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UNCERTAINTY IN HYPERGRAPHS

HUGO AIMAR, IVANA GÓMEZ, AND JOAQUÍN TOLEDO

Abstract. In this paper we explore an uncertainty principle on certain hypergraphs. Precisely we prove a multiplicative uncertainty inequality for regular hypergraphs of Kirchhoff type, based on the definition of reasonable position and momentum operators for a quantum formalism in hypergraphs.

1. Introduction

In [BK15] and [Kop15] a Fourier Analysis version of the uncertainty principle is established in the setting of finite weighted undirected graphs. The approach of John Benedetto and Paul Koprowski is based in the spectral theory for the graph Laplacian that provides the Fourier Analysis of signals defined on the vertex set of the graph (see also [AL13]). In recent years, in applications to data analysis, it became clear that interactions of order higher than two between nodes of a net, provide additional information on the system under analysis. See for example [GMV23]. Actually the basic definition of hypergraph as a covering of the vertex set by subsets, the hyperedges, of any cardinality can be found in [Ber73]. Since, as it is well known, see for example [Rod02], every hypergraph induces weighted matrix of adjacency of vertices, we can construct an undirected weighted graph induced by the hypergraph. So, we can apply the results in [BK15] to this graph to obtain an additive Fourier type uncertainty principle. Nevertheless, since an hypergraph is not completely determined by its adjacency matrix, the above reduction of an hypergraph to a graph is loosing relevant information regarding the higher order interactions that the hyperedges are recording. With the idea of introducing position and momentum operators on a hypergraph in order to recover a multiplicative uncertainty principle, we consider a special case of regular hypergraphs satisfying Kirchhoff type conditions. Instead of the Fourier Analysis induced by the spectra of the given hypergraph, our approach is based in a formula that resembles an integration by parts.

2. Weighted Hypergraphs. Basic definitions

A **hypergraph** is a pair $\mathscr{H} = (\mathscr{V}, \mathscr{E})$ where $\mathscr{V}$ is a finite set and $\mathscr{E}$ is a subfamily of the non empty parts of $\mathscr{V}$ that covers $\mathscr{V}$. The elements of $\mathscr{E}$ are called the hiperedges. In other words $\mathscr{E} \subset \mathcal{P}(\mathscr{V})$, $\bigcup_{e\in\mathscr{E}} = \mathscr{V}$ and for each $e \in \mathscr{E}$ we have that $e \neq \emptyset$. A **weighted**

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hypergraph is a pair $(\mathscr{V}, w)$ where $\mathscr{V}$ is finite and $w : \mathcal{P}(\mathscr{V}) \to \mathbb{R}^+ \cup \{0\}$ is a function satisfying $\bigcup_{\{a\subseteq\mathscr{V}:w(a)>0\}} a = \mathscr{V}$. As it is easy to see, with $\mathscr{E} = \{e \in \mathcal{P}(\mathscr{V}) : w(e) > 0\}$, we have that $(\mathscr{V}, \mathscr{E})$ is an hypergraph in the sense of the above definition.

Notice that every hypergraph $\mathscr{H} = (\mathscr{V}, \mathscr{E})$ can be seen as a weighted hypergraph taking $w(e) = 1$ if $e \in \mathscr{E}$ and $w(e) = 0$ when $e \notin \mathscr{E}$. Given a weighted hypergraph $\mathscr{H} = (\mathscr{V}, w)$ the **incidence function** of $\mathscr{H}$ is the function $h : \mathscr{V} \times \mathscr{E} \to \{0, 1\}$ defined by $h(v, e) = \chi_e(v)$, here χ_A denotes the indicator function of the set A. The **degree of** the **vertex** $v \in \mathscr{V}$ is given by $d(v) = \sum_{e\in\mathscr{E}} h(v, e)w(e)$. The **degree of** the **hyperedge** $e \in \mathscr{E}$ is defined as $\delta(e) = \sum_{v\in\mathscr{V}} h(v, e) = |e|$, where $|A|$ denotes the number of elements of the set A.

