanted to prove. □

The following lemma we need later.

Lemma 2.6. *Let $0 \leq \gamma < 1$ and we define $a = 2^{n(1-\gamma)+1}$. For a cube $R = Q(x_R, l_R)$ and a non-negative function $f \in L^1_{loc}$ whit support in R, there exist $k_0 \in \mathbb{Z}$ and $j_0 \in \mathbb{N}_0$ such that*

$$|2R_{j_0}|^{\gamma-1} \int_R f \leq a^{k_0} < |R_{j_0}|^{\gamma-1} \int_R f \leq a^{k_0+1}\,,$$

where $R_j = 2^j R = Q(x_R, 2^j l_R)$.

Proof. First, we consider $k_0 \in \mathbb{Z}$ such that

$$a^{k_0} < |R|^{\gamma-1} \int_R f \leq a^{k_0+1}\,.$$

Next, since $|R_j|^{\gamma-1} \int_R f$ tends to zero we can define

$$j_0 = \max\left\{ j \in \mathbb{N} \,:\, a^{k_0} < |R_j|^{\gamma-1} \int_R f \right\}.$$

The first inequality now is trivial and the proof of the lemma is complete. □

Remark 2.7. From now on, we will denote $\hat{R} = 2^{j_0}\tilde{R}$, where j_0 is as in the above proposition and $\tilde{R}$ is as in the Lemma 2.4.

In the next definition we introduce a local version of the dyadic grid considered in [2] (pag 31).

Definition 2.8. *Given a cube $Q \in \Omega$. We say that a collection of subcubes $\mathcal{D}$ is a dyadic grid of Q, denoted by $\mathcal{D}(Q)$, if*

i) If $P \in \mathcal{D}$ then $l_P = 2^{-k} l_Q$, for some $k \in \mathbb{N}_0$.
ii) If $P, R \in \mathcal{D}$ then $P \cap R \in \{\emptyset, P, R\}$.

IMAL PREPRINT # 2024-0075
ISSN 2451-7100
Publication date: December 27, 2024

iii) For each k we have $\left|Q \setminus \bigcup_{P\in\mathcal{D}_k} P\right| = 0.$

In the last item it means that the collection $\mathcal{D}_k = \{P \in \mathcal{D} : l_P = 2^{-k} l_Q\}$ forms a partition of Q, for every $k \in \mathbb{N}_0$, except perhaps for a set of measure zero.

In addition, we will need that the cubes of the grids belong to some family of well-inside cubes. The following lemma prove it.

Lemma 2.9. *Let $\sigma \in (0, 1]$ and $Q \in \mathcal{F}_\beta$, with $\beta \in \left(0, \frac{\sigma}{1+\sigma}\right)$. Then there exists $\theta \in (\beta, \sigma)$ such that $P \in \mathcal{F}_\theta$, for every cube $P \subset Q$.*

Proof. Let us consider $Q = Q(x, l_Q) \in \mathcal{F}_\beta$ and $P = P(y, l_P) \subset Q$. First, we note that

$$d(x, \Omega^c) \leq d(y, \Omega^c) + d(x, y) \leq d(y, \Omega^c) + l_Q < d(y, \Omega^c) + \beta d(x, \Omega^c)\,.$$

So

$$d(x, \Omega^c) < \frac{1}{1-\beta}\, d(y, \Omega^c)\,,$$

and

$$l_P \leq l_Q < \beta\, d(x, \Omega^c) < \frac{\beta}{1-\beta}\, d(y, \Omega^c)\,.$$

By defining $\theta = \dfrac{\beta}{1-\beta}$ we obtain that $\theta < \sigma$. With this, $P \in \mathcal{F}_\theta$ and the proof of the lemma is complete. □

As a consequence, not only the grid of Q but the grid of any subcube contained in Q all belongs to the same family.

Corollary 2.10. *Let $\sigma \in (0, 1]$. If $Q \in \mathcal{F}_\beta$ with $\beta \in \left(0, \frac{\sigma}{1+\sigma}\right)$, then there exists $\theta \in (\beta, \sigma)$ such that $\mathcal{D}(P) \subset \mathcal{F}_\theta$, for every $P \subset Q$.*

Now we present a necessary tool with which we will work later. Given a cube $Q = Q(x_Q, l_Q)$ we says that $Q_t = Q(x_t, l_Q)$, with $x_t = x_Q + 2\,t\,l_Q$, $t \in \{-1/3; 0; 1/3\}^n$ is the 1/3-translation of Q. The following observation exposes the relationship between a cube and the grids built on each translation of it.

Remark 2.11. Every 1/3-translation R_t of the cube $R = Q(x_0, \frac{3}{5} l_0)$ is contained in $Q(x_0, l_0)$. In fact, for each $y \in R_t$ we have

$$d(y, x_0) \leq d(y, x_t) + d(x_t, x_0) \leq \frac{3}{5} l_0 + 2\,\frac{1}{3}\,\frac{3}{5} l_0 = l_0\,.$$

3. Sparse approach for the Maximal operator

In order to construct an adequate sparse operator for our goal, we need the following result.

Theorem 3.1. *Let $\beta < 3/10$ and $R = Q(x_R, l_R) \in \mathcal{F}_\beta$, then every cube in the 1/3-translation dyadic grid $\mathcal{D}^k(R)$, whit $k = 1, \ldots, 3^n$, belongs to $\mathcal{F}_{5\beta/(3-5\beta)}$. Moreover, given any cube $Q_0 \subset R$, there exists k and $P \in \mathcal{D}^k(R)$ such that $Q_0 \subset P$ and $l_P \leq 3\, l_{Q_0}$.*

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

Proof. Taking into account the Remark 2.11 and since $R \in \mathcal{F}_\beta$, if we take $l_0 = \frac{5}{3} l_R$ we have that $l_0 < \frac{5}{3}\beta\rho(x_R)$ with witch $Q(x_R, l_0) \in \mathcal{F}_{\frac{5}{3}\beta}$, where $\frac{5}{3}\beta < \frac{1}{2}$. Then $\mathcal{D}^k(R)$ belongs to $\mathcal{F}_{5\beta/(3-5\beta)}$ by applying the Corollary 2.10. Finally, the rest of the proof is analogous to the proof of Theorem 3.1 in [2]. □

Now, we will consider a fractional maximal operator associated with a dyadic grid as follows.

Definition 3.2. *Let $0 \le \gamma < 1$ and $f \in L^1_{loc}(\Omega)$ and a cube $R \subset \Omega$. For the dyadic grid $\mathcal{D} = \mathcal{D}(R)$ we define the **dyadic fractional maximal operator** in every $x \in \Omega$ as*

$$M^{\gamma,\mathcal{D}} f(x) = \sup_{Q \in \mathcal{D}} \left(|Q|^{\gamma-1} \int_Q |f| \right) \chi_Q(x) .$$

Proposition 3.3. *Let $0 \le \gamma < 1$. There exists $\beta_0 \in (0,1)$ such that for every $0 < \beta < \beta_0$ and $f \in L^1_{loc}(\Omega)$ such that* $\operatorname{supp}(f) \subset R \in \mathcal{F}_\beta$ *there exists a positive constant $C = C(n,\gamma)$ such that for each $x \in S_\beta(R)$*

$$M_\beta^\gamma f(x) \le C \sum_{1 \le k \le 3^n} M^{\gamma,\mathcal{D}^k} f(x) ,$$

where $\mathcal{D}^k = \mathcal{D}^k(\hat{R})$ is the 1/3-translation of the grid $\mathcal{D}(\hat{R})$ and $\hat{R}$ is as in Remark 2.7.

Proof. Since $g(\beta) = 2^{j_0} \frac{\beta^2+3\beta}{1-\beta}$ is a increase and continuous function with $g(0) = 0$, we define β_0 such that $g(\beta_0) = 3/10$. Then, if $0 < \beta < \beta_0$, for each $x \in S_\beta(R)$ then $x \in Q \subset \hat{R}$ by Lemma 2.4. So, by Theorem 3.1 there exists k and $P \in \mathcal{D}^k = \mathcal{D}^k(\hat{R})$ such that $Q \subset P$ and $|P| \le 3^n |Q|$. So

$$|Q|^{\gamma-1} \int_Q |f| \le 3^{n(1-\gamma)} |P|^{\gamma-1} \int_P |f| \le C\, M^{\gamma,\mathcal{D}^k} f(x) \le C \sum_k M^{\gamma,\mathcal{D}^k} f(x) .$$

Then, we have

$$M_\beta^\gamma f(x) \le C \sum_k M^{\gamma,\mathcal{D}^k} f(x) ,$$

as we wanted to prove. □

Finally, in order to prove our first result, we will consider the following notion of sparse.

Definition 3.4. *Given a cube $Q_0 \in \Omega$ and a dyadic grid $\mathcal{D}(Q_0)$, we say that a subset $\mathcal{S} = \mathcal{S}(Q_0) \subset \mathcal{D}(Q_0)$ is a sparse set if*

$$\left| \bigcup_{\substack{P \in \mathcal{S} \\ P \subsetneq Q}} P \right| \le \frac{1}{2} |Q| ,$$

for every $Q \in \mathcal{S}$. Equivalently, if we define $E(Q) = Q \setminus (\cup P)$, where the union is as above, then the sets $E(Q)$ are disjoint and $|E(Q)| \ge \frac{1}{2}|Q|$.

IMAL PREPRINT # 2024-0075
ISSN 2451-7100
Publication date: December 27, 2024

Definition 3.5. *Let $0 \leq \gamma < 1$ and $f \in L^1_{loc}(\Omega)$. For a cube $R \subset \Omega$ and a sparse $\mathcal{S} \subset \mathcal{D}(R)$ we define a Sparse Maximal Fractional operator for any $x \in \Omega$ as*

$$L^{\gamma,\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \left(|Q|^{\gamma-1} \int_Q |f| \right) \chi_{E(Q)}(x) \,.$$

Proposition 3.6. *Let $0 \leq \gamma < 1$ and f be a bounded locally integrable function such that $\operatorname{supp}(f)$ is contained on a cube $R \subset \Omega$. We consider $\mathcal{D} = \mathcal{D}(\hat{R})$ the dyadic grid of $\hat{R}$, then there exists a sparse set $\mathcal{S} = \mathcal{S}(f) \subset \mathcal{D}(\hat{R})$ and a positive constant $C = C(n,\gamma)$ such that for all $x \in \hat{R}$ we have*

$$M^{\gamma,\mathcal{D}} f(x) \leq C\, L^{\gamma,\mathcal{S}} f(x) \,. \tag{3.7}$$

Proof. For $a = 2^{n(1-\gamma)+1}$ and $k \in \mathbb{Z}$ we define

$$\Omega_k = \left\{ x \in \hat{R} : M^{\gamma,\mathcal{D}} f(x) > a^k \right\} .$$

We observe that there exist k_0 and k_1 such that the last sets are not trivial. In fact, since the Maximal is bounded, we take k_1 the smaller k such that $M^{\mathcal{D}}_{\gamma} f(x) \leq a^{k_1}$, for all $x \in \hat{R}$. With this we get $\Omega_k = \emptyset$, for every $k \geq k_1$.

