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BOUNDEDNESS PROPERTIES OF THE BILINEAR FRACTIONAL INTEGRAL OPERATORS INDUCED BY HYPERMETRICS OF THIRD ORDER

HUGO AIMAR, IVANA GÓMEZ, AND JOAQUÍN TOLEDO

Abstract. We introduce a natural bilinear fractional integral type operator induced by a third order hypermetric on Ahlfors regular quasi-metric spaces. Given a quasi-metric space (X,d) the function $\rho(x,y,z)$, defined as the distance, in X^3, of (x,y,z) to the diagonal $\triangle_3=\{(x,x,x)\in X^3 : x\in X\}$ is said to be a third order hypermetric in X. When (X,d) is a Euclidean space or, more generally, when (X,d,μ) is η-Ahlfors regular for some η positive, the function $\rho(x,y,z)$ generates kernels for bilinear operators of the type $T^\gamma(f,g)(x)=\iint_{X\times X}\rho(x,y,z)^{-\gamma}f(y)g(z)d\mu(y)d\mu(z)$, for a given positive γ. In the setting of η-Ahlfors regular space, the power $-\gamma=-2\eta$ of $\rho(x,\cdot,\cdot)$ provides the natural singularity for this family of kernels. In this paper we consider the fractional integral rank $0<\gamma<2\eta$. We prove boundedness properties of the type $\|T^\gamma(f,g)\|_{p_3}\leq C\|f\|_{p_1}\|g\|_{p_2}$ for adequate values of the exponents p_1,p_2 and p_3. The proof is based on three upper bounds for $T^\gamma(f,g)$ in terms of the classical linear fractional Riesz operators $I_{\eta-\frac{\gamma}{2}}$, using the linear Hardy-Littlewood-Sobolev inequality.

1. Introduction

The classical linear fractional integral operator I_α, in particular its boundedness properties on Lebesgue space, has been extended to the multilinear setting. See for example [Gra92], [KS99], [Gra14]. On the other hand the boundedness of the linear Riesz operator I_α in Lebesgue spaces, that in the Euclidean setting can be found for example in [Ste70], was extended to metric measure spaces in [GCG04]. The main results in all these papers provide extensions of the Hardy-Littlewood-Sobolev inequality.

In [AGT26] we introduced the notion of hypermetric of order k on a quasi-metric space. The basic idea is provided by the fact that given two different points x and y in the metric space (X,d) we have that $d(x,y)$ is equivalent to $\rho(x,y)$ given as the distance of the pair $(x,y)\in X\times X$ to the diagonal $\triangle_2$ of $X\times X$ with respect to the metric $d^{(2)}((x,y);(x',y'))=\max\{d(x,x'),d(y,y')\}$. In fact, from the triangle inequality for d we easily see that $\frac{1}{2}d(x,y)\leq\rho(x,y)\leq d(x,y)$. When instead of a metric space we have a quasi-metric space (X,d) with triangular constant $\kappa\geq 1$, we have that $\frac{1}{2\kappa}d(x,y)\leq\rho(x,y)\leq d(x,y)$ for every $x,y\in X$. For a given quasi-metric space (X,d), we consider the

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product space $(X^3, d^{(3)})$ with $X^3 = X \times X \times X$ and $d^{(3)}(\boldsymbol{x}, \boldsymbol{y}) = \sup\{d(x_i, y_i), i = 0, 1, 2\}$, where $\boldsymbol{x} = (x_0, x_1, x_2)$ and $\boldsymbol{y} = (y_0, y_1, y_2)$. Set $\triangle_3 = \{(x, x, x) \in X^3 : x \in X\}$ to denote the diagonal of X^3. The **third order hypermetric** induced by d in X is defined by $\rho(\boldsymbol{x}) = \rho(x_0, x_1, x_2) = d^{(3)}(\boldsymbol{x}, \triangle_3)$. We shall use (x, y, z) to denote the points in X^3. Hence $\rho(x, y, z) = d^{(3)}\left((x, y, z), \triangle_3\right) = \inf_{u \in X} d^{(3)}\left((x, y, z), (u, u, u)\right)$.The function ρ defined in X^3 generates a wide family of kernels for bilinear operators of the type $\varphi\left(\rho(x, y, z)\right)$.