These basic definitions provide the ingredients for a notion of adjacency of two vertices in a weighted hypergraph. The function $A : \mathscr{V} \times \mathscr{V} \to \mathbb{R}^+ \cup \{0\}$ given by $A(v, \tilde{v}) = \sum_{e\in\mathscr{E}} h(v, e)h(\tilde{v}, e)w(e)$ is called the **adjacency function** of $\mathscr{H}$. Notice that when $w \equiv 1$ on $\mathscr{E}$, the adjacency function takes only integer values. In fact $A(v, \tilde{v}) = |\{e \in \mathscr{E} : v \text{ and } \tilde{v} \text{ belong to } e\}|$. It is clear that A is symmetric and that on the diagonal $A(v, v) = \sum_{e\in\mathscr{E}} h(v, e)h(v, e)w(e) = \sum_{e\in\mathscr{E}} h(v, e)w(e) = d(v)$ is the degree of the vertex v. Hence the modified version of A given by $\tilde{A}(v, \tilde{v}) = A(v, \tilde{v}) - d(v)\delta_{v,\tilde{v}}$, where $\delta_{v,\tilde{v}}$ denotes the Kronecker delta, is the adjacency function (matrix) of a weighted undirected graph on the set $\mathscr{V}$ of vertices with no loops. We say that the graph $\mathcal{G} = (\mathscr{V}, \tilde{A})$ is the graph on $\mathscr{V}$ induced by the hypergraph $\mathscr{H} = (\mathscr{V}, w)$ on $\mathscr{V}$. It is worthy noticing at this point that different hypergraphs $\mathscr{H}$ can produce the same induced graph. In some sense the above algorithm can be seen as a projector of hypergraphs onto weighted undirected graphs. Since in $\mathcal{G}$ we have the harmonic analysis induced by its Laplace operator, we also have the uncertainty results in [BK15]. Nevertheless some substantial properties of $\mathscr{H}$ are not inherited by $\mathcal{G}$.

We shall say that a hypergraph is **regular of degree** $k \geq 2$ (or k-regular) if $\mathscr{E} = \{e \in \mathcal{P}(\mathscr{V}) : w(e) > 0\} \subset \mathcal{P}_k(\mathscr{V}) = \{e \in \mathcal{P}(\mathscr{V}) : |e| = k\}$. Notice first that when $k = 2$ the class of regular hypergraphs of degree 2 is precisely the class of all weighted undirected graphs.

In what follows we consider a k-regular hypergraph on the vertex set $\mathscr{V}$ with $|\mathscr{V}| = N$ and N larger than $2(k-1)$. The wave functions ψ for a quantum formalism in $\mathscr{H} = (\mathscr{V}, w)$ will be here real functions with domain $\mathscr{V}$. We are looking for two operators P and Q acting on the wave functions ψ playing the roles of momentum and position operators in quantum mechanics. In doing so we realize that, so far, all the above definitions were given in, let us say, absolute terms regardless of any enumeration of the vertex set $\mathscr{V}$. An enumeration of the set $\mathscr{V}$ can be seen as the introduction of a coordinate system in the classical formalism of quantum mechanics in order to properly define position and velocity. It is a delicate craft the enumeration of the vertices of a given hypergraph.

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From now on we shall fix an enumeration of $\mathscr{V} = \{v_1, v_2, \dots, v_N\} \simeq \{1, 2, \dots, N\}$ and frequently we shall simply use i to denote vertex v_i, $i = 1, \dots, N$. Since the operators of position and momentum will be defined through this enumeration of $\mathscr{V}$, we emphasize on their dependence on this "coordinate system".

Since in our k-regular hypergraph $\mathscr{H} = (\mathscr{V}, w)$ we have introduced the enumeration of $\mathscr{V}$, this fact determines a prelation order of vertices in each $e \in \mathcal{P}_k(\mathscr{V})$. Let us precise this in the following definition.

Definition 2.1. For $j = 1, 2, \dots, k$ we define the functions $i_j : \mathcal{P}_k(\mathscr{V}) \to \mathscr{V} \simeq \{1, \dots, N\}$ by

$$\begin{aligned} i_1(e) =& \inf\{l : v_l \in e\}; \\ i_j(e) =& \inf\{l \notin \{i_1(e), i_2(e), \dots, i_{j-1}(e)\} : v_l \in e\}, 1 < j \leq k. \end{aligned}$$

Proposition 2.2. *With the above notation we have*
(a) $j \leq i_j(e) \leq N - k + j$, *for every* $j \in \{1, 2, \dots, k\}$ *and every* $e \in \mathcal{P}_k(\mathscr{V})$;
(b) $i_k(e) = \sup\{l : v_l \in e\}$.