On the other hand, by the Lemma 2.6 there exists $k_0 \in \mathbb{Z}$ such that

$$a^{k_0} < |\hat{R}|^{\gamma-1} \int_{\hat{R}} f \leq a^{k_0+1}$$

Moreover, for every $k \leq k_0$ we have that

$$a^k < |\hat{R}|^{\gamma-1} \int_{\hat{R}} f \,,$$

so, $\Omega_k = \hat{R}$ for $k \leq k_0$. Now, we define

$$\mathcal{S}_k = \left\{ Q \in \mathcal{D} :\ Q \text{ maximal } \wedge |Q|^{\gamma-1} \int_Q f > a^k \right\} .$$

and

$$\mathcal{S} = \bigcup_{k_0 \leq k < k_1} \mathcal{S}_k \,.$$

It is not difficult to see that $\Omega_k = \bigcup_{P \in \mathcal{S}_k} P$, for each k. Moreover, if $P \in \mathcal{S}_k$ and $P \neq \hat{R}$ then taking the father $\hat{P}$ of P we deduce

$$a^k < |P|^{\gamma-1} \int_P f \leq 2^{n(1-\gamma)} |\hat{P}|^{\gamma-1} \int_{\hat{P}} f \leq 2^{n(1-\gamma)} a^k \,. \tag{3.8}$$

On the other hand if $P = \hat{R}$ then (3.8) still holds by Lemma 2.6. We write two claims that we will prove later.

Claim 1: The cubes in $\mathcal{S}$ are nested. That is, if $P' \in \mathcal{S}_{k+1}$ and P' has a father in $\mathcal{D}$ (i.e. $P' \neq \hat{R}$) then there exists $P \in \mathcal{S}_k$ such that $P' \subsetneq P$.

Claim 2: For a fixed k and $P \in \mathcal{S}_k$ we have

$$\bigcup_{\substack{P' \subsetneq P \\ P' \in \mathcal{S}}} P' = \bigcup_{\substack{P' \subsetneq P \\ P' \in \mathcal{S}_{k+1}}} P' \,.$$

IMAL PREPRINT # 2024-0075
ISSN 2451-7100
Publication date: December 27, 2024

Now, we will see that $\mathcal{S}$ is sparse. In fact, for $k_0 \leq k < k_1$ we consider $P \in \mathcal{S}_k$, by using the **Claim 2** and (3.8) we deduce

$$\left|\bigcup_{\substack{P' \subsetneq P \\ P' \in \mathcal{S}}} P'\right| = \left|\bigcup_{\substack{P' \subsetneq P \\ P' \in \mathcal{S}_{k+1}}} P'\right| = \sum |P'| < \frac{1}{a^{k+1}} \sum |P'|^{\gamma} \int_{P'} f$$

$$\leq \frac{1}{a^{k+1}} |P|^{\gamma} \int_P f \leq \frac{2^{n(1-\gamma)}}{a} |P| = \frac{1}{2} |P| ,$$

so, the set satisfy the definition as we wanted to show.

Finally, we prove (3.7). The proof is analogous to prove of Proposition 3.5 in [2]. But we write the idea for the sake of completness. First, we observe that

$$\Omega_k \setminus \Omega_{k+1} = \bigcup_{P \in \mathcal{S}_k} E(P) = \bigcup_{P \in \mathcal{S}_k} \Bigg(P \setminus \bigcup_{\substack{P' \subsetneq P \\ P' \in \mathcal{S}_{k+1}}} P' \Bigg).$$

If $M^{\mathcal{D}}_{\gamma} f(x) = 0$ it is nothing to prove. On the other hand, we take $x \in \Omega_k \setminus \Omega_{k+1}$, that is $x \in E(P)$, for some $P \in \mathcal{S}_k$. Thus

$$M^{\gamma,\mathcal{D}} f(x) \leq a^{k+1} \leq a \left(|P|^{\gamma-1} \int_P f \right) \chi_{E(Q)}(x) = C \sum_{P \in \mathcal{S}} \left(|P|^{\gamma-1} \int_P f \right) \chi_{E(Q)}(x) ,$$

then

$$M^{\gamma,\mathcal{D}} f(x) \leq C\, L^{\gamma,\mathcal{S}} f(x) ,$$

as we wanted to prove.

It only remain to prove the claims made above. For the first, by maximality suppose that $P' \in \mathcal{S}_{k+1}$ (we note $P' = \tilde{R}$ implies $P' \in \mathcal{S}_{k_0}$, then there is nothing to prove) it must necessary happen that if $P' \subsetneq P$, for some $P \in \mathcal{D}$ then $P \in \mathcal{S}_k$. In fact, if this does not happen we get

$$|P|^{\gamma-1} \int_P f \leq a^k \quad \Rightarrow \quad |P'|^{\gamma-1} \int_{P'} f \leq 2^{n(1-\gamma)} a^k < a^{k+1} ,$$

which is a contradiction.

For the second claim, it is only needs to show for any $P \in \mathcal{S}_k$

$$\bigcup_{\substack{P' \subsetneq P \\ P' \in \mathcal{S}}} P' \subset \bigcup_{\substack{P' \subsetneq P \\ P' \in \mathcal{S}_{k+1}}} P' .$$

The other inclusion is trivial. Then, we consider $k_0 < k < k_1 - 1$ and let $x \in P' \subsetneq P$, for some $P' \in \mathcal{S}_{\tilde{k}}$. By the maximality it is clear that $\tilde{k} > k$. Now, by using the claim 1 there exists a finite sequence of cubes belonging to each $\mathcal{S}_k$ from $\tilde{k}$ to $k+1$ such that $P' \subsetneq \tilde{P}$ and $\tilde{P} \in \mathcal{S}_{k+1}$. Thus, since $P \cap \tilde{P} \neq \emptyset$ must be $\tilde{P} \subset P$ and the inclusion is proved. The case $k = k_0$ is analogous. For $k = k_1 - 1$, is an exercise to show that a cube $P' \in \mathcal{S}$ such that $P' \subsetneq P$ cannot exist. Then

$$\bigcup_{\substack{P' \subsetneq P \\ P' \in \mathcal{S}}} P' = \bigcup_{\substack{P' \subsetneq P \\ P' \in \mathcal{S}_{k+1}}} P' = \emptyset .$$

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

□

Remark 3.9. Let us observe that the previous proposition is actually valid for all translations de $D(\hat{R})$.

Proposition 3.10. *Let $0 \leq \gamma < 1$ and $0 < \tau < 1$. For $1 < p \leq q < \infty$ let us consider u and v weights belonging to $A^{\tau,\gamma}_{p,q,\psi}$, where ψ is a Young function such that $\overline{\psi} \in B_p$. There exists $\beta_1 < \tau$ such that for each locally integrable function f, with $\operatorname{supp}(f) \subset R \in \mathcal{F}_\beta$, with $\beta < \beta_1$, then*

$$\left\| M^\gamma_\beta f \right\|_{L^q(\Omega,u)} \leq C \left\| f \right\|_{L^p(\Omega,v)} .$$

Proof. For a given τ we consider $\beta_1 = \min\{\beta_0, \frac{3}{5}\frac{\tau}{1+2\tau}\}$, where β_0 is as in Proposition 3.3. Since $\beta_0 < 1/4$ and $\beta < \frac{3}{5}\frac{\tau}{1+2\tau}$ implies $\frac{5\beta}{3-5\beta} < \frac{\tau}{\tau+1}$, for the Theorem 3.1 and the Corollary 2.10 we obtain that the grids on each of the translations of the dilation $\hat{R}$ belongs to $\mathcal{F}_\tau$.

Now, in order to prove the result, Propositions 3.3 and 3.6 says that it is sufficient to prove the result for $L^{\gamma,\mathcal{S}}$, where $\mathcal{S}$ is a sparse in any of 1/3-translations of $\mathcal{D}(\hat{R}) \subset \mathcal{F}_\tau$.

Since the sets $E(Q)$ are disjoint we have that

$$\left(L^{\gamma,\mathcal{S}} f(x)\right)^q = \sum_{Q\in\mathcal{S}} \left(|Q|^{\gamma-1} \int_Q |f|\right)^q \chi_{E(Q)}(x) .$$

So, by using the generalized Hölder inequality, the hypothesis on the weights we estimates as follows

$$\begin{aligned}
\left\|L^{\gamma,\mathcal{S}}(f)\right\|^q_{L^q(\Omega,u)} &= \int_\Omega \left(L^{\gamma,\mathcal{S}} f(x)\right)^q u \\
&= \sum_{Q\in\mathcal{S}} \left(|Q|^{\gamma-1} \int_Q |f|\right)^q u(E(Q)) \\
&\leq C \sum_{Q\in\mathcal{S}} |Q|^{q\gamma+1-\frac{q}{p}} \left\|f v^{\frac{1}{p}}\right\|^q_{\overline{\psi},Q} \left\|v^{-\frac{1}{p}}\right\|^q_{\psi,Q} \left(|Q|^{-1}\int_Q u\right) |Q|^{\frac{q}{p}} \\
&= C \sum_{Q\in\mathcal{S}} \left(|Q|^{\gamma+\frac{1}{q}-\frac{1}{p}} \left\|u^{\frac{1}{q}}\right\|_{q,Q} \left\|v^{-\frac{1}{p}}\right\|_{\psi,Q}\right)^q \left\|f v^{\frac{1}{p}}\right\|^q_{\overline{\psi},Q} |Q|^{\frac{q}{p}} \\
&\leq C \sum_{Q\in\mathcal{S}} \left\|f v^{\frac{1}{p}}\right\|^q_{\overline{\psi},Q} |E(Q)|^{\frac{q}{p}} ,
\end{aligned}$$

where in the last inequality we use the fact that $|Q| \leq 2|E(Q)|$. Now, since $p \leq q$ and the boundedness of the Maximal Orlicz we conclude

$$\begin{aligned}
\left\|L^{\gamma,\mathcal{S}}(f)\right\|^q_{L^q(\Omega,u)} &\leq C \left(\sum_{Q\in\mathcal{S}} \int_{E(Q)} \left\|f v^{\frac{1}{p}}\right\|^p_{\overline{\psi},Q}\right)^{\frac{q}{p}} \\
&\leq C \left(\sum_{Q\in\mathcal{S}} \int_{E(Q)} \left(M_{\overline{\psi}}(f v^{\frac{1}{p}})\right)^p\right)^{\frac{q}{p}}
\end{aligned}$$

IMAL PREPRINT # 2024-0075
ISSN 2451-7100
Publication date: December 27, 2024

$$\begin{aligned}&\leq C\left\|M_{\overline{\psi}}\left(fv^{\frac{1}{p}}\right)\right\|^q_{L^p(\Omega)}\\ &\leq C\left\|fv^{\frac{1}{p}}\right\|^q_{L^p(\Omega)}=C\|f\|^q_{L^p(\Omega,v)}\,.\end{aligned}$$

Thus the proof of proposition is complete. □

Proposition 3.11. *Let $0\leq\gamma<1$, $0<\tau<1$ and $1<p\leq q<\infty$ Let u, v two weights belonging to $A^{\tau,\gamma}_{p,q,\psi}$, where ψ is a Young function such that $\overline{\psi}\in B_p$. Then there exists $\beta_2<\tau$ such that*

$$M^\gamma_\beta : L^p(\Omega,v)\to L^q(\Omega,u)\,,$$

for all $0<\beta<\beta_2$.