Recall that for $\eta > 0$ given, a space of homogeneous type (X, d, μ) is said to be η-Ahlfors regular, if there exist two constants $0 < a \leq A < \infty$ such that the inequalities $ar^\eta \leq \mu\left(B_d(x, r)\right) \leq Ar^\eta$ hold for every $x \in X$ and every $r > 0$. Here $B_d(x, r) = \{y \in X : d(x, y) < r\}$ is the d-ball centered at x with radius $r > 0$. The bilinear integral operators induced by ρ and φ have the general form

$$T_\varphi(f, g)(x) = \iint_{X \times X} \varphi(\rho(x, y, z)) f(y) g(z) d\mu(y) d\mu(z)$$

under adequate conditions on φ, f and g that guarantee the existence of the integral.

The next result provides a necessary and sufficient condition on a nonincreasing, nonnegative function $\varphi(t)$ of the positive variable t in order to have the convergence of the integral $\iint_{X \times X} \varphi(\rho(x, y, z)) d\mu(y) d\mu(z)$ on an η-Ahlfors regular space.

Lemma 1.1. *Let (X, d, μ) be an η-Ahlfors regular space. Let $\varphi : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ be a nonincreasing function. Then, there exist two constants $0 < C_1 \leq C_2 < \infty$ depending only on the geometric constants κ, a and A such that the inequalities*

$$C_1 \int_0^\infty \varphi(t) t^{2\eta - 1} dt \leq \iint_{X \times X} \varphi(\rho(x, y, z)) d\mu(y) d\mu(z) \leq C_2 \int_0^\infty \varphi(t) t^{2\eta - 1} dt$$

hold for every $x \in X$.

We shall prove the above result in Section 2. Taking, $\varphi(t) = t^{-\alpha} \chi_{(0,1)}(t)$, $\alpha > 0$, we see that $\iint_{\{(y,z) : \rho(x,y,z) < 1\}} \rho^{-\alpha}(x, y, z) d\mu(y) d\mu(z)$ is finite if and only if $0 < \alpha < 2\eta$. On the other hand, taking $\varphi(t) = \chi_{(0,1)}(t) + t^{-\beta} \chi_{[1,\infty)}(t)$ in Lemma 1.1, we also see that $\iint_{\{(y,z) : \rho(x,y,z) \geq 1\}} \rho^{-\beta}(x, y, z) d\mu(y) d\mu(z)$ is finite if and only if $\beta > 2\eta$. Hence the power 2η of $\frac{1}{\rho(x,y,z)}$, for $x \in X$, determines the singularity of this family of kernels. Thus the natural fractional integral bilinear operator induced by the hypermetric ρ is given by

$$T^\gamma(f, g)(x) = \iint_{X \times X} \frac{f(y) g(z)}{\rho^\gamma(x, y, z)} d\mu(y) d\mu(z)$$

with $0 < \gamma < 2\eta$.

The main result of this paper, that we shall prove in Section 3, concerning the boundedness in Lebesgue spaces of T^γ is the following.

Theorem 1.2. *Let (X, d, μ) be an η-Ahlfors regular space with $\eta > 0$, and ρ as before. Let $0 < \gamma < 2\eta$. Then, for every $p_1 > 1$, $p_2 > 1$ and p_3 such that $0 < \frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{2\eta - \gamma}{\eta}$,*

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there exists $C > 0$ *such that the inequality*

$$\|T^{\gamma}(f,g)\|_{p_3} \leq C\|f\|_{p_1}\|g\|_{p_2}$$

holds for every couple f, g *of measurable nonnegative functions.*

2. PROOF OF LEMMA 1.1

The results of this section are contained in [AGT26], we include them for the sake of completeness. The proof of Lemma 1.1 is based on the following control of the sections at $x \in X$, $E(x,r) = \{(y,z) \in X \times X : \rho(x,y,z) < r\}$, of the neighborhood of $\triangle_3$ given by $\{(x,y,z) : \rho(x,y,z) < r\}$, $r > 0$. With μ^2 we shall denote the product measure $\mu \times \mu$ on $X \times X$.