Proof. It follows readily from the definition of the functions i_j. □

The trivial fact reflected in Proposition 2.2 is that not every vertex can be the first in some hyperedge, neither can be the k^{th}. Nevertheless, since in applications, the number of vertices N is much larger than the order k of interactions that determines the order of regularity of $\mathscr{H}$, we may say that most of the vertices $\mathscr{V}$ can be the first, the second and the k^{th} of some hyperedge $e \in \mathcal{P}_k(\mathscr{V})$. This fact leads us to (non topological) definitions of boundary and interior of an enumerated k-regular hypergraph $\mathscr{H}$.

Definition 2.3. Given an enumerated k-regular hypergraph $\mathscr{H}$, we define its **boundary** and its **interior** respectively by

$$\begin{aligned} \partial(\mathscr{H}) =& \{v_l \in \mathscr{V} : 1 \leq l < k\} \cup \{v_l \in \mathscr{V} : N - k + 1 < l \leq N\} \quad \text{and} \\ \mathring{\mathscr{H}} =& \mathscr{V} \setminus \partial(\mathscr{H}) = \{v_l \in \mathscr{V} : k \leq l \leq N - k + 1\}. \end{aligned}$$

As a simple example consider $\mathscr{V}$ as the set of the first one hundred positive integers, $\mathscr{V} = \{1, 2, \dots, 100\}$. The interactions take place only when three integers are consecutive, then $k = 3$, $w(\{j, j+1, j+2\}) = 1$, $j = 1, \dots, 98$ and $w(e) = 0$ for any other $e \in \mathcal{P}(\mathscr{V})$. Then, $\partial(\mathscr{H}) = \{1, 2, 99, 100\}$ and $\mathring{\mathscr{H}} = \{j \in \mathbb{N} : 3 \leq j \leq 98\}$.

The weight w defined on the hyperedges of $\mathscr{H}$ produces a measure on the family of all the subsets of $\mathcal{P}_k(\mathscr{V})$. Given a subset E of $\mathcal{P}_k(\mathscr{V})$, define $\mu(E) = \sum_{e \in E} w(e)$. With this measure we may and shall consider the distribution functions of the functions $i_j : \mathcal{P}_k(\mathscr{V}) \to \{1, \dots, N\}$ given by $d_j : \{1, \dots, N\} \to \mathbb{R}^+ \cup \{0\}$, with $d_j(n) = \mu(i_j^{-1}(n)) = \mu(\{e \in \mathcal{P}_k(\mathscr{V}) : i_j(e) = n\})$, for $n \in \{1, \dots, N\}$ and $j = 1, \dots, k$. It is worthy noticing

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that for simple hypergraphs, that is when w takes only the values zero and one, the function $d_j(n)$ counts the number of hyperedges having n as its j^{th} element.

3. Some Calculus on k-regular hypergraphs

In this section we aim to introduce some elementary notions of gradient of real functions defined on the vertex set a given hypergraph, allowing for a Leibniz type formula. We also prove a kind of integration by parts formula that shall be the key to prove our main result in Section 4. The starting point for our definition of gradient is that provided for the case of graphs by Benedetto and Koprowski ([BK15]). If $\mathcal{G} = (\mathscr{V}, w)$ is a weighted, undirected, enumerated graph, the derivative, defined in [BK15], of a function f in $\mathscr{V}$ is the function defined on the edges $\mathcal{P}_2(\mathscr{V})$ by

$$Df(e) = \sqrt{w(e)}(f(v_j) - f(v_i))$$

where $e = \{i, j\}$ and $j > i$.

In what follows we shall use the notation $\{\boldsymbol{E}_j : j = 1, \dots, k-1\}$ for the canonical basis of $\mathbb{R}^{k-1}$. Precisely $\boldsymbol{E}_j = (0, \dots, 1, 0, \dots, 0)$ are the $k-1$ tuples with a 1 in the position j and zeros in all the other components.

Definition 3.1. Let $\mathscr{H} = (\mathscr{V}, w)$ be an enumerated k-regular hypergraph. Let f and h be two real functions defined on $\mathscr{V}$. The gradient of f with respect to h is the $k-1$ dimensional vector valued function of the hyperedges given by

$$\boldsymbol{\nabla}_h f(e) = \frac{1}{2}\sum_{j=1}^{k-1} \left[h(i_{j+1}(e)) + h(i_j(e))\right]\left[f(i_{j+1}(e)) - f(i_j(e))\right]\boldsymbol{E}_j.$$

Notice that when $k = 2$ and $h \equiv 1$, we recover the operator $Df(e)$ in [BK15] except for the factor $\sqrt{w(e)}$.