Proof. Let us consider $f\in L^p(\Omega,v)$ and $\{R_j\}$ be a decomposition for Ω as in Lemma 2.2. We consider $\hat{R}_j$ as in the Remark 2.7. Then, if we take $\beta_2=\beta_1/5$, where β_1 is provided by the Proposition 3.10, for each $\beta\leq\beta_2$ we estimate as follows

$$\begin{aligned}\int_\Omega\left(M^\gamma_\beta f\right)^q u&=\sum_j\int_{R_j}\left(M^\gamma_\beta f\right)^q u\\ &=\sum_j\int_{R_j}\left(M^\gamma_\beta\left(f\chi_{S_\beta(R_j)}\right)\right)^q u\\ &\leq C\sum_j\left(\int_\Omega\left(f\chi_{S_\beta(R_j)}\right)^q v\right)^{q/p}\\ &\leq C\left(\int_\Omega\Big(\sum_j\chi_{S_\beta(R_j)}\Big)f^q v\right)^{q/p}\\ &\leq C\left(\int_\Omega f^q v\right)^{q/p},\end{aligned}$$

where in the last inequality Lemma 2.3 was applied. □

Let us introduce the following maximal operator. For $0\leq\gamma<1$, $0<\theta<\beta<1$, and $f\in L^1_{\text{loc}}(\Omega)$ we define for each $x\in\Omega$

$$M^\gamma_{(\theta,\beta]}f(x)=\sup_{Q\in\mathcal{F}_\beta\setminus\mathcal{F}_\theta}\left(|Q|^{\gamma-1}\int_Q|f|\right)\chi_Q(x)\,.$$

For this operator we prove the desired estimate.

Proposition 3.12. *Lets $0\leq\gamma<1$, $0<\theta<\beta<1$ and $1<p\leq q<\infty$. If u,v are two weights such that $(u,v)\in A^{\tau,\gamma}_{p,q}$, for some $\tau\in(\beta,1)$ then*

$$M^\gamma_{(\theta,\beta]} : L^p(\Omega,v)\to L^q(\Omega,u)\,.$$

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

Proof. Let $f \in L^p(\Omega, v)$. For each $x \in \Omega$ we can take a cube Q_x such that $x \in Q_x \in \mathcal{F}_\beta \setminus \mathcal{F}_\theta$ and

$$M^{\gamma}_{(\theta,\beta]} f(x) \leq 2\, |Q_x|^{\gamma - 1} \int_{Q_x} |f| \,.$$

Let us take $t \in \mathbb{N}$ large enough such that $2^{-t} \leq \theta/5$ and the covering $\{Q_j\}$ given by Lemma 2.2. Thus

$$\Omega = \bigcup_j Q_j = \bigcup_j Q_j(x_j, l_j)\,.$$

Now, if $x \in Q_x \bigcap Q_j \neq \emptyset$, for some j, we claim that there exists a bigger cube, namely $\tilde{Q}_x$ such that $Q_j \subset \tilde{Q}_x \in \mathcal{F}_\tau$. In fact, if we denote $Q_x = Q(x_Q, l_Q)$ and $Q_j = Q(x_j, l_j)$ we have in a similar way as in (2.5)

$$d(x_j, \Omega^c) \leq \frac{1+\beta}{1-\beta}\, d(x_Q, \Omega^c)\,.$$

So, for each $z \in Q_j$, taking into account the properties of the covering and the size of the cube Q_x we can deduce that

$$\begin{aligned} d(x_Q, z) &\leq d(x_Q, x) + d(x, x_j) + d(x_j, z) \\ &\leq l_Q + 2 l_j \\ &\leq l_Q + 2^{-t}\, d(x_j, \Omega^c) \\ &\leq l_Q + 2^{-t} \frac{1+\beta}{1-\beta}\, d(x_Q, \Omega^c) \\ &\leq \left(1 + 2^{-t} \frac{1+\beta}{\theta\,(1-\beta)}\right) l_Q = \xi\, l_Q\,. \end{aligned}$$

Note that it is possible to take t large enough at the beginning of the proof such that $\beta\,\xi < \tau$, then the inclusion $Q_j \subset Q(x_Q, \xi\, l_Q)$ and the fact that $Q(x_Q, \xi\, l_Q) \in \mathcal{F}_\tau$ is evident. Now, in order to prove the proposition, we will denote $\tilde{Q}_x = Q(x_Q, \xi\, l_Q)$ and since $\Omega = \cup_j Q_j$ we have that

$$\begin{aligned} &\int_\Omega \left(M^{\gamma}_{(\theta,\beta]} f\right)^q u \\ &= \sum_j \int_{Q_j} \left(M^{\gamma}_{(\theta,\beta]} f\right)^q u \\ &\leq C \sum_j \int_{Q_j} |Q_x|^{q(\gamma-1)} \left(\int_{Q_x} |f|\right)^q u \\ &\leq C \sum_j \int_{Q_j} |Q_x|^{q(\gamma-1)} \left(\int_{Q_x} |f|^p v\right)^{q/p} \left(\int_{Q_x} v^{-\frac{p'}{p}}\right)^{q/p'} u \\ &\leq C \sum_j \left(\int_{S_\beta(Q_j)} |f|^p v\right)^{q/p} \int_{Q_j} |\tilde{Q}_x|^{q(\gamma-1)} \left(\int_{\tilde{Q}_x} v^{-\frac{p'}{p}}\right)^{q/p'} u(\tilde{Q}_x) \frac{u}{u(\tilde{Q}_x)} \\ &= C \sum_j \left(\int_{S_\beta(Q_j)} |f|^p v\right)^{q/p} \int_{Q_j} \left(|\tilde{Q}_x|^{\gamma + \frac{1}{q} - \frac{1}{p}} \left\| u^{\frac{1}{q}} \right\|_{q, \tilde{Q}_x} \left\| v^{-\frac{1}{p}} \right\|_{p', \tilde{Q}_x}\right)^q \frac{u}{u(\tilde{Q}_x)}\,. \end{aligned}$$

IMAL PREPRINT # 2024-0075
ISSN 2451-7100
Publication date: December 27, 2024

Now, since $(u,v)\in A^{\tau;\gamma}_{p,q}$ then the last integral is bounded. Thus, we estimates in a usual way for obtain

$$\int_\Omega \left(M^\gamma_{(\theta,\beta]}f\right)^q u \le C\left(\sum_j \int_{S_\beta(Q_j)} |f|^p v\right)^{q/p} \le C\left(\int_\Omega |f|^p v\right)^{q/p},$$

as we wanted to prove. □

Now, we will proof our first result.

Proof of the Theorem 1.8. For a given τ we consider $0<\beta<\tau<1$ and we let β_2 as in the Proposition 3.11. Now, for $\theta<\beta_2$ we can disarm the operator as follows

$$M^\gamma_\beta f(x) \le M^\gamma_\theta f(x) + M^\gamma_{(\theta,\beta]} f(x).$$

So, by the Lemma 1.6 we can apply the Propositions 3.11 and 3.12 to obtain

$$\left\|M^\gamma_\beta f\right\|_{L^q(\Omega,u)} \le \|M^\gamma_\theta f\|_{L^q(\Omega,u)} + \left\|M^\gamma_{(\theta,\beta]} f\right\|_{L^q(\Omega,u)} \le C\,\|f\|_{L^p(\Omega,v)},$$

as we wanted to prove. □

4. Sparse approach for the Fractional integral operator

In this section we will prove the Theorem 1.9. We start with the following definitions. As in the last section R be a cube in Ω.

Definition 4.1. *Let $0<\gamma<1$ and $f\in L^1_{loc}(\Omega)$ with $\operatorname{supp}(f)\subset R\subset\Omega$. For the dyadic grid $\mathcal{D}=\mathcal{D}(R)$ we define the local fractional integral operator in $x\in\Omega$ as*

$$I^{\gamma,\mathcal{D}} f(x) = \sum_{Q\in\mathcal{D}} \left(|Q|^{\gamma-1}\int_Q |f|\right)\chi_Q(x).$$

Definition 4.2. *Let $0<\gamma<1$, $f\in L^1_{loc}(\Omega)$ such that $\operatorname{supp}(f)\subset R$ and a sparse $\mathcal{S}\subset\mathcal{D}(R)$. Then, for any $x\in R$ we define a sparse fractional integral operator as*

$$I^{\gamma,\mathcal{S}} f(x) = \sum_{Q\in\mathcal{S}} \left(|Q|^{\gamma-1}\int_Q f\right)\chi_Q(x).$$

Proposition 4.3. *Let $0<\gamma<1$. There exists β_0 such that for every $0<\beta<\beta_0$ and $f\in L^1_{loc}(\Omega)$ such that $\operatorname{supp}(f)\subset R\in\mathcal{F}_\beta$ the following estimates holds*

$$I^\gamma_\beta f(x) \le C \sum_{1\le k\le 3^n} I^{\gamma,\mathcal{D}^k} f(x),$$

for every $x\in R$, for some positive constant $C=C(n,\gamma)$, where $\mathcal{D}^k=\mathcal{D}^k(\hat{R})$ is the 1/3-translation of the grid $\mathcal{D}(\hat{R})$ and $\hat{R}$.

IMAL PREPRINT # 2024-0075
ISSN 2451-7100
Publication date: December 27, 2024

Proof. With out lost of generality we consider a non-negative function f. Let us take β_0 as in the Proposition 3.3. We will suppose that $R \in \mathcal{F}_\beta$ where $\beta < \beta_0$. For a fixed $x \in R$ we denote $Q_j^\beta = Q(x, 2^{-j}\beta\, d(x,\Omega^c))$, $j \in \mathbb{N}_0$ and estimate as follow

$$I_\beta^\gamma f(x) = \sum_{j=0}^{\infty} \int_{Q_j^\beta \setminus Q_{j+1}^\beta} \frac{f(y)}{|x-y|^{n(1-\gamma)}}\, dy \le C \sum_j \left(2^{-j}\beta\, d(x,\Omega^c)\right)^{n(\gamma-1)} \int_{Q_j^\beta} f\,.$$

In an analogous way as in the Proposition 3.3, for each j, there are some $k \in \{1, \dots, 3^n\}$ and a cube $Q_k \in \mathcal{D}^k = \mathcal{D}^k(\hat{R})$ such that $Q_j^\beta \subset Q_k$ and

$$2^{-j}\beta\, d(x,\Omega^c) = l(Q_j^\beta) \le l(Q_k) \le 3\, l(Q_j^\beta) < 2^{-j+2}\beta\, d(x,\Omega^c)\,. \tag{4.4}$$

Now, taking into account that $l(Q_k) = 2^{j_0 - i}\tilde{\beta}\, d(x_R, \Omega^c)$ for some i (see Lemma 2.4), there are a finite number of possibilities for $l(Q_k)$ regardless of β and x. In fact, we recall that for each $x \in Q(x_R, \tilde{\beta}\rho(x_R))$ we get that

$$(1-\beta) d(x_R, \Omega^c) \le d(x,\Omega^c) \le (1+\beta) d(x_R,\Omega^c)$$

Return to (4.4) and by dividing the expression by $\tilde{\beta}\, d(x_R,\Omega^c)$ we obtain that

$$2^{-j-4} < 2^{-j}\frac{(1-\beta)^2}{\beta+3} \le 2^{j_0-i} \le 2^{-j+2}\frac{1-\beta^2}{\beta+3} < 2^{-j}\,,$$

where we use that $\beta < \beta_0 < 1/2$ implies $\frac{1}{16} < \frac{(1-\beta)^2}{\beta+3}$ and $\frac{1-\beta^2}{\beta+3} < \frac{1}{4}$.