Lemma 2.1. *Let* (X,d,μ) *be an* η*-Ahlfors regular space with geometric constants* κ*,* a *and* A*. Let* ρ *be the hypermetric of third order induced by* d*. Then, for every* $x \in X$ *and every* $r > 0$*, we have,*

(2.1.a) $B_d(x,r) \times B_d(x,r) \subset E(x,r) \subset B_d(x,2\kappa r) \times B_d(x,2\kappa r)$*, and*
(2.1.b) $a^2r^{2\eta} \leq \mu^2\left(E(x,r)\right) \leq (2\kappa)^{2\eta}A^2r^{2\eta}$*.*

Proof. Notice that *(2.1.b)* follows from *(2.1.a)* and the η-Ahlfors character of (X,d,μ). The first inclusion in *(2.1.a)* follows from the fact that if $(y,z) \in B_d(x,r) \times B_d(x,r)$, then $\rho(x,y,z) \leq \max\{d(x,x), d(x,y), d(x,z)\} < r$. Hence $(y,z) \in E(x,r)$. To prove the second inclusion in *(2.1.a)*, take now $(y,z) \in E(x,r)$. Then $\rho(x,y,z) < r$. So that, there exists $u \in X$ such that, $\max\{d(x,u), d(y,u), d(z,u)\} < r$. Hence $d(y,x) \leq \kappa(d(y,u) + d(u,x)) < 2\kappa r$ and $d(z,x) \leq \kappa(d(z,u) + d(u,x)) < 2\kappa r$, and $(y,z) \in B_d(x,2\kappa r) \times B_d(x,2\kappa r)$. □

Proof of Lemma 2.1. Set

$$J(x) := \iint_{X\times X} \varphi(\rho(x,y,z))d\mu(y)d\mu(z).$$

Take $x \in X$ and $r > 0$, then, since $\mu^2(\{(x,x)\}) = 0$, for $\lambda > 1$

$$J(x) = \sum_{j\in\mathbb{Z}} \iint_{\lambda^j \leq \rho(x,y,z) < \lambda^{j+1}} \varphi(\rho(x,y,z))d\mu(y)d\mu(z).$$

Hence from the nonincreasing condition on φ and *(2.1.a)* in Lemma 2.1,

$$\begin{aligned} J(x) &\leq \sum_{j\in\mathbb{Z}} \varphi\left(\lambda^j\right) \iint_{\lambda^j \leq \rho(x,y,z) < \lambda^{j+1}} d\mu(y)d\mu(z) \\ &\leq \sum_{j\in\mathbb{Z}} \varphi\left(\lambda^j\right) \mu^2\left(E(x,\lambda^{j+1})\right) \\ &\leq A^2(2\kappa)^{2\eta} \sum_{j\in\mathbb{Z}} \varphi\left(\lambda^j\right) \lambda^{2\eta(j+1)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda^{2\eta} A^2 (2\kappa)^{2\eta}}{\log \lambda} \sum_{j\in\mathbb{Z}} \int_{\lambda^{j-1}}^{\lambda^j} \varphi\left(\lambda^j\right) \lambda^{2\eta j} \frac{dt}{t} \\
&\leq \frac{\lambda^{4\eta} A^2 (2\kappa)^{2\eta}}{\log \lambda} \sum_{j\in\mathbb{Z}} \int_{\lambda^{j-1}}^{\lambda^j} \varphi\left(t\right) t^{2\eta-1} dt \\
&= \left(\frac{\lambda^{4\eta}}{\log \lambda}\right) A^2 (2\kappa)^{2\eta} \int_0^\infty \varphi(t) t^{2\eta-1} dt. \qquad (2.1)
\end{aligned}
$$

Let us consider now the lower bound for $J(x)$. From Lemma 2.1 we have that for $\lambda > 2\kappa$,

$$E(x,\lambda^{j+1}) \setminus E(x,\lambda^j) \supset [B_d(x,\lambda^{j+1}) \times B_d(x,\lambda^{j+1})] \setminus [B_d(x,2\kappa\lambda^j) \times B_d(x,2\kappa\lambda^j)].$$