Proposition 3.2. *Let $\mathscr{H} = (\mathscr{V}, w)$ be an enumerated k-regular hypergraph. Then the operator that applies the pair of functions (f, g) into the vector functions on the hyperedges $\boldsymbol{\nabla}_g f$ is bilinear. Moreover,*

$$\boldsymbol{\nabla}_1(fg) = \boldsymbol{\nabla}_g f + \boldsymbol{\nabla}_f g,$$

where $\boldsymbol{\nabla}_1 = \boldsymbol{\nabla}_h$ with $h \equiv 1$ in $\mathscr{V}$.

Proof. The linearity in f for g fixed and the linearity in g for f fixed are clear. Let us check the Leibniz like formula,

$$\begin{aligned}\boldsymbol{\nabla}_1(fg) &= \sum_{j=1}^{k-1}\left[(fg)(i_{j+1}(e)) - (fg)(i_j(e))\right]\boldsymbol{E}_j\\ &= \sum_{j=1}^{k-1}\left[f(i_{j+1}(e)) - f(i_j(e))\right]g(i_{j+1}(e))\boldsymbol{E}_j + \sum_{j=1}^{k-1}\left[g(i_{j+1}(e)) - g(i_j(e))\right]f(i_j(e))\boldsymbol{E}_j.\end{aligned}$$

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Since $fg = gf$, then $\boldsymbol{\nabla}_1(fg) = \boldsymbol{\nabla}_1(gf)$ that can be computed as above interchanging f and g. So that, we also have

$$\boldsymbol{\nabla}_1(fg) = \sum_{j=1}^{k-1} \left[g(i_{j+1}(e)) - g(i_j(e))\right] f(i_{j+1}(e))\boldsymbol{E}_j + \sum_{j=1}^{k-1} \left[f(i_{j+1}(e)) - f(i_j(e))\right] g(i_j(e))\boldsymbol{E}_j$$

hence, adding the two formulas above, we have the desired identity. $\square$

In the family of all the subsets of $\mathcal{P}_k(\mathscr{V})$ we have the measure μ defined in Section 2, by $\mu(E) = \sum_{e\in E} w(e)$ for $E \subseteq \mathcal{P}_k(\mathscr{V})$. Since the gradient of a function f defined on $\mathscr{V}$ is a vector field on $\mathcal{P}_k(\mathscr{V})$ we may compute its integral with respect to μ and we may look for a kind of fundamental theorem of calculus in terms of integrals of f on $\partial(\mathscr{H})$ and on $\mathring{\mathscr{H}}$. In order to properly and briefly write the result let us first introduce two relevant signed measures on $\mathscr{V}$. For each $j = 1, 2, \ldots, k-1$, set

$$\delta_j(n) = d_{j+1}(n) - d_j(n) = \mu(i_{j+1}^{-1}(n)) - \mu(i_j^{-1}(n)),$$

and

$$\Delta_j(V) = \sum_{n\in V} \delta_j(n), \quad V \subseteq \mathscr{V}.$$

For every $j = 1, \ldots, k-1$ define the following function on the set of vertices,

$$m_j(n) = \begin{cases} -d_j(j), & n = j; \\ d_{j+1}(N-k+j+1), & n = N-k+j+1; \\ \delta_j(n), & j+1 \le n \le k-1; \\ \delta_j(n), & N-k+2 \le n \le N-k+j; \\ 0 & \text{otherwise.} \end{cases}$$

It is simple to check that each m_j is supported in $\partial(\mathscr{H})$. Set M_j to denote the measure on $\mathscr{V}$, supported on $\partial(\mathscr{H})$, given by $M_j(V) = \sum_{n\in V} m_j(n)$. With this notation we have the following result.