Then

$$\begin{aligned} I_\beta^\gamma f(x) &\le C \sum_j \left(2^{-j}\beta\, d(x,\Omega^c)\right)^{n(\gamma-1)} \int_{Q_j^\beta} f \\ &\le C \sum_j \sum_k \sum_{\substack{Q \in \mathcal{D}^k \\ 2^{-j-4} \le 2^{j_0-i} \le 2^{-j}}} \left(|Q|^{\gamma-1}\int_Q f\right)\chi_Q(x) \\ &\le C \sum_k \sum_{Q\in\mathcal{D}^k} \left(|Q|^{\gamma-1}\int_Q f\right)\chi_Q(x) \\ &= C\sum_k I^{\gamma,\mathcal{D}^k} f(x)\,. \end{aligned}$$

Then the proof of proposition is complete. □

Proposition 4.5. *Let $0 < \gamma < 1$ and f be a bounded locally integrable function in Ω such that $\operatorname{supp}(f) \subset R \in \mathcal{F}_\beta$ the following estimates holds.*

$$I^{\gamma,\mathcal{D}} f(x) \le C\, I^{\gamma,\mathcal{S}} f(x)\,.$$

for every $x \in S_\beta(R)$, for some positive constant $C = C(n,\gamma)$, where $\mathcal{D}^k = \mathcal{D}^k(\hat{R})$ is the 1/3-translation of the grid $\mathcal{D}(\hat{R})$ and $\hat{R}$.

Proof. The argument is similar to that used in Proposition 3.6 and can be considered as an adaptation to the local context of Proposition 3.6 in [2]. Let $a = 2^{n+1}$ and for each $k \in \mathbb{Z}$ we consider

$$\mathcal{Q}_k = \left\{ Q \in \mathcal{D} : a^k < \frac{1}{|Q|}\int_Q f \le a^{k+1} \right\}.$$

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

As above, in this context there exists k_0 and k_1 such that $\mathcal{S}_k = \hat{R}$, for $k \le k_0$ and $\mathcal{S}_k = \emptyset$, for $k \ge k_1$. Thus we consider again a subset of $\mathcal{D}$ as a sparse set given by

$$\mathcal{S} = \bigcup_{k_0 \le k < k_1} \mathcal{S}_k\,,$$

where

$$\mathcal{S}_k = \left\{ Q \in \mathcal{D} :\ Q \text{ maximal } \wedge \frac{1}{|Q|}\int_Q f > a^k \right\}.$$

So, for $x \in R$ we can estimate in an analogous way as in [2]. More precisely

$$\begin{aligned} I^{\gamma,\mathcal{D}} f(x) &= \sum_{k<k_1} \sum_{Q \in \mathcal{Q}_k} \left(|Q|^{\gamma-1} \int_Q f \right) \chi_Q(x) \\ &\le \sum_{k<k_1} a^{k+1} \sum_{P \in \mathcal{S}_k} \sum_{\substack{Q \in \mathcal{Q}_k \\ Q \subseteq P}} |Q|^\gamma \chi_Q(x) \end{aligned}$$

Now, for a fixed $k < k_1$ and $P \in \mathcal{S}_k$ we have that the last inner sum

$$\sum_{\substack{Q \in \mathcal{Q}_k \\ Q \subseteq P}} |Q|^\gamma \chi_Q(x) = \sum_{r=0}^{\infty} \sum_{\substack{Q \in \mathcal{Q}_k \\ Q \subseteq P \\ l(Q) = 2^{-r} l(P)}} |Q|^\gamma \chi_Q(x) = \frac{1}{1 - 2^{n\gamma}} |P|^\gamma \chi_P(x)\,.$$

Then, recall that $\mathcal{S}_k = \hat{R}$, for every $k \le k_0$ we get

$$\begin{aligned} I^{\gamma,\mathcal{D}} f(x) &\le C\, a \sum_{k<k_1} \sum_{P \in \mathcal{S}_K} |P|^\gamma a^k \chi_P(x) \\ &= C \left(|\hat{R}|^\gamma \chi_{\hat{R}}(x) \sum_{k \le k_0} a^k + \sum_{k_0<k<k_1} \sum_{P \in \mathcal{S}_k} |P|^\gamma a^k \chi_P(x) \right) \\ &= C \left(|\hat{R}|^\gamma a^{k_0} \chi_{\hat{R}}(x) + \sum_{k_0<k<k_1} \sum_{P \in \mathcal{S}_k} |P|^\gamma a^k \chi_P(x) \right) \\ &= C \sum_{P \in \mathcal{S}} \left(|P|^{\gamma-1} \int_P f \right) \chi_P(x) \\ &= C\, I^{\gamma,\mathcal{S}} f(x)\,, \end{aligned}$$

as we wanted to prove. □

Remark 4.6. As in Remark 3.9 we note that the previous proposition is actually valid for all translations de $D(\hat{R})$.

Proposition 4.7. *Let $0 < \gamma < 1$ and $0 < \tau < 1$. For $1 < p \le q < \infty$ let us consider u and v weights belonging to $A^{\tau,\gamma}_{p,q,\phi,\psi}$, where ϕ, ψ are Young function such that $\overline{\phi} \in B_{q'}$ and $\overline{\psi} \in B_p$. Then there exists $\beta_1 < \tau$ such that for every $\beta < \beta_1$ and $f \in L^1_{loc}$, with $\operatorname{supp}(f) \subset R \in \mathcal{F}_\beta$ we have that*

$$\left\| I^\gamma_\beta f \right\|_{L^q(R,u)} \le C\, \|f\|_{L^p(\Omega,v)}\,.$$

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

Proof. We can assume that f is non-negative and by the monotone convergence theorem we may also assume that f is bounded and has compact support. Thus, if we take β_1 provided in the Proposition 3.10 by applying the Propositions 4.3 and 4.5 it will suffice to prove the result for the sparse operator $I^{\gamma,\mathcal{S}}$, where $\mathcal{S}$ is an appropriate sparse depending on f and its support. For this, we take $h \in L^{q'}(R,u)$ with $\|h\|_{L^{q'}(R,u)} = 1$ then by taking $g = h\chi_R$ for $\beta < \beta_1$ we get that

$$\begin{aligned}
&\int_R I^{\gamma,\mathcal{S}} f(x)g(x)u(x)dx \\
&\quad \le \sum_{Q\in\mathcal{S}} |Q|^{\gamma-1}\Big(\int_Q f\Big)\Big(\int_Q g\,u\Big) \\
&\quad = \sum_{Q\in\mathcal{S}} |Q|^{\gamma+1}\Big(\frac{1}{|Q|}\int_Q f\,v^{\frac{1}{p}}v^{-\frac{1}{p}}\Big)\Big(\frac{1}{|Q|}\int_Q g\,u^{\frac{1}{q'}}u^{\frac{1}{q}}\Big) \\
&\quad \le C\sum_{Q\in\mathcal{S}} |Q|^{\gamma+1}\Big(\left\|fv^{\frac{1}{p}}\right\|_{\overline{\psi},Q}\left\|v^{-\frac{1}{p}}\right\|_{\psi,Q}\Big)\Big(\left\|gu^{\frac{1}{q'}}\right\|_{\overline{\phi},Q}\left\|u^{\frac{1}{q}}\right\|_{\phi,Q}\Big) \\
&\quad = C\sum_{Q\in\mathcal{S}} \Big(|Q|^{\gamma+\frac{1}{q}-\frac{1}{p}}\left\|u^{\frac{1}{q}}\right\|_{\phi,Q}\left\|v^{-\frac{1}{p}}\right\|_{\psi,Q}\Big)\left\|fv^{\frac{1}{p}}\right\|_{\overline{\psi},Q}\left\|gu^{\frac{1}{q'}}\right\|_{\overline{\phi},Q}|Q|^{\frac{1}{p}+\frac{1}{q'}} \\
&\quad \le C\sum_{Q\in\mathcal{S}} \Big(\left\|fv^{\frac{1}{p}}\right\|_{\overline{\psi},Q}|E(Q)|^{\frac{1}{p}}\Big)\Big(\left\|gu^{\frac{1}{q'}}\right\|_{\overline{\phi},Q}|E(Q)|^{\frac{1}{q'}}\Big),
\end{aligned}$$

where in the last inequality we use the conditions on the weights and the definition of sparse set. So, since $p' > q'$ by using a standard argument we can continue the estimation as follow

$$\begin{aligned}
&\int_R I^{\gamma,\mathcal{S}} f(x)g(x)u(x)dx \\
&\quad \le C\left(\sum \left\|fv^{\frac{1}{p}}\right\|^p_{\overline{\psi},Q}|E(Q)|\right)^{\frac{1}{p}}\left(\sum \left\|gu^{\frac{1}{q'}}\right\|^{p'}_{\overline{\phi},Q}|E(Q)|^{\frac{p'}{q'}}\right)^{\frac{1}{p'}} \\
&\quad \le C\left(\sum \left\|fv^{\frac{1}{p}}\right\|^p_{\overline{\psi},Q}|E(Q)|\right)^{\frac{1}{p}}\left(\sum \left\|gu^{\frac{1}{q'}}\right\|^{q'}_{\overline{\phi},Q}|E(Q)|\right)^{\frac{1}{q'}} \\
&\quad = C\left(\sum \int_{E(Q)}\left\|fv^{\frac{1}{p}}\right\|^p_{\overline{\psi},Q}dx\right)^{\frac{1}{p}}\left(\sum \int_{E(Q)}\left\|gu^{\frac{1}{q'}}\right\|^{q'}_{\overline{\phi},Q}dx\right)^{\frac{1}{q'}} \\
&\quad \le C\left(\sum \int_{E(Q)}\left(M_{\overline{\psi}}\left(fv^{\frac{1}{p}}\right)(x)\right)^p dx\right)^{\frac{1}{p}}\left(\sum \int_{E(Q)}\left(M_{\overline{\phi}}\left(gu^{\frac{1}{q'}}\right)(x)\right)^{q'} dx\right)^{\frac{1}{q'}} \\
&\quad \le C\,\|f\|_{L^p(\Omega,v)}\,\|h\|_{L^{q'}(R,u)}\,.
\end{aligned}$$

Finally, taking the supremum over all such functions h we conclude

$$\left\|I^{\gamma}_{\beta} f\right\|_{L^q(R,u)} \le C\,\|f\|_{L^p(\Omega,v)}\,.$$

and the proof is complete. □

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

Proposition 4.8. *Let $0 < \gamma < 1$ and $0 < \tau < 1$. For $1 < p \leq q < \infty$ let us consider u and v weights belonging to $A_{p,q,\phi,\psi}^{\tau,\gamma}$, where ϕ, ψ are Young function such that $\overline{\phi} \in B_{q'}$ and $\overline{\psi} \in B_p$. Then there exists $\beta_2 < \tau$ such that*

$$I_\beta^\gamma : L^p(\Omega, v) \to L^q(\Omega, u)\,,$$

for each $f \in L^1_{loc}$ and every $\beta \leq \beta_2$.