Now,

$$
\begin{aligned}
J(x) &\geq \sum_{j\in\mathbb{Z}} \varphi\left(\lambda^{j+1}\right) \mu^2 \left(E(x,\lambda^{j+1}) \setminus E(x,\lambda^j)\right) \\
&\geq \sum_{j\in\mathbb{Z}} \varphi\left(\lambda^{j+1}\right) \mu^2 \left([B_d(x,\lambda^{j+1}) \times B_d(x,\lambda^{j+1})] \setminus [B_d(x,2\kappa\lambda^j) \times B_d(x,2\kappa\lambda^j)]\right).
\end{aligned}
$$

From the Ahlfors character of the space,

$$
\begin{aligned}
&\mu^2 \left([B_d(x,\lambda^{j+1}) \times B_d(x,\lambda^{j+1})] \setminus [B_d(x,2\kappa\lambda^j) \times B_d(x,2\kappa\lambda^j)]\right) \\
&\qquad = \mu^2 \left(B_d(x,\lambda^{j+1}) \times B_d(x,\lambda^{j+1})\right) - \mu^2 \left(B_d(x,2\kappa\lambda^j) \times B_d(x,2\kappa\lambda^j)\right) \\
&\qquad \geq a^2 (\lambda^{j+1})^{2\eta} - A^2 (2\kappa\lambda^j)^{2\eta} \\
&\qquad = \lambda^{2\eta j} \left[a^2 \lambda^{2\eta} - A^2 2^{2\eta} \kappa^{2\eta}\right].
\end{aligned}
$$

Hence, if we choose $\lambda = \dfrac{\left(1 + A^2 (2\kappa)^{2\eta}\right)^{\frac{1}{2\eta}}}{a^{\frac{1}{\eta}}}$, we get

$$
\begin{aligned}
J(x) &\geq \sum_{j\in\mathbb{Z}} \varphi(\lambda^{j+1}) \lambda^{2\eta j} \\
&\geq \frac{1}{(\lambda^{2\eta})^2} \frac{1}{\log \lambda} \sum_{j\in\mathbb{Z}} \int_{\lambda^{j+1}}^{\lambda^{j+2}} \varphi(t) t^{2\eta} \frac{dt}{t} \\
&\geq \frac{1}{\lambda^{4\eta} \log \lambda} \int_0^\infty \varphi(t) t^{2\eta-1} dt.
\end{aligned}
$$

This inequality together with (2.1), gives the result. □

3. PROOF OF THEOREM 1.2

The next elementary geometric observation will be useful at describing the type regions for the operator T^γ.

Lemma 3.1. *Let $0 < \sigma < 1$ be given and*

$$\Omega_\sigma = \{(r,s) \in (0,1)^2 : r + s > 2\sigma\}.$$

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Set $A_\sigma = (\sigma,1)^2$, $B_\sigma = \{(r,s) \in (0,1)^2 : s > \sigma \text{ and } 2\sigma < r+s < 1+\sigma\}$ *and* $C_\sigma = \{(r,s) \in (0,1)^2 : r > \sigma \text{ and } 2\sigma < r+s < 1+\sigma\}$ *(see Figure 1). Then*

$$\Omega_\sigma = A_\sigma \cup B_\sigma \cup C_\sigma.$$

Proof. Notice first that $A_\sigma \cup B_\sigma \cup C_\sigma \subset \Omega_\sigma$. Let us now prove that Ω_σ is covered by $A_\sigma \cup B_\sigma \cup C_\sigma$. In fact, if $(r,s) \notin (\sigma,1)^2$ but $(r,s) \in (0,1)^2$ and $r+s > 2\sigma$, then $0 < r \leq \sigma$ or $0 < s \leq \sigma$. Assume that $0 < r \leq \sigma$. Hence since $r+s > 2\sigma$, we necessarily have that $s > \sigma$. Also $r+s \leq \sigma + s < \sigma + 1$. Hence $(r,s) \in B_\sigma$. In case $0 < s \leq \sigma$, we see in the same way that $(r,s) \in C_\sigma$. □

