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A $T1$ CRITERION FOR SCHRÖDINGER–CALDERÓN–ZYGMUND OPERATORS WITH EXPONENTIAL DECAY

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Abstract. We establish the boundedness of exponential Schrödinger–Calderón–Zygmund operators on weighted $\mathrm{BMO}^\alpha_\rho(w)$ spaces via a $T1$ criterion, where the weights belong to classes that capture the exponential decay of the operators, and ρ is a critical radius function. Specifically, we prove that the boundedness of such an operator T on $\mathrm{BMO}^\alpha_\rho(w)$ is equivalent to a natural oscillation condition on $T1$ over sub-critical balls. The weight classes considered, introduced in connection with ρ, include and extend the classical A^ρ_p weights, and are well-adapted to the exponential decay of the kernels. As applications, we derive weighted endpoint estimates for several operators associated to the generalized Schrödinger operator $\mathcal{L}_\mu = -\Delta + \mu$, including Riesz transforms, Laplace transform-type multipliers, maximal operators for the heat and Poisson semigroups, Littlewood–Paley functions and fractional integral operators. When $d\mu(x) = V(x)dx$, the results above extend the known endpoint estimates to larger classes of weights.

1. Introduction and main result

A classical result in harmonic analysis, due to David and Journé ([15]), establishes that a Calderón–Zygmund operator T is bounded on $L^2(\mathbb{R}^d)$ if and only if $T1$ and T^*1 belong to $\mathrm{BMO}(\mathbb{R}^d)$. This $T1$ theorem has since been extended in various directions, including endpoint estimates on BMO spaces and their weighted counterparts. In the context of Schrödinger operators, where the geometry is governed by a critical radius function ρ, analogues of these spaces and operators have been developed, and $T1$-type criteria have been obtained (see, for instance, [7, 12, 21]).

The operators arising in the Schrödinger setting present an additional feature compared with the classical case, as their kernels may exhibit not only the standard Calderón–Zygmund size and smoothness conditions, but also an extra decay depending on the ratio $|x-y|/\rho(x)$. This decay reflects the influence of the potential on the geometry and requires weight classes adapted to it. In [7], a $T1$ criterion was established for operators with polynomial decay, with weights in the class A^ρ_p introduced in [11], on weighted regularity spaces $\mathrm{BMO}^\alpha_\rho(w)$ (see definition below).

The aim of this article is to give a $T1$ criterion for the boundedness on $\mathrm{BMO}^\alpha_\rho(w)$ for exponential Schrödinger–Calderón–Zygmund operators, defined in [14], as well as for a fractional version of these operators introduced here, where the weights are associated with a critical radius function ρ belonging to the class introduced in [1], and are larger than A^ρ_p.

Before stating our main theorem, we begin by introducing some notation and definitions; further details will be provided in the following sections.

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By a **critical radius function** we will mean a function $\rho : \mathbb{R}^d \to [0,\infty)$ for which there exist constants $k_0, C_0 \geq 1$ such that

$$C_0^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C_0\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}, \tag{1.1}$$

for $x, y \in \mathbb{R}^d$. This function behaves like a constant at a local scale, since it is easy to check that

$$C_0^{-1}2^{-k_0}\rho(x) \leq \rho(y) \leq C_0 2^{\frac{k_0}{k_0+1}}\rho(x), \tag{1.2}$$

when $|x-y| \leq \rho(x)$. On the other hand, at a global scale it can neither increase nor decrease faster than a polynomial depending on the normalized distance $|x-y|/\rho(x)$.

In the context of Schrödinger operators with potential V, a specific function ρ can be defined that satisfies (1.1) and shows how V influences the geometry around each point. Nevertheless, the abstract formulation above allows one to develop the theory independently of the particular potential.

A ball of the form $B(x,\rho(x))$, with $x \in \mathbb{R}^d$, will be a **critical ball**. When the radius $r \leq \rho(x)$ we will say the ball $B(x,r)$ is **sub-critical**, and when $r > \rho(x)$ we will call it **super-critical**. We denote by $\mathcal{B}_\rho$ the family of all sub-critical balls, that is,

$$\mathcal{B}_\rho = \left\{B(x,r) : x \in \mathbb{R}^d, r \leq \rho(x)\right\}. \tag{1.3}$$

From (1.2), note that even though the radii can be comparable when $|x-y| \leq \rho(x)$, it does not necessarily follow that $|x-y| \leq \rho(y)$. In fact, if $y \in B(x,\rho(x))$ (a critical ball), then $x \in B(y, C_0 2^{k_0}\rho(y))$, which is super-critical since $C_0, k_0 \geq 1$.