Proof. Let $f \in L^p(\Omega, v)$ and $\{R_j\}$ be a decomposition for Ω as in Lemma 2.2. We consider $\hat{R}_j$ as in the Remark 2.7. Thus, for $\beta_2 = \beta_1/5$ by applying the Proposition 4.7 we get

$$\begin{aligned}
\int_\Omega \left(I_\beta^\gamma f\right)^q u &= \sum_j \int_{R_j} \left(I_\beta^\gamma f\right)^q u \\
&= \sum_j \int_{R_j} \left(I_\beta^\gamma \left(f\chi_{S_\beta(R_j)}\right)\right)^q u \\
&\leq C \sum_j \left(\int_\Omega \left(f\chi_{S_\beta(R_j)}\right)^q v\right)^{q/p} \\
&\leq C \left(\int_\Omega \left(\sum_j \chi_{S_\beta(R_j)}\right) f^q v\right)^{q/p} \\
&\leq C \left(\int_\Omega f^q v\right)^{q/p},
\end{aligned}$$

for every $\beta \leq 1/50$ and the proof is complete. □

Proof of Theorem 1.9. Let $0 < \gamma < 1$. If $0 < \beta \leq 1/50$ by applying the Proposition 4.8 it is nothing to prove here. On the other hand, for $0 < 1/50 < \beta < \tau < 1$, $f \in L^p(\Omega, v)$ and each $x \in \Omega$ we estimates as follows

$$\begin{aligned}
|I_\beta^\gamma f(x)| &= \left|\int_{Q(x,\beta d(x,\Omega^c))} \frac{f(y)}{|x-y|^{n(1-\gamma)}} dy\right| \\
&\leq \left|I_{\beta_1}^\gamma f(x)\right| + \left|\int_{Q(x,\beta d(x,\Omega^c))\setminus Q(x,\beta_0 d(x,\Omega^c))} \frac{f(y)}{|x-y|^{n(1-\gamma)}} dy\right| \\
&\leq \left|I_{\beta_1}^\gamma f(x)\right| + CM_{(\beta_0,\beta]}^\gamma f(x)\,.
\end{aligned}$$

Finally, by the Propositions 4.8 and 3.12 we can obtain that

$$\left\|I_\beta^\gamma f\right\|_{L^q(\Omega,u)} \leq \left\|I_{\beta_1}^\gamma f\right\|_{L^q(\Omega,u)} + C\left\|M_{(\beta_1,\beta]}^\gamma f\right\|_{L^q(\Omega,u)} \leq C\left\|f\right\|_{L^p(\Omega,v)}\,.$$

□

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

5. Applications

Interior Sobolev's type estimates. Let us consider the m-Laplacian operator, denoted as Δ^m, where the notation means composing the Laplacian operator m times in Ω.

For U, a solution of the problem $\Delta^m U = f$, many estimates are already known in the context of weighted Sobolev spaces. In particular, in the context of local weights, in [5], the author considers a version of weighted Sobolev spaces that takes into account the distance to the boundary. That is, if $\rho(x) = d(x, \Omega^c)$, then they define

$$W^{k,\,p}_{\rho,\,\omega}(\Omega) = \left\{ f \in L^1_{loc}(\Omega) \ : \ \|f\|_{W^{k,\,p}_{\rho,\,\omega}(\Omega)} = \sum_{|\alpha|\le k} \left\| \rho^{|\alpha|}\, D^\alpha\, f \right\|_{L^p(\Omega,\omega)} < \infty \right\}.$$

Our first application is a two weights estimates for the solution U as above.

Theorem 5.1. *Let $1 < p \le q < \infty$. For a pair of weights (u, v) that satisfy the hypotheses of Theorem (1.8), and additionally $u \in A^\beta_q$ and U a solution of $\Delta^m U = f$ in Ω, we have:*

$$\|U\|_{W^{2m,\,q}_{\rho,\,u}(\Omega)} \le C \left(\|U\|_{L^p(\Omega,v)} + \left\| \rho^{2m}\, f \right\|_{L^p(\Omega,v)} \right)$$

Proof. Since $u \in A^\beta_q$, we have

$$\|U\|_{W^{2m,q}_{\rho,u}(\Omega)} \le C \left(\|U\|_{L^q(\Omega,u)} + \left\| \rho^{2m} f \right\|_{L^q(\Omega,u)} \right).$$

Then, by the Lebesgue differentiation theorem and the Theorem (1.8),

$$\|U\|_{L^q(\Omega,u)} \le \|M_\beta U\|_{L^q(\Omega,u)} \le C\, \|U\|_{L^p(\Omega,v)}\,.$$

Note that the pairs of weights $(u\rho^{2mq}, v\rho^{2mp})$ also belong to the class $A^\tau_{p,q,\psi}$. In fact, if $Q \in \mathcal{F}_\tau$ is a cube centered at x_0, we have $\rho(x) \approx \rho(x_0)$, for all $x \in Q$, where the constant depends only on τ. Thus,

$$\left\| u^{\frac{1}{q}} \rho^{2m} \right\|_{q,Q} \left\| v^{-\frac{1}{p}} \rho^{-2m} \right\|_{\psi,Q} \le C\rho(x_0)^{2m} \rho(x_0)^{-2m} \left\| u^{\frac{1}{q}} \right\|_{q,Q} \left\| v^{-\frac{1}{p}} \right\|_{\psi,Q}.$$

Then, by applying the Theorem (1.8) again we conclude

$$\left\| \rho^{2m} f \right\|_{L^q(\Omega,u)} = \|f\|_{L^q(\Omega,u\rho^{2mq})} \le \|M_\beta f\|_{L^q(\Omega,u\rho^{2mq})} \le C \left\| \rho^{2m} f \right\|_{L^p(\Omega,v)}.$$

With this the proof of the theorem is complete. □

Theorem 5.2. *Let $0 < \tau < 1$ y $1 < p \le q < \infty$. Let us consider two weights u and v satisfying the hypothesis of Theorem 1.9. Then*

$$\|\rho\, g\|_{L^q(\Omega,u)} \le C\, \|g\|_{W^{1,\,p}_{\rho,\,v}(\Omega)}\,.$$

for every $g \in W^{1,\,p}_{\rho,\,v}(\Omega)$.

Proof. For any $x \in \Omega$, in [6, Theorem 5.3], the authors proved that

$$|g(x)| \le C \left(\rho(x)^{-1}\, I^{1/n}_\beta |g|(x) + I^{1/n}_\beta |\nabla\, g|(x) \right),$$

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

holds for every $g \in C^1(\Omega)$, where $\beta \in (0,1)$. Now, given $\tau \in (0,1)$ it is easy to see that $(u,v) \in A^{\tau,\gamma}_{p,q,\phi,\psi}$ implies $(u\rho^q,\, v\rho^p) \in A^{\tau,\gamma}_{p,q,\phi,\psi}$. Then, for $\beta < \tau$, by applying the Theorem 1.9 we have

$$\begin{aligned}
\|\rho\, g\|_{L^q(\Omega,u)} &\leq C\left(\left\|I_\beta^{1/n}|g|\right\|_{L^q(\Omega,u)} + \left\|\rho\, I_\beta^{1/n}|\nabla g|\right\|_{L^q(\Omega,u)}\right)\\
&= C\left(\left\|I_\beta^{1/n}|g|\right\|_{L^q(\Omega,u)} + \left\|I_\beta^{1/n}|\nabla g|\right\|_{L^q(\Omega,u\rho^q)}\right)\\
&\leq C\left(\|g\|_{L^p(\Omega,v)} + \|\nabla g\|_{L^p(\Omega,v\rho^p)}\right)\\
&= C\,\|g\|_{W^{1,\,p}_{\rho,\,v}(\Omega)}\,.
\end{aligned}$$

Equivalently, we conclude that $W^{1,\,p}_{\rho,\,v}(\Omega) \subset L^q(\Omega, u\rho^q)$ and the proof is done. □

Schödinger type operators. We will analyze the behavior of the Schödinger type Fractional Integral acting on weighted Lebesgue spaces with different weights satisfying a *bump* type condition in both factors. We now provide a description of the context. This operators was considered by Shen in [11] and they are defined as

$$L_\mu := -\Delta + \mu\,,$$

where μ is a non-negative Radon measure on $\mathbb{R}^n$ with the following properties: there exists positive constants δ_μ, C_μ and D_μ such that

$$\mu(B(x,r)) \leq C_\mu\left(\frac{r}{R}\right)^{d-2+\delta_\mu}\mu(B(x,R))\,; \tag{5.3}$$

and

$$\mu(B(x,2r)) \leq D_\mu\left(\mu(B(x,R)) + r^{d-2}\right)\,, \tag{5.4}$$

for all $x \in \mathbb{R}^n$ and $0 < r < R$.

Thus, by using the semigroup theory we can considered the Riesz potential as

$$I_\mu^\alpha := L_\mu^{-\frac{\alpha}{2}}\,,$$

for $0 < \alpha \leq 2$. In [1] the author introduce a class of weights w for which the I_μ^α is bounded from $L^p(w)$ to $L^v(w^{v/p})$ for the case $0 < \alpha \leq 2$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{v} = \frac{1}{p} - \frac{\alpha}{n}$.

For the next result we require the following concept. We denote $d_\mu(x,y)$ the Agmon distance for the measure μ defined through

$$d_\mu(x,y) = d_{\rho_\mu}(x,y) = \inf_\gamma \int_0^1 \rho_\mu(\gamma(t))^{-1}|\gamma'(t)|dt\,,$$

where the infimum is taken over all absolutely continuous $\gamma : [0,1] \to \mathbb{R}^n$ with $\gamma(0) = x$ and $\gamma(1) = y$ and

$$\rho_\mu(x) = \sup\left\{r > 0 \,:\, \frac{\mu(B(x,r))}{r^{d-s}} \leq 1\right\}\,, \qquad x \in \mathbb{R}^n\,.$$

We will denote ρ instead ρ_μ for the next. It have the properties of a know called critical radius function. A cube of the form $Q(x,\rho(x))$ is called critical cube. If $\sigma > 0$ and $x,\, y \in \sigma Q$, where Q is a critical cube, then $\rho(x) \leq C_\sigma \rho(y)$. As a consequence of it we have the fundamental decomposition of the space.