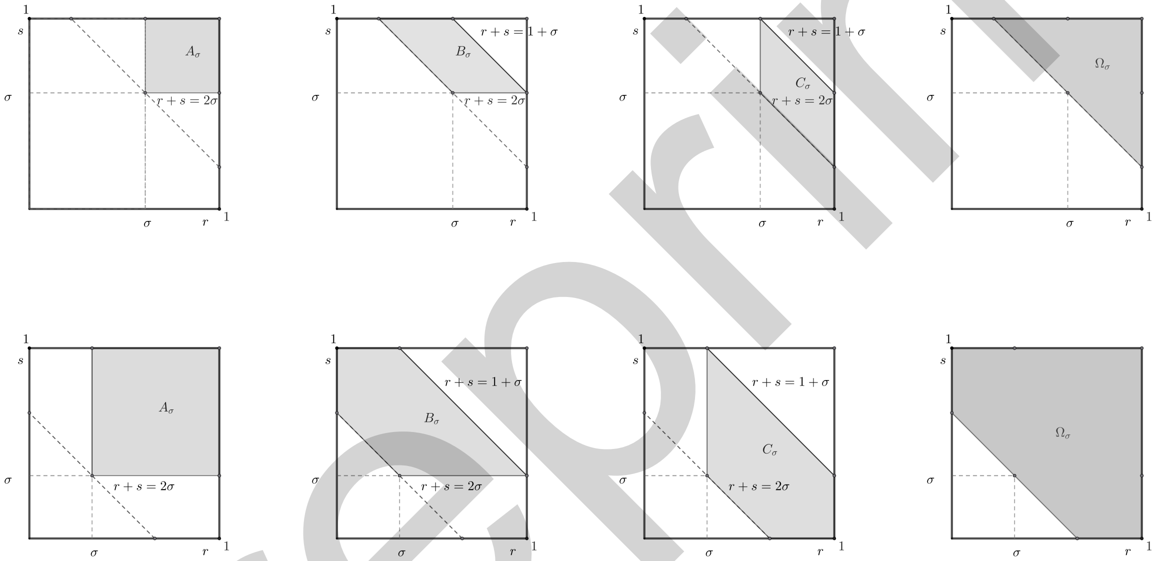

Figure 1. Pictures of A_σ, B_σ, C_σ and Ω_σ. The first row depicts the four regions when $\frac{1}{2} < \sigma < 1$. The second row when $0 < \sigma < \frac{1}{2}$.

The next result provides the three basic estimates of the kernel of the operator T^γ.

Proposition 3.2. *Let* (X,d,μ) *be an* η*-Ahlfors regular space with* $\eta > 0$*. Let* $X^3, d^{(3)}, \triangle_3$ *and* ρ *be defined as in the above section. Then,*

(3.2.a) the kernel $\rho^{-\gamma}(x,y,z)$ *is bounded above by*

$$(2\kappa)^\gamma \min\left\{d(x,y)^{-\frac{\gamma}{2}}d(x,z)^{-\frac{\gamma}{2}}, d(x,y)^{-\frac{\gamma}{2}}d(y,z)^{-\frac{\gamma}{2}}, d(x,z)^{-\frac{\gamma}{2}}d(z,y)^{-\frac{\gamma}{2}}\right\}$$

for every $(x,y,z) \in X^3$ *and every* $\gamma > 0$*;*

(3.2.b) for f *and* g *two measurable nonnegative real functions and* $0 < \gamma < 2\eta$*, we have* $T^\gamma(f,g)(x)$ *is bounded above by*

$$(2\kappa)^\gamma \min\left\{\left(I_{\eta-\frac{\gamma}{2}}f(x)\right)\left(I_{\eta-\frac{\gamma}{2}}g(x)\right), I_{\eta-\frac{\gamma}{2}}\left(fI_{\eta-\frac{\gamma}{2}}g\right)(x), I_{\eta-\frac{\gamma}{2}}\left(gI_{\eta-\frac{\gamma}{2}}f\right)(x)\right\}$$

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for every $x \in X$*, where* I_α *denotes the linear fractional integral operator of order* α *in the space* (X, d, μ)*.*