Given a weight w, i.e., a nonnegative locally integrable function, and a number $0 \leq \alpha < 1$, we say that a measurable function f belongs to $\mathrm{BMO}^\alpha_\rho(w)$ if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and satisfies

$$\frac{1}{w(B)}\int_B |f - f_B| \leq C|B|^{\alpha/d}, \quad \text{for every ball } B = B(x,r) \tag{1.4}$$

and

$$\frac{1}{w(B)}\int_B |f| \leq C|B|^{\alpha/d}, \qquad \text{for every ball } B = B(x,r) \text{ with } r \geq \rho(x). \tag{1.5}$$

Here, f_B stands for the average of f over the ball B. As usual, the norm $\|f\|_{\mathrm{BMO}^\alpha_\rho(w)}$ is defined as the smallest constants in (1.4) and (1.5). When $w \equiv 1$, we will denote this space by $\mathrm{BMO}^\alpha_\rho(\mathbb{R}^d)$.

Indeed, since the validity of (1.5) for some ball B implies (1.4) for the same ball, it is sufficient to ask for the first condition to hold only on balls $B = B(x,r)$ with $r < \rho(x)$. Moreover, by virtue of [9, Proposition 3], if w is a doubling weight, then we might ask condition (1.5) to be true only on critical balls.

We now define the classes of singular and fractional operators we will consider, which are included in the classes studied in [4, 7, 8]. The singular case ($\nu = 0$) was already presented in our previous article [14].

Definition 1.1. For $0 \leq \nu < d$, $\frac{d}{d-\nu} < s < \infty$ and $0 < \delta \leq 1$ we say that T is an **exponential Schrödinger–Calderón–Zygmund operator of** (ν, s, δ) **type with parameters** c **and** m if

(i) T is of weak type $\left(s', \frac{s'd}{d-s'\nu}\right)$;

(ii) T has an associated kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ in the sense that

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy, \qquad f \in L^{s'}(\mathbb{R}^d) \text{ and } x \notin \mathrm{supp}(f),$$

where the kernel K satisfies the following conditions: there exists a constant C such that

$$\left(\frac{1}{R^d}\int_{R<|x_0-y|\leq 2R} |K(x,y)|^s dy\right)^{\frac{1}{s}} \leq \frac{C}{R^{d-\nu}}\exp\left(-c\left(1+\frac{R}{\rho(x)}\right)^m\right), \tag{1.6}$$

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whenever $|x-x_0|<R/2$, and

$$\left(\frac{1}{R^d}\int_{R<|x_0-y|\leq 2R}|K(x,y)-K(x_0,y)|^s dy\right)^{\frac{1}{s}}\leq \frac{C}{R^{d-\nu}}\left(\frac{r}{R}\right)^{\delta}, \tag{1.7}$$

for every $|x-x_0|<r\leq\rho(x_0)$ and $r<R/2$.

The limiting case $s=\infty$ is defined by pointwise estimates as we will see below.

Definition 1.2. For $0\leq\nu<d$ and $0<\delta\leq 1$ we say that T is an **exponential Schrödinger–Calderón–Zygmund operator of (ν,∞,δ) type with parameters c and m** if

(i) T is of weak type $(1,\frac{d}{d-\nu})$;

(ii) T has an associated kernel $K:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ in the sense that

$$Tf(x)=\int_{\mathbb{R}^d}K(x,y)f(y)dy,\qquad f\in L^1(\mathbb{R}^d)\text{ and }x\notin\operatorname{supp}(f),$$

where the kernel K satisfies the following conditions: there exists a constant C such that

$$|K(x,y)|\leq\frac{C}{|x-y|^{d-\nu}}\exp\left(-c\left(1+\frac{|x-y|}{\rho(x)}\right)^{m}\right), \tag{1.8}$$

for every $x\neq y$, and

$$|K(x,y)-K(x_0,y)|\leq\frac{C}{|x-y|^{d-\nu}}\left(\frac{|x-x_0|}{|x-y|}\right)^{\delta}, \tag{1.9}$$

for every $|x-y|>2|x-x_0|$.