IMAL PREPRINT # 2024-0075
ISSN 2451-7100
Publication date: December 27, 2024

Proposition 5.5 ([4], [3]). *There exists a sequence of points $\{x_j\}_{j\in\mathbb{N}} \in \mathbb{R}^n$ that satisfies the following two properties,*

(1) $\mathbb{R}^n = \cup_{j\in\mathbb{N}} Q(x_j, \rho(x_j))$.
(2) *There exists C, $N_1 > 0$ such that for every $\sigma \geq 1$*

$$\sum_{j\in\mathbb{N}} \chi_{Q(x_j,\rho(x_j))} \leq C\,\sigma^{N_1}\,.$$

Lemma 5.6. *Let Q and $\tilde{Q}$ be two balls such that $\tilde{Q} \subset Q$, then*

$$\|f\|_{\eta,\tilde{Q}} \leq \frac{|Q|}{|\tilde{Q}|}\,\|f\|_{\eta,Q}\,.$$

Proof. Let $\tilde{Q} \subset Q$ and suppose that $\|f\|_{\eta,Q} < \infty$. By the convexity of η we have that

$$\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} \eta\Big(\frac{|\tilde{Q}|}{|Q|}\frac{|f|}{\|f\|_{\eta,Q}}\Big) \leq \frac{1}{|Q|}\int_Q \eta\Big(\frac{|f|}{\|f\|_{\eta,Q}}\Big) = 1\,,$$

and the proof of lemma is complete. □

The following two lemmas are the key for our application. The constant δ is greater than 2 and depend of the critical radius function. Their proofs can be found in [1, Lemma 2.4 and Lemma 2.5].

Lemma 5.7. *Let $\rho : \mathbb{R}^n \to (0,\infty)$ be a critical radius function. Let $r > 0$ and $x \in \mathbb{R}^n$. Suppose that $r \leq 2$. Then*

$$B(x, r\rho(x)) \subset B_\rho(x, \delta r)\,.$$

Suppose instead that $r > 2$. Then

$$B(x, r\rho(x)) \subset B_\rho\Big(x, \delta(1+r)^{k_0+1}\Big)\,.$$

Lemma 5.8. *Let $\rho : \mathbb{R}^n \to (0,\infty)$ be a critical radius function. There exists a constant $A_0 > 1$, dependent on ρ only through B_0 and k_0, such that for all $x \in \mathbb{R}^n$ and $0 < r \leq \delta$,*

$$B_\rho(x, r) \subset B(x, A_0 r\rho(x))\,.$$

Also, for $x\mathbb{R}^n$ and $r > \delta$,

$$B_\rho(x, r) \subset B\Big(x, ((r\delta)^{k_0+1}?1)\rho(x)\Big)\,.$$

Our application is now exposed.

Theorem 5.9. *Let μ be a non-negative Radon measure on $\mathbb{R}^n$ satisfying* (5.3) *and* (5.4). *Let $n \geq 3$, $q > \frac{n}{2}$, $0 < \alpha < 2$ and $1 < p \leq q < \infty$. Let us consider Young functions ψ and ϕ such that $\overline{\psi} \in B_p$ and $\overline{\phi} \in B_{q'}$. Then*

$$I_\mu^\alpha : L^p(\mathbb{R}^n, v) \to L^q(\mathbb{R}^n, u)\,,$$

for every pair of weights (u, v) that satisfy

$$\sup_{B_\rho} |B_\rho|^{\frac{\alpha}{n}-\frac{1}{p}+\frac{1}{q}} \left\|\frac{u^{1/q}}{\mathrm{e}^{cr/q}}\right\|_{\phi,B_\rho} \left\|\frac{v^{-1/p}}{\mathrm{e}^{cr/p'}}\right\|_{\psi,B_\rho} < \infty\,, \tag{5.10}$$

for some $c > 0$, where B_ρ is the ball with center x and radius $r > 0$ with the metric d_ρ.

IMAL PREPRINT # 2024-0075
ISSN 2451-7100
Publication date: December 27, 2024

Proof. The fractional integral operators are well-known to have the following representation (see [7, p. 286])

$$I_\mu^\alpha|f|(x) = \int_{\mathbb{R}^n} \mathcal{K}_\mu^\alpha(x,y)\,|f(y)|\,dy\,.$$

For the proof we consider the known decomposition in the local and global components of the operator. That is, for each $x\in\mathbb{R}^n$

$$I_\mu^\alpha|f|(x) = I_\mu^\alpha(|f|\chi_{Q(x,\rho(x))})(x) + I_\mu^\alpha(|f|\chi_{Q(x,\rho(x))^c})(x)\,.$$

So, it will to be sufficient to prove the result for each of one. For the local part we use an estimation proved in Lemma 4.5 in [1], then

$$\begin{aligned}\left|I_\mu^\alpha(|f|\chi_{Q(x,\rho(x))})(x)\right| &\le C\int_{Q(x,\rho(x))}|\mathcal{K}_\mu^\alpha(x,y)|\,|f(y)|\,dy\\ &\le C\int_{Q(x,\rho(x))}\frac{|f(y)|}{|x-y|^{n-\alpha}}dy\,. \qquad (5.11)\end{aligned}$$

Let us consider $\{Q_j\}$ the critical cubes provided by Proposition 5.5 and for the properties of ρ we can take $\sigma\ge 1$ such that $\cup_{x\in Q_j}Q(x,\rho(x))\subset\sigma Q_j=\tilde{Q}_j$. For each j, we define $\Omega_j=\theta\tilde{Q}_j$, with $\theta>1$ to be determinate later. In this context, we can to prove that the critical cubes belongs to a certain local family of well-inside cubes in Ω_j. More precisely, we have that $Q_x\in\mathcal{F}_{1/(\theta-1)}$, for every $x\in Q_j$. In fact since $\rho(x)\le C_\sigma\,\rho(x_j)$, where $C_\sigma\ge 1$ by denoting x_j the center of Q_j we can estimate

$$d(x,\Omega_j^c)\ge d(x_j,\Omega_j^c)-d(x,x_j)>(\theta C_\sigma-1)\rho(x_j)>C_\sigma\,(\theta-1)\rho(x_j)\ge(\theta-1)\rho(x)\,,$$

that is $Q(x,\rho(x))\subset Q(x,\frac{1}{(\theta-1)}d(x,\Omega_j^c))$ and following the estimate (5.11) we get for $x\in Q_j$

$$\left|I_\mu^\alpha(|f|\chi_{Q(x,\rho(x))})(x)\right|\le C\int_{Q(x,\frac{1}{(\theta-1)}d(x,\Omega_j^c))}\frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy = C\,I^{\alpha/n}_{\frac{1}{(\theta-1)},j}|f|(x)\,,$$

where $I^{\alpha/n}_{\frac{1}{(\theta-1)},j}$ is the local fractional integral operator defined in Ω_j (see Definition 1.2).

Now, in order to use the Proposition 4.7 we must show that Q_j belongs to the appropriate local family of well inside cubes in Ω_j and that the pair of weights satisfy the hypothesis of the aforementioned proposition.

For the first we take θ large enough such that $\frac{1}{\theta-1}\le\beta_1$ (see Proposition 4.7). In second place, for $0<\tau<1$ and a cube $Q=Q(x_Q,l_Q)\in\mathcal{F}_\tau(\Omega_j)$ applying the Lemmas 5.6, 5.7 and 5.8 it can be seen that

$$|Q|^{\frac{\alpha}{n}-\frac{1}{p}+\frac{1}{q}}\left\|u^{1/q}\right\|_{\phi,Q}\left\|v^{-1/p}\right\|_{\psi,Q}<\infty\,,$$

Then $(u,v)\in A^{\tau,\alpha/n}_{p,q,\phi,\psi}$ and the estimation can be continue

$$\begin{aligned}\int_{\mathbb{R}^n}\left|I_\mu^\alpha(|f|\chi_{Q(x,\rho(x))})(x)\right|^q u(x)\,dx &= \sum_j\int_{Q_j}\left|I_\mu^\alpha(|f|\chi_{Q(x,\rho(x))})(x)\right|^q u(x)\,dx\\ &\le C\sum_j\int_{Q_j}\left|I^{1/2}_{\alpha/n,j}|f|(x)\right|^q u(x)\,dx\end{aligned}$$

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

$$
\begin{aligned}
&\leq C \sum_j \Big(\int_{\Omega_j} |f(x)|^p \, v(x)\, dx \Big)^{\frac{q}{p}} \\
&\leq C \bigg(\int_{\mathbb{R}^n} |f(x)|^p \Big(\sum_j \chi_{4\beta Q_j}(x) \Big) v(x)\, dx \bigg)^{\frac{q}{p}} \\
&\leq C \, \|f\|^q_{L^p(v)} \, ,
\end{aligned}
$$

where in the last inequality we use the Proposition 5.5.

For the global part, that is $I^\alpha_\mu(|f|\chi_{Q(x,\rho(x))^c})(x)$ let us consider the balls $B_j := B(x_j, \rho(x_j))$, with $j \in \mathbb{N}$, as in Proposition 5.5, $B_{\rho,x} := B_\rho(x, A_0^{-1})$, $B_{\rho,j} := B_\rho(x_j, \beta + A_0^{-1})$, where β and A_0 are constants that dependent of the critical radius function ρ. (see [1], pg. 24).

By an argument of duality, for a function g such that $\|g\|_{L^{q'}(u)} = 1$, we can estimates as follow

$$
\begin{aligned}
&\big\| I^\alpha_\mu(|f|\chi_{Q(x,\rho(x))^c})(x) \big\|_{L^q(u)} \\
&\qquad = \int_{\mathbb{R}^n} I^\alpha_\mu(|f|\chi_{Q(x,\rho(x))^c})(x)\, g(x)\, u(x)\, dx \\
&\qquad \leq \sum_{j\in\mathbb{N}} \int_{Q_j} I^\alpha_\mu(|f|\chi_{Q(x,\rho(x))^c})(x)\, g(x)\, u(x)\, dx \\
&\qquad \leq \sum_{j\in\mathbb{N}} \int_{Q_j} \left(\int_{Q(x,\rho(x))^c} \mathcal{K}^\alpha_\mu(x,y)\, |f(y)|\, dy \right) g(x)\, u(x)\, dx \\
&\qquad \leq \sum_{j\in\mathbb{N}} \int_{Q_j} \left(\int_{B^c_{\rho,x}} \mathcal{K}^\alpha_\mu(x,y)\, |f(y)|\, dy \right) g(x)\, u(x)\, dx \\
&\qquad = \sum_{j\in\mathbb{N}} \int_{Q_j} \left(\sum_{k=1}^\infty \int_{(k+1)B_{\rho,x}\setminus kB_{\rho,x}} \mathcal{K}^\alpha_\mu(x,y)\, |f(y)|\, dy \right) g(x)\, u(x)\, dx \, .
\end{aligned}
$$

In the last inequality, since $\beta \geq 2$ and $A_0 > 1$, by Lemma 5.8 with $r = A_0^{-1}$, we have that $B(x, \rho_(x))^c \subset B^c_{\rho,x}$. Then, for each $j \in \mathbb{N}$, $k \in \mathbb{N}$, $x \in B_j$ and $y \in (k+1)B_{\rho,x} \setminus kB_{\rho,x}$, we have the following estimate for the kernel

$$
\mathcal{K}^\alpha_\mu(x,y) \leq C\, e^{-\delta k} \rho(x)^{\alpha-d} \, ,
$$

for some δ small enough. Moreover, we note that $(k+1)B_{\rho,x} \subset B_{\rho,j}$ for all $x \in B_j$.