Proof. Let us first check (3.2.a). Given $(x, y, z) \in X^3$ and $\varepsilon > 0$, there exists $u_\varepsilon \in X$ such that

$$\rho(x, y, z) + \varepsilon \geq d^{(3)}\left((x, y, z), (u_\varepsilon, u_\varepsilon, u_\varepsilon)\right) = \max\{d(x, u_\varepsilon), d(y, u_\varepsilon), d(z, u_\varepsilon)\}.$$

Now from triangle inequality, $d(x, y) \leq \kappa\left(d(x, u_\varepsilon) + d(u_\varepsilon, y)\right) \leq 2\kappa\rho(x, y, z) + 2\kappa\varepsilon$; $d(y, z) \leq \kappa\left(d(y, u_\varepsilon) + d(u_\varepsilon, z)\right) \leq 2\kappa\rho(x, y, z) + 2\kappa\varepsilon$, and $d(x, z) \leq \kappa\left(d(x, u_\varepsilon) + d(u_\varepsilon, z)\right) \leq 2\kappa\rho(x, y, z) + 2\kappa\varepsilon$. Hence

$$\rho^{-\frac{\gamma}{2}}(x, y, z) \leq (2\kappa)^{-\frac{\gamma}{2}} \min\left\{d(x, y)^{-\frac{\gamma}{2}}, d(x, z)^{-\frac{\gamma}{2}}, d(y, z)^{-\frac{\gamma}{2}}\right\}.$$

So that the square of the left hand side of the above inequality is bounded by the product of any two of the three numbers on the right. In other words

$$\rho^{-\gamma}(x, y, z) \leq (2\kappa)^{\gamma} \min\left\{d(x, y)^{-\frac{\gamma}{2}} d(x, z)^{-\frac{\gamma}{2}}, d(x, y)^{-\frac{\gamma}{2}} d(y, z)^{-\frac{\gamma}{2}}, d(x, z)^{-\frac{\gamma}{2}} d(z, y)^{-\frac{\gamma}{2}}\right\},$$

as desired.

Let us prove (3.2.b). Given $f \geq 0$ and $g \geq 0$, from (3.2.a) we have

$$\begin{aligned} T^{\gamma}(f, g) &= \iint_{X \times X} \frac{f(y) g(z)}{\rho^{\gamma}(x, y, z)} d\mu(y) d\mu(z) \\ &\leq (2\kappa)^{\gamma} \left(\int_{y \in X} \frac{f(y)}{d^{\frac{\gamma}{2}}(x, y)} d\mu(y)\right) \left(\int_{z \in X} \frac{g(z)}{d^{\frac{\gamma}{2}}(x, z)} d\mu(z)\right) \\ &= (2\kappa)^{\gamma} \left(I_{\eta - \frac{\gamma}{2}} f\right)(x) \left(I_{\eta - \frac{\gamma}{2}} g\right)(x); \end{aligned}$$

$$\begin{aligned} T^{\gamma}(f, g) &= \iint_{X \times X} \frac{f(y) g(z)}{\rho^{\gamma}(x, y, z)} d\mu(y) d\mu(z) \\ &\leq (2\kappa)^{\gamma} \int_{y \in X} \frac{f(y)}{d^{\frac{\gamma}{2}}(x, y)} \left(\int_{z \in X} \frac{g(z)}{d^{\frac{\gamma}{2}}(y, z)} d\mu(z)\right) d\mu(y) \\ &= (2\kappa)^{\gamma} I_{\eta - \frac{\gamma}{2}} \left(f I_{\eta - \frac{\gamma}{2}} g\right)(x); \end{aligned}$$

and

$$\begin{aligned} T^{\gamma}(f, g) &= \iint_{X \times X} \frac{f(y) g(z)}{\rho^{\gamma}(x, y, z)} d\mu(y) d\mu(z) \\ &\leq (2\kappa)^{\gamma} \int_{z \in X} \frac{g(z)}{d^{\frac{\gamma}{2}}(x, z)} \left(\int_{y \in X} \frac{f(y)}{d^{\frac{\gamma}{2}}(z, y)} d\mu(y)\right) d\mu(z) \\ &= (2\kappa)^{\gamma} I_{\eta - \frac{\gamma}{2}} \left(g I_{\eta - \frac{\gamma}{2}} f\right)(x). \end{aligned}$$