Proposition 3.3. *Let $\mathscr{H} = (\mathscr{V}, w)$ be an enumerated k-regular hypergraph. Let f be a real function defined on the set of vertices $\mathscr{V}$. Then, with the notation introduced above, we have the identity*

$$\int_{\mathcal{P}_k(\mathscr{V})} \boldsymbol{\nabla}_1 f d\mu = \sum_{j=1}^{k-1} \left(\int_{\partial(\mathscr{H})} f dM_j + \int_{\mathring{\mathscr{H}}} f d\Delta_j \right) \boldsymbol{E}_j.$$

Proof. Notice first that for $j = 1, 2, \ldots, k-1$ we can obtain two disjoint partitions of $\mathcal{P}_k(\mathscr{V})$ using the functions i_j and i_{j+1}. In fact

$$\mathcal{P}_k(\mathscr{V}) = \bigcup_{n=j+1}^{N-k+j+1} i_{j+1}^{-1}(n), \quad i_{j+1}^{-1}(n) \cap i_{j+1}^{-1}(m) = \emptyset$$

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for $n \neq m$. Also

$$\mathcal{P}_k(\mathscr{V}) = \bigcup_{n=j}^{N-k+j} i_j^{-1}(n), \quad i_j^{-1}(n) \cap i_j^{-1}(m) = \emptyset$$

for $n \neq m$. Hence

$$\begin{aligned}
\int_{\mathcal{P}_k(\mathscr{V})} \boldsymbol{\nabla}_1 f d\mu &= \sum_{e\in\mathcal{P}_k(\mathcal{V})} w(e) \sum_{j=1}^{k-1} [f(i_{j+1}(e)) - f(i_j(e))] \boldsymbol{E}_j \\
&= \sum_{j=1}^{k-1} \boldsymbol{E}_j \left(\sum_{e\in\mathcal{P}_k(\mathcal{V})} w(e) f(i_{j+1}(e)) - \sum_{e\in\mathcal{P}_k(\mathcal{V})} w(e) f(i_j(e)) \right) \\
&= \sum_{j=1}^{k-1} \boldsymbol{E}_j \left(\sum_{n=j+1}^{N-k+j+1} \sum_{e\in i_{j+1}^{-1}(n)} w(e) f(n) - \sum_{n=j}^{N-k+j} \sum_{e\in i_j^{-1}(n)} w(e) f(n) \right) \\
&= \sum_{j=1}^{k-1} \boldsymbol{E}_j \left(\sum_{n=j+1}^{N-k+j+1} f(n)\mu(i_{j+1}^{-1}(n)) - \sum_{n=j}^{N-k+j} f(n)\mu(i_j^{-1}(n)) \right) \\
&= \sum_{j=1}^{k-1} \boldsymbol{E}_j \left(\sum_{n=j+1}^{N-k+j+1} f(n) d_{j+1}(n) - \sum_{n=j}^{N-k+j} f(n) d_j(n) \right) \\
&= \sum_{j=1}^{k-1} \boldsymbol{E}_j \Bigg[\left(\sum_{n=j+1}^{k-1} f(n) d_{j+1}(n) + \sum_{n=N-k+2}^{N-k+j+1} f(n) d_{j+1}(n) \right) \\
&\quad - \left(\sum_{n=j}^{k-1} f(n) d_j(n) + \sum_{n=N-k+2}^{N-k+j} f(n) d_j(n) \right) + \sum_{n=k}^{N-k+1} f(n) \left(d_{j+1}(n) - d_j(n) \right) \Bigg] \\
&= \sum_{j=1}^{k-1} \left(\int_{\partial(\mathscr{H})} f dM_j + \int_{\mathring{\mathscr{H}}} f d\Delta_j \right) \boldsymbol{E}_j .
\end{aligned}$$

□

Propositions 3.2 and 3.3 can be used to obtain a formula of integration by parts that is contained in the next statement.

Proposition 3.4. *Let $\mathscr{H} = (\mathscr{V}, w)$ be an enumerated k-regular hypergraph and let f and g be two functions defined on $\mathscr{V}$, then*

$$\int_{\mathcal{P}_k(\mathscr{V})} \boldsymbol{\nabla}_g f(e) d\mu(e) = -\int_{\mathcal{P}_k(\mathscr{V})} \boldsymbol{\nabla}_f g(e) d\mu(e) + \sum_{j=1}^{k-1} \left(\int_{\partial(\mathscr{H})} fg dM_j + \int_{\mathring{\mathscr{H}}} fg d\Delta_j \right) \boldsymbol{E}_j .$$

Proof. Since from Proposition 3.2

$$\boldsymbol{\nabla}_1 (fg)(e) = \boldsymbol{\nabla}_g f + \boldsymbol{\nabla}_f g,$$