For any $1<s\leq\infty$, we may refer to the above operators as *exponential SCZOs of (ν,s,δ) type*, for short. For these types of operators, we will specify in Section 2 the meaning of Tf when $f\in\mathrm{BMO}_\rho^\alpha(w)$.

Remark 1.3. Note that if K satisfies conditions (1.8) and (1.9), then it also satisfies (1.6) and (1.7) for all $1<s<\infty$, with the same parameters c,m and δ.

Remark 1.4. Although Definition 1.1 only requires the weak type (s',s') when $\nu=0$, it follows from [4, Proposition 6 and Theorem 5] that exponential SCZOs of $(0,s,\delta)$ type are actually bounded on $L^p(\mathbb{R}^d)$ for every $s'<p<\infty$. Moreover, exponential SCZOs of $(0,s,\delta)$ type are bounded on $L^p(w)$ for every $s'<p<\infty$ and every weight $w\in A^{\rho}_{p/s'}$, the class of weights introduced in [11]. Consequently, exponential SCZOs of $(0,\infty,\delta)$ type are bounded on $L^p(\mathbb{R}^d)$ for every $1<p<\infty$ and on $L^p(w)$ for every $1<p<\infty$ and every weight $w\in A^{\rho}_{p}$.

Remark 1.5. Clearly, exponential SCZOs of $(0,s,\delta)$ type are exponential SCZOs of (s,δ) type as considered in [14, Definition 2.6] for $1<s<\infty$. However, we observe that in [14, Definition 2.7] we gave a slightly different definition for exponential SCZOs of $(0,\infty,\delta)$ type. In spite of that, from Remarks 1.3 and 1.4, if T is an exponential SCZO of $(0,\infty,\delta)$ type, it satisfies [14, Definition 2.7] and all the results already obtained in [14] still hold for this new class.

Remark 1.6. Notice that if condition (1.7) holds for some $0<\delta\leq 1$, it is also valid for any $\delta'\in(0,\delta)$.

In this article, we will be dealing with one of the classes of weights studied in [1], which captures the exponential decay of the operators under consideration.

For $1<p<\infty$ and $c,m\geq 0$ we say a weight $w\in H^{\rho,m}_{p,c}$ if there exists a constant $C>0$ for which

$$\left(\int_B w\right)^{\frac{1}{p}}\left(\int_B w^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}}\leq C|B|\exp\left(c\left(1+\frac{r}{\rho(x)}\right)^{m}\right),$$

for every ball $B=B(x,r)$ with $x\in\mathbb{R}^d$ and $r>0$.

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When $p = 1$, for $c, m \geq 0$ we denote by $H^{\rho,m}_{1,c}$ the collection of weights w such that

$$\int_B w \leq C|B| \exp\left(c\left(1+\frac{r}{\rho(x)}\right)^m\right) \inf_B w,$$

for each ball $B = B(x, r)$, and some constant $C > 0$ independent of B.

We will also consider a class of doubling weights, and reverse–Hölder classes adapted to this context (already considered in [14]); they will be denoted by $D^{\rho}_{\kappa,c}$ with $\kappa \geq 1$ and $c > 0$ and $RH^{\rho}_{\eta,c}$ with $c > 0$ and $\eta > 1$, respectively. Their precise definitions will be provided in Section 2.

For simplicity, we denote by $E^{\rho,m}_{s,c_1,c_2}$ to the class of weights

$$E^{\rho,m}_{s,c_1,c_2} = \bigcup_{\eta > s'} H^{\rho,m}_{s/\eta',c_1} \cap RH^{\rho,m}_{\eta,c_2},$$

where $1 < s < \infty$, $c_1, c_2, m \geq 0$. And we will write

$$E^{\rho,m}_{\infty,c_1,c_2} = \bigcup_{s>1} E^{\rho,m}_{s,c_1,c_2}.$$

We are now ready to state the central theorem of this work.