□

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Proposition 3.3. *Let (X, d, μ) be an η-Ahlfors regular space with $\eta > 0$ and ρ as before. Let $0 < \gamma < 2\eta$ be given. Set $\sigma = \frac{2\eta-\gamma}{2\eta}$ and $\Pi_\sigma = \{(r, s, t) : r + s - t = 2\sigma\}$. Then, with the notation of Lemma 3.1 we have that*

(3.3.a) for every $(p_1, p_2; p_3)$ such that $\left(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}\right) \in \Pi_\sigma$ and $\left(\frac{1}{p_1}, \frac{1}{p_2}\right) \in A_\sigma$, there exists $C > 0$ such that

$$\left\| \left(I_{\eta-\frac{\gamma}{2}} f\right) \left(I_{\eta-\frac{\gamma}{2}} g\right) \right\|_{p_3} \leq C \|f\|_{p_1} \|g\|_{p_2}$$

for $f \in L^{p_1}(X, \mu)$ and $g \in L^{p_2}(X, \mu)$;

(3.3.b) for every $(p_1, p_2; p_3)$ such that $\left(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}\right) \in \Pi_\sigma$ and $\left(\frac{1}{p_1}, \frac{1}{p_2}\right) \in B_\sigma$, there exists $C > 0$ such that

$$\left\| I_{\eta-\frac{\gamma}{2}} \left(f I_{\eta-\frac{\gamma}{2}} g\right) \right\|_{p_3} \leq C \|f\|_{p_1} \|g\|_{p_2}$$

for $f \in L^{p_1}(X, \mu)$ and $g \in L^{p_2}(X, \mu)$;

(3.3.c) for every $(p_1, p_2; p_3)$ such that $\left(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}\right) \in \Pi_\sigma$ and $\left(\frac{1}{p_1}, \frac{1}{p_2}\right) \in C_\sigma$, there exists $C > 0$ such that

$$\left\| I_{\eta-\frac{\gamma}{2}} \left(g I_{\eta-\frac{\gamma}{2}} f\right) \right\|_{p_3} \leq C \|f\|_{p_1} \|g\|_{p_2}$$

for $f \in L^{p_1}(X, \mu)$ and $g \in L^{p_2}(X, \mu)$.

Proof. Let us first prove *(3.3.a)*. Take $\left(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}\right) \in \Pi_\sigma$ with $\left(\frac{1}{p_1}, \frac{1}{p_2}\right) \in A_\sigma$. So that

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = 2\sigma \tag{3.1}$$

$$1 > \frac{1}{p_1} > \sigma \tag{3.2}$$

$$1 > \frac{1}{p_2} > \sigma. \tag{3.3}$$

From the first inequality in (3.2) we have that $p_1 > 1$, from the second, $\frac{1}{\pi_1} := \frac{1}{p_1} - \sigma = \frac{1}{p_1} - \frac{\eta - \frac{\gamma}{2}}{\eta} > 0$. From (3.3), the same argument shows that $\frac{1}{\pi_2} := \frac{1}{p_2} - \sigma = \frac{1}{p_2} - \frac{\eta - \frac{\gamma}{2}}{\eta} > 0$. Moreover, from (3.1) we have

$$\frac{1}{\pi_1} + \frac{1}{\pi_2} = \frac{1}{p_1} - \sigma + \frac{1}{p_2} - \sigma = \frac{1}{p_1} + \frac{1}{p_2} - 2\sigma = \frac{1}{p_3}.$$

Hence, we can apply first Hölder inequality and then Hardy-Littlewood-Sobolev twice to obtain

$$\begin{aligned} \left\| \left(I_{\eta-\frac{\gamma}{2}} f\right) \left(I_{\eta-\frac{\gamma}{2}} g\right) \right\|_{p_3} &\leq \left\| I_{\eta-\frac{\gamma}{2}} f \right\|_{\pi_1} \left\| I_{\eta-\frac{\gamma}{2}} g \right\|_{\pi_2} \\ &\leq C \|f\|_{p_1} \|g\|_{p_2}, \end{aligned}$$

as desired.