$$
\begin{aligned}
&\big\| I^\alpha_\mu(|f|\chi_{Q(x,\rho(x))^c})(x) \big\|_{L^q(u)} \\
&\leq C \sum_{j\in\mathbb{N}} \int_{B_j} \rho(x)^{\alpha-d} \left(\sum_{k=1}^\infty e^{-\delta k} \int_{(k+1)B_{\rho,j}} |f(y)|\, dy \right) g(x)\, u(x)\, dx \\
&= C \sum_{j\in\mathbb{N}} \left(\sum_{k=1}^\infty e^{-\delta k} \int_{(k+1)B_{\rho,j}} |f(y)|\, dy \right) \int_{B_j} \rho(x)^{\alpha-d}\, g(x)\, u(x)\, dx \\
&\leq C \sum_{j\in\mathbb{N}} \rho(x_j)^{\alpha-d} \left(\sum_{k=1}^\infty e^{-\delta k} \int_{(k+1)B_{\rho,j}} |f(y)|\, dy \right) \left(\int_{B_{\rho,j}} g(x)\, u(x)\, dx \right) ,
\end{aligned}
$$

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

where in the last inequality we use $\rho(x) \simeq \rho(x_j)$, for all $x \in B_j$ and the inclusion $B_j \subset B_{\rho,j}$ (see Lemma 5.7). Thus, by using the generalized Hölder inequality in both parenthesis for adequate young functions we estimate as follows

$$\begin{aligned}
&\left\| I_\mu^\alpha(|f|\chi_{Q(x,\rho(x))^c})(x)\right\|_{L^q(u)} \\
&\le C \sum_{j\in\mathbb{N}} \rho(x_j)^{\alpha-d}\left(\sum_{k=1}^\infty e^{-\delta k}\,|(k+1)B_{\rho,j}|\,\|fv^{1/p}\|_{\overline{\psi},(k+1)B_{\rho,j}}\|v^{-1/p}\|_{\psi,(k+1)B_{\rho,j}}\right)\\
&\qquad\qquad \times\left(|B_{\rho,j}|\,\|gu^{1/q'}\|_{\overline{\phi},B_{\rho,j}}\|u^{1/q}\|_{\phi,B_{\rho,j}}\right)\\
&\le C \sum_{j\in\mathbb{N}} \rho(x_j)^{\alpha+d}\left(\sum_{k=1}^\infty e^{-\delta k}\,(k+1)^{d(k_0+1)}\right.\\
&\qquad\qquad \left.\times\|fv^{1/p}\|_{\overline{\psi},(k+1)B_{\rho,j}}\|gu^{1/q'}\|_{\overline{\phi},B_{\rho,j}}\|v^{-1/p}\|_{\psi,(k+1)B_{\rho,j}}\|u^{1/q}\|_{\phi,B_{\rho,j}}\right).
\end{aligned}$$

The next step is to make use the bump condition about the pair of weights. More precisely, by the Lemmas 5.6, 5.7 and 5.8 again and the hypothesis on weights we get that

$$\begin{aligned}
&\|v^{-1/p}\|_{\psi,(k+1)B_{\rho,j}}\|u^{1/q}\|_{\phi,B_{\rho,j}}\\
&\le C\,\frac{|(k+1)B_{\rho,j}|}{|B_{\rho,j}|}\|v^{-1/p}\|_{\psi,(k+1)B_{\rho,j}}\|u^{1/q}\|_{\phi,(k+1)B_{\rho,j}}\\
&\le C\,\frac{|(k+1)B_{\rho,j}|^{-\frac{\alpha}{n}}}{|B_{\rho,j}|}|(k+1)B_{\rho,j}|^{\frac1p+1-\frac1q}\ \mathrm{e}^{c(k+1)(\beta+A_0^{-1})\left(\frac1q+\frac1{p'}\right)}\\
&\le C\,\rho(x_j)^{-\alpha-n}\,|(k+1)B_{\rho,j}|^{\frac1p+\frac1{q'}}\ \mathrm{e}^{c(k+1)(\beta+A_0^{-1})\left(\frac1q+\frac1{p'}\right)}\ .
\end{aligned}$$

Now, returning to the original account we have that

$$\begin{aligned}
&\left\| I_\mu^\alpha(|f|\chi_{Q(x,\rho(x))^c})(x)\right\|_{L^q(u)}\\
&\le C\sum_{j\in\mathbb{N}}\left(\sum_{k=1}^\infty (k+1)^{d(k_0+1)}\ \mathrm{e}^{-\delta k\,+\,c(k+1)(\beta+A_0^{-1})\left(\frac1q+\frac1{p'}\right)}\right.\\
&\qquad\qquad\left.\times\,|(k+1)B_{\rho,j}|^{\frac1p+\frac1{q'}}\ \|fv^{1/p}\|_{\overline{\psi},(k+1)B_{\rho,j}}\|gu^{1/q'}\|_{\overline{\phi},B_{\rho,j}}\right)\\
&\le C\sum_{k=1}^\infty (k+1)^{d(k_0+1)}\ \mathrm{e}^{-\delta k\,+\,c(k+1)(\beta+A_0^{-1})\left(\frac1q+\frac1{p'}\right)}\\
&\qquad\qquad\times\left(\sum_{j\in\mathbb{N}}|(k+1)B_{\rho,j}|^{\frac1p+\frac1{q'}}\ \|fv^{1/p}\|_{\overline{\psi},(k+1)B_{\rho,j}}\|gu^{1/q'}\|_{\overline{\phi},B_{\rho,j}}\right)\\
&\le C\sum_{k=1}^\infty (k+1)^{2\,d(k_0+1)}\ \mathrm{e}^{-\delta k\,+\,c(k+1)(\beta+A_0^{-1})\left(\frac1q+\frac1{p'}\right)}\\
&\qquad\qquad\times\left(\sum_{j\in\mathbb{N}}|(k+1)B_{\rho,j}|^{\frac1p}\ |B_{\rho,j}|^{\frac1{q'}}\ \|fv^{1/p}\|_{\overline{\psi},(k+1)B_{\rho,j}}\|gu^{1/q'}\|_{\overline{\phi},B_{\rho,j}}\right),
\end{aligned}$$

where in the last inequality we use $|(k+1)B_{\rho,j}| \le C\,(k+1)^{d(k_0+1)}\,|B_{\rho,j}|$. For a fixed k we apply the discrete version of the Hölder inequality with $p>1$ and since

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

$q' < p'$ and taking into account that $B_{\rho,j} \subset B_{k,j} \doteq B(x_j, \delta\,(k+1)^{(k_0+1)}\rho(x_j))$, with $\delta = \left(\beta(\beta + A_0^{-1})\right)^{(k_0+1)}$, we can get

$$
\begin{aligned}
&\sum_{j\in\mathbb{N}} |(k+1)B_{\rho,j}|^{\frac{1}{p}}\, |B_{\rho,j}|^{\frac{1}{q'}}\, \|fv^{1/p}\|_{\overline{\psi},(k+1)B_{\rho,j}} \|gu^{1/q'}\|_{\overline{\phi},B_{\rho,j}} \\
&\le \left(\sum_{j\in\mathbb{N}} |(k+1)B_{\rho,j}|\, \|fv^{1/p}\|^{p}_{\overline{\psi},(k+1)B_{\rho,j}}\right)^{1/p} \left(\sum_{j\in\mathbb{N}} |B_{\rho,j}|^{\frac{p'}{q'}}\, \|gu^{1/q'}\|^{p'}_{\overline{\phi},B_{\rho,j}}\right)^{1/p'} \\
&\le \left(\sum_{j\in\mathbb{N}} |(k+1)B_{\rho,j}|\, \|(f\chi_{k,j})v^{1/p}\|^{p}_{\overline{\psi},(k+1)B_{\rho,j}}\right)^{1/p} \left(\sum_{j\in\mathbb{N}} |B_{\rho,j}|\, \|(g\chi_{k,j})u^{1/q'}\|^{q'}_{\overline{\phi},B_{\rho,j}}\right)^{1/q'} \\
&\le \left(\sum_{j\in\mathbb{N}} \int_{\mathbb{R}^n} \|(f\chi_{k,j})v^{1/p}\|^{p}_{\overline{\psi},(k+1)B_{\rho,j}}\, dx\right)^{1/p} \left(\sum_{j\in\mathbb{N}} \int_{\mathbb{R}^n} \|(g\chi_{k,j})u^{1/q'}\|^{q'}_{\overline{\phi},B_{\rho,j}}\, dx\right)^{1/q'} \\
&\le \sum_{j\in\mathbb{N}} \left(\int_{\mathbb{R}^n} \left(\mathrm{M}_{\tilde{\psi}}((f\chi_{k,j})v^{1/p})(x)\right)^{p} dx\right)^{1/p} \sum_{j\in\mathbb{N}} \left(\int_{\mathbb{R}^n} \left(\mathrm{M}_{\tilde{\varphi}}((g\chi_{k,j})u^{1/q'})(x)\right)^{q'} dx\right)^{1/q'} .
\end{aligned}
$$

Since the Agmon metric is continuous (see Lemma 2.3 in [1]) we can consider the averages that de definition of the Orlicz Maximal involves and since both Young functions ϕ and ψ are such that $\overline{\phi} \in B_{q'}$ and $\overline{\psi} \in B_p$ conditions, we can apply the boundedness in L^p and $L^{q'}$ respectively (see [9]).

So, since $\|g\|_{L^{q'}(u)} = 1$ we conclude that

$$
\begin{aligned}
&\sum_{j\in\mathbb{N}} |(k+1)B_{\rho,j}|^{\frac{1}{p}}\, |B_{\rho,j}|^{\frac{1}{q'}}\, \|fv^{1/p}\|_{\overline{\psi},(k+1)B_{\rho,j}} \|gu^{1/q'}\|_{\overline{\phi},B_{\rho,j}} \\
&\le \sum_{j\in\mathbb{N}} \left(\int_{\mathbb{R}^n} (f\chi_{k,j})(x)^p\, v(x)\, dx\right)^{1/p} \sum_{j\in\mathbb{N}} \left(\int_{\mathbb{R}^n} (g\chi_{k,j})(x)^{q'} u(x)\, dx\right)^{1/q'} \\
&\le \left(\int_{\mathbb{R}^d} \left(\sum_{j\in\mathbb{N}} \chi_{B_{k,j}}(x)\right) f(x)^p\, v(x)\, dx\right)^{1/p} \left(\int_{\mathbb{R}^d} \left(\sum_{j\in\mathbb{N}} \chi_{\hat{B}_j}(x)\right) g(x)^{q'}\, u(x)\, dx\right)^{1/q'} \\
&\le C(d, k_0, \beta, N_1)\, (k+1)^{N_1(k_0+1)}\, \|f\|_{L^p(v)} \|g\|_{L^{q'}(u)} \\
&\le C\, (k+1)^{N_1(k_0+1)}\, \|f\|_{L^p(v)} \,.
\end{aligned}
$$

where

$$
B_{k,j} \doteq B(x_j, \delta\,(k+1)^{(k_0+1)}\rho(x_j))\,; \qquad \hat{B}_j \doteq B(x_j, \delta\,\rho(x_j))\,,
$$

with $\delta = \left(\beta(\beta + A_0^{-1})\right)^{(k_0+1)}$ and N_1 is provided by Proposition 5.5. So,

$$
\begin{aligned}
&\left\| I^{\alpha}_{\mu}(|f|\chi_{Q(x,\rho(x))^c})(x)\right\|_{L^q(u)} \\
&\qquad \le C\, \|f\|_{L^p(v)} \sum_{k=1}^{\infty} (k+1)^{(2d+N_1)(k_0+1)}\; \mathrm{e}^{-\delta k \,+\, c(k+1)(\beta+A_0^{-1})\left(\frac{1}{q}+\frac{1}{p'}\right)} \,.
\end{aligned}
$$

Finally, the result for the global part of the operator is valid if we choose the constant c such that

$$
-\delta k \,+\, c(k+1)(\beta + A_0^{-1})\left(\frac{1}{q} + \frac{1}{p'}\right) < 0\,,
$$

IMAL PREPRINT # 2024-0075
ISSN 2451-7100
Publication date: December 27, 2024

or equivalent $8ck\beta - \delta k < 0 \quad \Leftrightarrow c < \frac{\delta}{8\beta}$, independent of p and q.