Theorem 1.7. *Let T be an exponential SCZO of (ν, s, δ) type with parameters c and m, for some $0 \leq \nu < d$, $\frac{d}{d-\nu} < s < \infty$ and $0 < \delta \leq 1$. Let $0 \leq \alpha < \delta$ and $1 \leq \kappa < \frac{\delta-\alpha-\nu}{d} + 1$. Then, the following statements are equivalent.*

(i) There exists a constant C such that, for every ball $B = B(x_0, r)$, with $x_0 \in \mathbb{R}^d$ and $0 < r \leq \frac{1}{2}\rho(x_0)$, the function $T1$ satisfies

$$\frac{1}{|B|^{1+\nu/d}} \int_B |T1(y) - (T1)_B| dy \leq C \left(\frac{r}{\rho(x_0)}\right)^{\alpha+d(\kappa-1)}, \tag{1.10}$$

when $\alpha > 0$ or $\kappa > 1$, or

$$\frac{1}{|B|^{1+\nu/d}} \int_B |T1(y) - (T1)_B| dy \leq C \log^{-1}\left(\frac{\rho(x_0)}{r}\right), \tag{1.11}$$

when $\alpha = 0$ and $\kappa = 1$.

(ii) T is bounded from $\mathrm{BMO}^{\alpha}_{\rho}(w)$ into $\mathrm{BMO}^{\alpha+\nu}_{\rho}(w)$ for every weight $w \in E^{\rho,m}_{s,c_1,c_2} \cap D^{\rho,m}_{\kappa,c_3}$ with $c_1, c_2, c_3 \geq 0$ such that $(c_2 + c_3) < c\left(1 - \frac{d(\kappa-1)+\alpha+\nu}{\delta}\right)(4C_0)^{-m}$. The operator norm depends on w only through the parameters of the classes $E^{\rho,m}_{s,c_1,c_2}$ and $D^{\rho,m}_{\kappa,c_3}$.

(iii) T is bounded from $\mathrm{BMO}^{\alpha}_{\rho}(w)$ into $\mathrm{BMO}^{\alpha+\nu}_{\rho}(w)$ for every weight of the form $w(x) = |x - x_0|^{d(\kappa-1)}$, $x_0 \in \mathbb{R}^d$, with operator norm independent of x_0.

It is easy to see from Remark 1.3 and Theorem 1.7 that for $s = \infty$ we get the following consequence.

Corollary 1.8. *Let T be an exponential SCZO of (ν, ∞, δ) type with parameters c and m, for some $0 \leq \nu < d$ and $0 < \delta \leq 1$. Let $0 \leq \alpha < \delta$ and $1 \leq \kappa < \frac{\delta-\alpha-\nu}{d} + 1$. Then, the boundedness of T from $\mathrm{BMO}^{\alpha}_{\rho}(w)$ into $\mathrm{BMO}^{\alpha+\nu}_{\rho}(w)$ is equivalent to either condition (i) or condition (iii) from Theorem 1.7, whenever $w \in E^{\rho,m}_{\infty,c_1,c_2} \cap D^{\rho,m}_{\kappa,c_3}$ with $c_1, c_2, c_3 \geq 0$ such that $(c_2 + c_3) < c\left(1 - \frac{d(\kappa-1)+\alpha+\nu}{\delta}\right)(4C_0)^{-m}$.*

Remark 1.9. Clearly, if (1.10) holds for some $\alpha > 0$ or $\kappa > 1$, then (1.11) follows, yielding the boundedness of the operator on $\mathrm{BMO}_{\rho}(w)$ (i.e., $\alpha = 0$). However, the significance of (1.11) when $\alpha = 0$ and $\kappa = 1$ lies in ensuring $\mathrm{BMO}_{\rho}(w)$ boundedness even for operators that fail (1.10).

Remark 1.10. The results above can be applied to linear operators related to the generalized Schrödinger semigroup. However, non-linear operators in this setting, such as maximal or Littlewood-Paley functions, can be viewed as linear operators in appropriate Banach spaces. By using the corresponding norms in Theorem 1.7 and Corollary 1.8, we obtain valid vector-valued versions of these results.

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The paper is organized as follows. In Section 2 we collect some important definitions and intermediate results to fully understand the theorem stated above. In Section 3 we prove Theorem 1.7, as well as some auxiliary results related to it. Finally, in Section 4 we establish the $T1$ condition for several operators associated to the generalized Schrödinger operator $\mathcal{L}_\mu$, defined in terms of a certain measure μ studied in [22] and [1], and we also apply it to the study of some operators associated to a potential V.