In order to prove *(3.3.b)*, take $\left(\frac{1}{p_1},\frac{1}{p_2},\frac{1}{p_3}\right)\in\Pi_\sigma$ with $\left(\frac{1}{p_1},\frac{1}{p_2}\right)\in B_\sigma$. Recall that the conditions are the following

$$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=2\sigma \tag{3.4}$$

$$\frac{1}{p_1}+\frac{1}{p_2}>2\sigma \tag{3.5}$$

$$\frac{1}{p_1}+\frac{1}{p_2}<1+\sigma \tag{3.6}$$

$$\frac{1}{p_2}>\sigma \tag{3.7}$$

Let us proceed to introduce two new Lebesgue exponents given in terms of $\left(\frac{1}{p_1},\frac{1}{p_2},\frac{1}{p_3}\right)$. Set $\frac{1}{q_1}=\frac{1}{p_3}+\sigma$ and $\frac{1}{q_2}=\frac{1}{p_2}-\sigma$. Notice first that $1<q_1<\infty$. In fact, it is clear that $q_1<\infty$. Let us show that $q_1>1$. From (3.4) and (3.6) we see that $\frac{1}{q_1}=\frac{1}{p_3}+\sigma=\frac{1}{p_1}+\frac{1}{p_2}-2\sigma+\sigma=\frac{1}{p_1}+\frac{1}{p_2}-\sigma<1$. Let us check that $q_2>1$. From (3.7) we see that $\frac{1}{q_2}=\frac{1}{p_2}-\sigma>0$. Now $\frac{1}{p_2}<\frac{1}{p_1}+\frac{1}{p_2}<1+\sigma$, from (3.6), hence $0<\frac{1}{q_2}=\frac{1}{p_2}-\sigma<1$. Now we observe that p_1, q_1, and q_2 are related by the Hölder identity through $\frac{1}{q_1}=\frac{1}{p_1}+\frac{1}{q_2}$. In fact $\frac{1}{q_1}-\frac{1}{q_2}=\frac{1}{p_3}+\sigma-\frac{1}{p_2}+\sigma=\frac{1}{p_3}-\frac{1}{p_2}+2\sigma=\frac{1}{p_1}$, form (3.4). Now p_3 and q_1, on one hand, and q_2 and p_2, on the other, are related by the Hardy-Littlewood-Sobolev formulae for the exponents in the η dimensional space (X,d,μ). To wit

$$\frac{1}{q_1}-\frac{\eta-\frac{\gamma}{2}}{\eta}=\frac{1}{q_1}-\sigma=\frac{1}{p_3},$$

and

$$\frac{1}{p_2}-\frac{\eta-\frac{\gamma}{2}}{\eta}=\frac{1}{p_2}-\sigma=\frac{1}{q_2}.$$

Hence the following estimates are obtained by applying twice the Hardy-Littlewood-Sobolev theorem and once Hölder inequality, with constant C that may change from to line,

$$\begin{aligned}\left\|I_{\eta-\frac{\gamma}{2}}\left(fI_{\eta-\frac{\gamma}{2}}g\right)\right\|_{p_3}&\leq C\|fI_{\eta-\frac{\gamma}{2}}g\|_{q_1}\\&\leq C\|f\|_{p_1}\|I_{\eta-\frac{\gamma}{2}}g\|_{q_2}\\&\leq C\|f\|_{p_1}\|g\|_{p_2},\end{aligned}$$

as desired.

In order to prove *(3.3.c)* we only have to observe that when $\left(\frac{1}{p_1},\frac{1}{p_2},\frac{1}{p_3}\right)\in\Pi_\sigma$ and $\left(\frac{1}{p_1},\frac{1}{p_2}\right)\in C_\sigma$ condition (3.4), (3.5) and (3.6) are symmetric with respect to the variables $r=\frac{1}{p_1}$ and $s=\frac{1}{p_2}$. For (3.7), notice that now the condition becomes $\frac{1}{p_1}>\sigma$. Hence the result follows as in the case *(3.3.b)*. □

Now the proof of Theorem 1.2 follows readily from Lemma 3.1, Proposition 3.2 and Proposition 3.3.

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