□

6. Examples

The Theorem 1.8 is an improvement over the analogous result found in [10] in the sense that the weights do not need to satisfy a doubling property. In this direction the following example presents a pair of weights belonging to $A_{p,q,\psi}^{\gamma,\beta}$, where one of them is not doubling.

Example 6.1. *Let $\Omega = \mathbb{R}^+$ and $\beta \in (0,1)$. Let $1 < p \leq q < \infty$, $0 \leq \gamma < 1$ such that $\gamma - \frac{1}{p} + \frac{1}{q} = 0$ and ψ be a Young function. Then the pair $u(x) = 1$ and $v(x) = e^{-e^{1/x}}$ belongs to $A_{p,q,\psi}^{\gamma,\beta}$. Moreover, v is not doubling.*

Let us consider $Q \in \mathcal{F}_\beta$. First, we note that $\left\|u^{1/q}\right\|_{q,Q} = 1$ and $\left\|v^{1/p'}\right\|_\infty \leq 1$. Now, since $\gamma - \frac{1}{p} + \frac{1}{q} = 0$, if we take $\lambda_0 = \max\{1, \psi(1)\}$ then by the Young function's properties we get

$$\frac{1}{|Q|}\int_Q \psi\left(\frac{v^{1/p'}}{\lambda_0}\right) \leq \psi\left(\frac{\left\|v^{\frac{1}{p'}}\right\|_\infty}{\lambda_0}\right) \leq 1,$$

So, $\left\|v^{1/p'}\right\|_{p',Q} \leq \lambda_0$, consequently $(u,v) \in A_{p,q,\psi}^{\gamma,\beta}$.

On the other hand, we will see that v is not doubling. Suppose that this not occur and let us take $x > 0$ and $r = \frac{\beta}{2}x$. If we denote $B = B(x,r)$, $B_1 = B(x+\frac{r}{4}, \frac{r}{4})$ and $B_2 = B(x + \frac{3r}{4}, \frac{r}{4})$, it is easy to see that $B_2 \subset 3B_1$ and it follows that

$$v(B_2) \leq v(3B_1) \leq Cv(B_1).$$

Furthermore, since v is increasing:

$$v(B_2) \geq Cxe^{-e^{\frac{4}{(4+\beta)x}}}$$

$$v(B_1) \leq v(B(x,r)) = v(B(x, 4\frac{r}{4})) \leq Cv(B(x, \frac{r}{4})) \leq Cxe^{-e^{\frac{8}{(8+\beta)x}}}$$

From which we conclude that

$$e^{e^{\frac{8}{(8+\beta)x}} - e^{\frac{4}{(4+\beta)x}}} \leq C$$

Since the left side tends to infinity as x approaches zero, we obtain a contradiction from assuming that v is doubling.

In Theorem 1.8 we require that $(u,v) \in A_{p,q,\psi}^{\gamma,\tau}$ for to prove the boundedness of M_β^γ for every $\beta < \tau$. One question that arises is whether the classes of weights satisfies certain openness condition on β. More precisely, $(u,v) \in A_{p,q,\psi}^{\gamma,\beta}$ implies $(u,v) \in A_{p,q,\psi}^{\gamma,\beta+\varepsilon}$. The answer is negative, ever in the particular case of $\psi(t) = t^q$, as can be seen in the following example.

IMAL PREPRINT # 2024-0075

ISSN 2451-7100
Publication date: December 27, 2024

Example 6.2. *Let $\Omega = \mathbb{R}^+$, $0 < \beta < \tau < 1$, $1 < p \leq q\infty$ and $0 \leq \gamma < 1$ such that $\gamma = \frac{1}{p} - \frac{1}{q}$ We consider the weights $u(y) = e^{\frac{1-\beta}{1+\beta} q\, y}$ and $v(y) = e^{p\, y}$. Then $(u, v) \in A^{\gamma,\beta}_{p,q}$ but however $(u, v) \notin A^{\gamma,\tau}_{p,q}$.*

Let us consider $x > 0$ and $l > 0$ such that $Q = (x-l, x+l) \subset \big((1-\beta)x, (1+\beta)x\big)$, that is $Q \in \mathcal{F}_\beta$. For each $y \in Q$ we have that

$$u(y) \leq \mathrm{e}^{\frac{1-\beta}{1+\beta}\,(x+l)\,q}\,, \qquad v(y)^{-p'/p} \leq \mathrm{e}^{-(x-l)\,p'}\,.$$

So

$$\left(\frac{1}{|Q|}\int_Q u\right)^{\frac{1}{q}} \left(\frac{1}{|Q|}\int_Q v^{-\frac{p'}{p}}\right)^{\frac{1}{p'}} \leq e^{\frac{1-\beta}{1+\beta}\,(x+l)} e^{-(x-l)} \leq 1\,.$$

Since $\gamma - \frac{1}{p} + \frac{1}{q} = 0$ we get that $(u, v) \in A^{\gamma,\beta}_{p,q}$.

Now we will see that $(u, v) \notin A^{\gamma,\tau}_{p,q}$. For this, we will consider a critical cube $Q = \big((1-\tau)x, (1+\tau)x\big)$ for some $x > 0$. Then $|Q| = 2\tau\, x$ and

$$\int_Q u = \int_{(1-\tau)x}^{(1+\tau)x} e^{\frac{1-\beta}{1+\beta} q\, y} dy = \frac{1+\beta}{1-\beta}\frac{1}{q}\left(e^{\frac{1-\beta}{1+\beta}(1+\tau)q\, x} - e^{\frac{1-\beta}{1+\beta}(1-\tau)q\, x}\right)\,;$$

$$\int_Q v^{-\frac{p'}{p}} = \int_{(1-\tau)x}^{(1+\tau)x} e^{-p'\, y} dy = \frac{1}{p'}\left(e^{-(1-\tau)p'\, x} - e^{-(1+\tau)p'\, x}\right)\,.$$

It is an exercise to show that for a given parameters $a > b$ the inequalities

$$\mathrm{e}^{ax} - \mathrm{e}^{bx} > \frac{1}{2}\,\mathrm{e}^{ax}\,, \qquad \mathrm{e}^{-bx} - \mathrm{e}^{-ax} > \frac{1}{2}\,\mathrm{e}^{-bx}\,,$$

holds for every x large enough.

With all this, taking into account that $|Q|^{\frac{1}{q}+\frac{1}{p'}} = C\, x^{1-\gamma}$, for every sufficiently large x we can estimates as follow

$$\left(\frac{1}{|Q|}\int_Q u\right)^{\frac{1}{q}} \left(\frac{1}{|Q|}\int_Q v^{-\frac{p'}{p}}\right)^{\frac{1}{p'}} \geq \frac{C}{|Q|^{\frac{1}{q}+\frac{1}{p'}}}\,\mathrm{e}^{\frac{1-\beta}{1+\beta}(1+\tau)\,x}\,\mathrm{e}^{-(1-\tau)\,x} = C\frac{\mathrm{e}^{Mx}}{x^{1-\gamma}}\,.$$

where $M > 0$ since the function $\phi(t) = \frac{1-t}{1+t}$ is strictly decreasing. Finally, since $\gamma - \frac{1}{p} + \frac{1}{q} = 0$, by considering $x \to +\infty$ we can conclude that $(u, v) \notin A^{\gamma,\tau}_{p,q}$.

Although we have seen that the inclusion of the families of weights $A^{\gamma,\beta}_{p,q}$ can be strict, it is also true that in some particular cases, for certain pairs of weights u, v belonging to a class $A^{\gamma,\beta}_{p,q}$, they can also satisfy the condition $A^{\gamma,\beta+\varepsilon}_{p,q}$. For this implication to hold, it is sufficient to require that at least one of the weights doubles.

Example 6.3. *Let $1 < p \leq q < \infty$, $0 < \beta < \tau < 1$, and a pair of weights $(u, v) \in A^{\gamma,\beta}_{p,q}$. If at least one of the weights is doubling, then $(u, v) \in A^{\gamma,\tau}_{p,q}$.*

IMAL PREPRINT # 2024-0075
ISSN 2451-7100
Publication date: December 27, 2024

References

[1] Julian Bailey. Weights of exponential growth and decay for Schrödinger-type operators. *J. Funct. Anal.*, 281(1): Paper N$^{\underline{o}}$. 108996, 93, 2021.

[2] David Cruz-Uribe and OFS. *"Two weight inequalities for fractional integral operators and commutators"*. Advanced Courses of Mathematical Analysis VI, (2017), 25–85.

[3] J. Dziubanski and J. Zienkiewicz. Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality. *Rev. Mat. Iberoam.* 15 (1999) 279–296.

[4] Bruno Bongioanni and Eleonor Harboure and Oscar Salinas, Classes of weights related to Schrödinger operators. *J. Math. Anal. Appl.* 373 (2011) 563–579.

[5] Eleonor Harboure, Oscar Salinas, and Beatriz Viviani. Local maximal function and weights in a general setting. *Math. Ann.*, 358(3-4):609–628, 2014.

[6] Eleonor Harboure, Oscar Salinas, and Beatriz Viviani. Local fractional and singular integrals on open subsets. *J.d'Analyse Math.*, 138(1):301–324, 2019.

[7] Kato, T. Perturbation Theory for Linear Operators. *Springer Science & Business Media*, 1995.

[8] Carlos Pérez. On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p-spaces with different weights. *Proc. London Math.* Soc 71, (1995) 135–157.

[9] Gladis Pradolini and Oscar Salinas. Maximal operators on spaces of homogeneous type. *Proc. Amer. Math. Soc.*, 132(2):435–441, 2004.

[10] Mauricio Ramseyer and Oscar Salinas and Marisa Toschi. Two-weight boundedness for local fractional maximal and applications. *Springer; European Journal of Mathematics*, 9, 109, 2023, 33.

[11] Zhongwei Shen. *"On fundamental solutions of generalized schrödinger operators"*. Journal of Functional Analysis, 167(2):521–564, 1999.

Instituto de Matemática Aplicada del Litoral, CONICET-UNL, Colectora Ruta Nac. N° 168, Paraje El Pozo - 3000 Santa Fe - Argentina.

Email address: salinas@santafe-conicet.gov.ar
Email address: mramseyer@santafe-conicet.gov.ar
Email address: jsotto@santafe-conicet.gov.ar
Email address: mtoschi@santafe-conicet.gov.ar