Throughout this paper C and c will always denote positive constants that may change in each occurrence. Also by $A \lesssim B$ will indicate that there exists a constant $c > 0$ such that $A \leq cB$ is satisfied and $A \sim B$ will denote that, at the same time, $A \lesssim B$ and $B \lesssim A$ hold.

2. Preliminaries

2.1. **Weights associated to a critical radius function.** When dealing with operators associated to a critical radius function, we may first mention the work of Bongioanni, Harboure and Salinas [11], where the authors introduced the class A_p^ρ of weights for $1 \leq p < \infty$ to obtain the weighted L^p-boundedness of many operators in the Schrödinger setting.

For $1 < p < \infty$, a weight w is said to be in A_p^ρ if $w \in A_p^{\rho,\theta}$ for some $\theta \geq 0$. This means that there exist $C > 0$ and some $\theta \geq 0$ such that

$$\left(\int_B w\right)^{\frac{1}{p}} \left(\int_B w^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}} \leq C|B| \left(1 + \frac{r}{\rho(x)}\right)^\theta,$$

for every ball $B = B(x,r)$ with $x \in \mathbb{R}^d$ and $r > 0$.

Similarly, when $p = 1$, a weight w belongs to A_1^ρ if $w \in A_1^{\rho,\theta}$ for some $\theta \geq 0$; equivalently, if there exist $C > 0$ and $\theta \geq 0$ such that

$$\frac{1}{|B|}\int_B w \leq C\left(1 + \frac{r}{\rho(x)}\right)^\theta \inf_B w,$$

for every ball $B = B(x,r)$.

Observe that when $B \in \mathcal{B}_\rho$, the family defined in (1.3), the polynomial term can be neglected, and one has that A_p^ρ resembles the Muckenhoupt A_p class for sub-critical balls. However, on super-critical balls, the A_p^ρ weights can be larger than the A_p weights (and indeed, $A_p \subsetneq A_p^\rho$ for every $1 \leq p < \infty$, as shown in [11, p. 564]). In view of this remark, it will also be useful to have at our disposal the classes of local weights introduced in [6]. We say that $w \in A_p^{\rho,\text{loc}}$ if there exists a constant $C > 0$ such that

$$\left(\int_B w\right)^{\frac{1}{p}} \left(\int_B w^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}} \leq C|B|,$$

for every ball $B \in \mathcal{B}_\rho$, and for $p = 1$ if

$$\int_B w \leq C|B| \inf_B w,$$

holds for every $B \in \mathcal{B}_\rho$.

As we have established in [14, Proposition 3.1], for every $1 \leq p < \infty$ and $c > 0$,

$$A_p^\rho \subsetneq H_{p,c}^\rho \subseteq A_p^{\rho,\text{loc}}, \tag{2.1}$$

where $H_{p,c}^\rho := \bigcup_{m\geq 0} H_{p,c}^{\rho;m}$. That is, the class $H_{p,c}^\rho$ is strictly larger than the one given in [11], but smaller than the local class of weights $A_p^{\rho,\text{loc}}$.

We now recall the definitions of the doubling and reverse–Hölder weight classes from [14].

Definition 2.1. Let $\kappa \geq 1$ and $c, m \geq 0$. We denote by $D_{\kappa,c}^{\rho;m}$ the class of weights w such that there exists a constant C for which

$$w(B(x,R)) \leq C\left(\frac{R}{r}\right)^{d\kappa} \exp\left(c\left(1 + \frac{R}{\rho(x)}\right)^m\right) w(B(x,r)),$$

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for all $x \in \mathbb{R}^d$ and $r \leq R$.

Remark 2.2. It is easy to see that $H^{\rho,m}_{p,c} \subseteq D^{\rho,m}_{p,cp}$ for any $1 \leq p < \infty$ and $c, m \geq 0$.