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integrating both sides on $\mathcal{P}_k(\mathscr{V})$ with respect to $d\mu$, we get

$$\int_{\mathcal{P}_k(\mathscr{V})} \boldsymbol{\nabla}_g f d\mu = -\int_{\mathcal{P}_k(\mathscr{V})} \boldsymbol{\nabla}_f g d\mu + \int_{\mathcal{P}_k(\mathscr{V})} \boldsymbol{\nabla}_1(fg) d\mu.$$

Now, applying Proposition 3.3 with fg instead of f, we get the result. □

All the gradients above are vector fields in the set of hyperedges. Their components can be regarded as "partial derivatives". We shall further use the notation

$$D_h^j f(e) = \frac{1}{2}\left[h(i_{j+1}(e)) + h(i_j(e))\right]\left[f(i_{j+1}(e)) - f(i_j(e))\right],$$

for the j^{th} component of $\boldsymbol{\nabla}_h f(e)$, $j = 1, \dots, k-1$. This notation give natural coordinate-wise versions of Propositions 3.2, 3.3 and 3.4.

In the next section we shall make use of the result contained in the next statement.

Lemma 3.5. *Let $\mathscr{H} = (\mathscr{V}, w)$ be an enumerated k-regular hypergraph. Let ν be the positive measure on $\mathscr{V}$ given by $\nu(V) = \frac{1}{2}\sum_{n\in V}\sum_{j=1}^{k-1}[d_{j+1}(n) + d_j(n)]$ for $V \subseteq \mathscr{V}$. Set $\mathcal{I} : \mathscr{V} \to \{1, \dots, N\}$ to denote the index function of each vertex in the given enumeration of $\mathscr{H}$, $\mathcal{I}(v_i) = i$. Then for every nonnegative function g on $\mathscr{V}$, we have*

(a) $D_g^j \mathcal{I}(e) \geq 0$, for every $j = 1, \dots, k-1$ and every $e \in \mathcal{P}_k(\mathscr{V})$;

(b)

$$\sum_{j=1}^{k-1}\int_{\mathcal{P}_k(\mathscr{V})} D_g^j \mathcal{I} d\mu \geq \int_{\mathscr{V}} g d\nu.$$

Proof. Notice first that for fixed $j = 1, \dots, k-1$ and $e \in \mathcal{P}_k(\mathscr{V})$ we have that

$$\begin{aligned} D_g^j \mathcal{I}(e) &= \frac{1}{2}\left[g(i_{j+1}(e)) + g(i_j(e))\right]\left[\mathcal{I}(i_{j+1}(e)) - \mathcal{I}(i_j(e))\right] \\ &= \frac{1}{2}\left[g(i_{j+1}(e)) + g(i_j(e))\right]\left[i_{j+1}(e) - i_j(e)\right] \geq \frac{1}{2}\left[g(i_{j+1}(e)) + g(i_j(e))\right]. \end{aligned}$$

In order to prove *(b)* we use the last inequality and the partitions of $\mathcal{P}_k(\mathscr{V})$ introduced in the proof of Proposition 3.3,

$$\begin{aligned} \sum_{j=1}^{k-1}\int_{\mathcal{P}_k(\mathscr{V})} D_g^j \mathcal{I} d\mu &\geq \frac{1}{2}\sum_{j=1}^{k-1}\int_{e\in\mathcal{P}_k(\mathscr{V})} \left[g(i_{j+1}(e)) + g(i_j(e))\right] d\mu(e) \\ &= \frac{1}{2}\sum_{j=1}^{k-1}\int_{e\in\mathcal{P}_k(\mathscr{V})} g(i_{j+1}(e)) d\mu(e) + \frac{1}{2}\sum_{j=1}^{k-1}\int_{e\in\mathcal{P}_k(\mathscr{V})} g(i_j(e)) d\mu(e) \\ &= \frac{1}{2}\sum_{j=1}^{k-1}\left[\sum_{n=1}^{N}\int_{e\in i_{j+1}^{-1}(n)} g(n) d\mu(e) + \sum_{n=1}^{N}\int_{e\in i_j^{-1}(n)} g(n) d\mu(e)\right] \\ &= \frac{1}{2}\sum_{j=1}^{k-1}\sum_{n=1}^{N} g(n)[\mu(i_{j+1}^{-1}(n)) + \mu(i_j^{-1}(n))] \end{aligned}$$

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$$= \sum_{n=1}^{N} g(n) \frac{1}{2} \sum_{j=1}^{k-1} [d_{j+1}(n) + d_j(n)]$$
$$= \int_{\mathscr{V}} g d\nu.$$

□

4. Wave functions, position and momentum operators on k-regular hypergraphs

So far, given a weighted hypergraph $\mathscr{H} = (\mathscr{V}, w)$, we have two positive measure spaces $(\mathcal{P}_k(\mathscr{V}), \mu)$ and $(\mathscr{V}, \nu)$. Hence in both of them we have the natural Hilbert spaces $L^2(\mathcal{P}_k(\mathscr{V}), \mu)$ and $L^2(\mathscr{V}, \nu)$. Let us denote with single angular brackets $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\mathscr{V}, \nu)$ and with $\| \cdot \|$ the induced norm. The scalar product in $L^2(\mathcal{P}_k(\mathscr{V}), \mu)$ is denoted by double angular brackets $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and the corresponding norm $|\!|\!| \cdot |\!|\!|$.

A **wave function** is a real function ψ belonging to the unit sphere of $L^2(\mathscr{V}, \nu)$ vanishing on $\partial(\mathscr{H})$. Precisely $\psi : \mathscr{V} \to \mathbb{R}$, $\|\psi\|^2 = 1$ and $\psi(n) = 0$ for every $n \in \partial(\mathscr{H})$.

Given a wave function ψ, the **position operator** acting on ψ is the $k-1$-dimensional vector field defined on the hyperedges or, more generally, on $\mathcal{P}_k(\mathscr{V})$ by

$$\boldsymbol{Q}\psi(e) = \sum_{j=1}^{k-1} Q_j \psi(e) \boldsymbol{E}_j,$$

with

$$Q_j \psi(e) = \frac{1}{2}[i_{j+1}(e) + i_j(e)][\psi(i_{j+1}(e)) + \psi(i_j(e))].$$

The **momentum operator** applied to the wave function is the $k-1$-dimensional vector field defined on $\mathcal{P}_k(\mathscr{V})$ by

$$\boldsymbol{P}\psi(e) = \boldsymbol{\nabla}_1 \psi(e) = \sum_{j=1}^{k-1} P_j \psi(e) \boldsymbol{E}_j,$$

with

$$P_j \psi(e) = \psi(i_{j+1}(e)) - \psi(i_j(e)) = D_1^j \psi(e).$$

Notice that both, $\boldsymbol{P}$ and $\boldsymbol{Q}$, can be seen as operators applying functions from $L^2(\mathscr{V}, \nu)$ into functions $(L^2(\mathcal{P}_k(\mathscr{V}), \mu))^{k-1}$, which is a Hilbert space with the inner product

$$\langle\!\langle \boldsymbol{F}, \boldsymbol{G} \rangle\!\rangle = \sum_{j=1}^{k-1} \langle\!\langle F_j, G_j \rangle\!\rangle = \sum_{j=1}^{k-1} \int_{\mathcal{P}_k(\mathscr{V})} F_j G_j d\mu$$

and

$$|\!|\!| \boldsymbol{F} |\!|\!|^2 = \langle\!\langle \boldsymbol{F}, \boldsymbol{F} \rangle\!\rangle = \sum_{j=1}^{k-1} \int_{\mathcal{P}_k(\mathscr{V})} F_j^2 d\mu.$$

Let us observe that in some sense the position and momentum operators resemble the classical analogues. In fact, the difference of the values of ψ in consecutive vertices of a

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hyperedge, mimics the derivative that is the essential definition of moment. On the other hand, a simple way to regard the classical momentum in one dimension, is given by a multiplication operator on the wave function $\psi(x)$ with multiplier x.

5. The main result

Let us start by going back to the result in Proposition 3.4 taking as g any function on $\mathscr{V}$ vanishing on $\partial(\mathscr{H})$ and $f = \mathcal{I}$ the index function $\mathcal{I}(v_i) = i$.

Proposition 5.1. *Assume that g vanishes on $\partial(\mathscr{H})$. Then*

$$\int_{\mathcal{P}_k(\mathscr{V})} \boldsymbol{\nabla}_g \mathcal{I}(e) d\mu(e) = -\int_{\mathcal{P}_k(\mathscr{V})} \left(\frac{1}{2} \sum_{j=1}^{k-1} [i_{j+1}(e) + i_j(e)][g(i_{j+1}(e)) - g(i_j(e))] \boldsymbol{E}_j \right) d\mu(e)$$
$$+ \sum_{n \in \mathring{\mathscr{H}}} n g(n) \sum_{j=1}^{k-1} (d_{j+1}(n) - d_j(n)) \boldsymbol{E}_j.$$

Proof. Follows directly form Proposition 3.4, taking $f = \mathcal{I}$. □

When $\mathscr{H}$ is actually a simple graph we have only two distribution functions $d_1(n)$ and $d_2(n)$. The equation $d_1(n) = d_2(n)$ means that the number of edges "arriving" to the vertex n equals the number of edges "departing" from n. For weighted graphs it can be seen as the first Kirchhoff law for circuits. This remark suggests the next definition.

Definition 5.2. Let $\mathscr{H} = (\mathscr{V}, w)$ be an enumerated k-regular hypergraph. We shall say that $\mathscr{H}$ is of Kirchhoff (respectively sub-Kirchhoff) type if and only if for every $n \in \mathring{\mathscr{H}}$ we have that $d_j(n)$ is constant as a function of $j \in \{1, \dots, k\}$ (respectively $d_j(n) \geq d_{j+1}(n)$ for every j).

In the example introduced in Section 2 with $\mathscr{V} = \{1, \dots, 100\}$, $k = 3$ and $w(\{j, j+1, j+2\}) = 1$ and $w = 0$ otherwise, since $\mathring{\mathscr{H}} = \{3, 4, \dots, 98\}$, we see that $d_j(n) = 1$ for $j = 1, 2, 3$ and $n \in \mathring{\mathscr{H}}$. So that this is a Kirchhoff type hypergraph.

Under the hypothesis of sub-Kirchhoff type for k-regular hypergraphs, the coordinate-wise version of Proposition 5.1 and $g \geq 0$, we obtain the following result.

Corollary 5.3. *Assume that $g \geq 0$ vanishes on $\partial(\mathscr{H})$ and that $\mathscr{H}$ is of sub-Kirchhoff type. Then, for each $j = 1, \dots, k-1$ we have*

$$\int_{\mathcal{P}_k(\mathscr{V})} D_g^j \mathcal{I}(e) d\mu(e) \leq -\int_{\mathcal{P}_k(\mathscr{V})} \frac{1}{2} [i_{j+1}(e) + i_j(e)][g(i_{j+1}(e)) - g(i_j(e))] d\mu(e).$$

Proof. Notice that, since $g \geq 0$ and from the sub-Kirchhoff property of $\mathscr{H}$, the second term in the right hand side of the formula in Proposition 5.1 is negative or vanishes. □

We are finally in position to state and prove our main result.

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Theorem 5.4. *Let $\mathscr{H} = (\mathscr{V}, w)$ be a k-regular sub-Kirchhoff hypergraph. Then, the inequality*

$$\|\boldsymbol{P}\psi\| \|\boldsymbol{Q}\psi\| \geq 1$$

holds for every wave function ψ defined on the vertex set $\mathscr{V}$.

Proof. From *(b)* in Lemma 3.5, with $g = \psi^2$

$$1 = \int_{\mathscr{V}} \psi^2 d\nu \leq \sum_{j=1}^{k-1} \int_{\mathcal{P}_k(\mathscr{V})} D^j_{\psi^2}\mathcal{I} d\mu.$$

Hence, applying Corollary 5.3 with $g = \psi^2$ we obtain

$$\begin{aligned}
1 &\leq -\sum_{j=1}^{k-1} \int_{e\in\mathcal{P}_k(\mathscr{V})} \frac{1}{2}[i_{j+1}(e) + i_j(e)][\psi^2(i_{j+1}(e)) - \psi^2(i_j(e))] d\mu(e) \\
&= -\sum_{j=1}^{k-1} \int_{e\in\mathcal{P}_k(\mathscr{V})} \frac{1}{2}[i_{j+1}(e) + i_j(e)][\psi(i_{j+1}(e)) + \psi(i_j(e))][\psi(i_{j+1}(e)) - \psi(i_j(e))] d\mu(e) \\
&= -\sum_{j=1}^{k-1} \int_{e\in\mathcal{P}_k(\mathscr{V})} Q_j\psi(e) P_j\psi(e) d\mu(e) \\
&\leq \|\boldsymbol{P}\psi\| \|\boldsymbol{Q}\psi\|,
\end{aligned}$$

where the last inequality follows from Schwartz inequality in $(L^2(\mathcal{P}_k(\mathscr{V}), \mu))^{k-1}$. □

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