Remark 2.3. The condition $w \in D^{\rho,m}_{\kappa,c}$ says, in particular, that w is doubling on critical balls. Indeed, for some $C = C(d, \kappa, c, m)$, we have

$$w(B(x, 2\rho(x))) \leq Cw(B(x, \rho(x))).$$

We also note that, for any $x \in \mathbb{R}^d$, $r > 0$ and $j \in \mathbb{N}$, there exists $C > 0$ such that

$$w(B(x, 2^j r)) \leq C2^{d\kappa j} \exp\left(c\left(1 + \frac{2^j r}{\rho(x)}\right)^m\right) w(B(x, r)), \tag{2.2}$$

which, in the case $r = \rho(x)$ gives

$$w(B(x, 2^j\rho(x))) \leq C2^{d\kappa j} \exp\left(c\left(1 + 2^j\right)^m\right) w(B(x, \rho(x))).$$

Reverse–Hölder classes adapted to this context are defined as follows.

Definition 2.4. Let $\eta > 1$ and $c, m \geq 0$. We denote by $RH^{\rho,m}_{\eta,c}$ the set of weights w satisfying

$$\left(\frac{1}{|B|}\int_B w^\eta\right)^{\frac{1}{\eta}} \leq C \exp\left(c\left(1 + \frac{r}{\rho(x)}\right)^m\right)\left(\frac{1}{|B|}\int_B w\right),$$

for every ball $B = B(x, r)$, and some $C > 0$ independent of B.

2.2. **Spaces of functions.** In the introduction we have already defined the spaces $\mathrm{BMO}^\alpha_\rho(w)$.

The first property that we present below shows us that it is enough to verify condition (1.5) only on critical balls when the weight belongs to the class $A^{\rho,\mathrm{loc}}_p$.

Lemma 2.5 ([6, Proposition 3.2])**.** *Let $w \in A^{\rho,\mathrm{loc}}_p$ for some $1 \leq p < \infty$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. If*

$$A := \sup_{x\in\mathbb{R}^d} \frac{1}{w(B(x, \rho(x)))\rho(x)^\alpha} \int_{B(x,\rho(x))} |f| < \infty,$$

then there exists a constant C such that

$$\sup_{x\in\mathbb{R}^d,\ r\geq\rho(x)} \frac{1}{w(B(x, r))r^\alpha} \int_{B(x,r)} |f| < CA.$$

The following lemma gives us a property of norm equivalences for the space $\mathrm{BMO}^\alpha_\rho(w)$ for $A^{\rho,\mathrm{loc}}_p$ weights. As usual, the notation p' stands for the conjugate exponent of p.

Lemma 2.6 ([6, Lemma 3.3])**.** *Let $0 \leq \alpha < 1$, $w \in A^{\rho,\mathrm{loc}}_p$, $1 < q \leq p'$ and $f \in \mathrm{BMO}^\alpha_\rho(w)$. Then,*

$$\frac{1}{|B|^{\alpha/d}}\left(\frac{1}{w(B)}\int_B |f|^q w^{1-q}\right)^{\frac{1}{q}} \leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)},$$

for all $B = B(x, r)$ with $r \geq \rho(x)$, and

$$\frac{1}{|B|^{\alpha/d}}\left(\frac{1}{w(B)}\int_B |f - f_B|^q w^{1-q}\right)^{\frac{1}{q}} \leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)},$$

for all $B = B(x, r)$ with $r \leq \rho(x)$.

By virtue of (2.1), both lemmas above can be applied to weights in the $H^{\rho,m}_{p,c}$ classes for any $c, m \geq 0$ and the corresponding values of p. This fact will be useful in the proof of Theorem 1.7.

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2.3. **Schrödinger-Calderón-Zygmund operators of exponential decay.** We will need to make some considerations about the meaning of Tf being T an exponential SCZO of (ν, s, δ) type with parameters c and m, for $0 \le \nu < d$, $1 < s \le \infty$ and $0 < \delta \le 1$, and f a function in $\mathrm{BMO}^\alpha_\rho(w)$.

Let $x_0 \in \mathbb{R}^d$ and $R \ge \rho(x_0)$. We define

$$Tf(x) = T(f\chi_{B(x_0,2R)})(x) + \int_{B(x_0,2R)^c} K(x,y)f(y)dy, \qquad x \in B(x_0,R). \tag{2.3}$$

The first term on the right hand side makes sense if w satisfies the hypothesis of Theorem 1.7 since in that case $f\chi_{B(x_0,2R)} \in L^{s'}(\mathbb{R}^d)$ for $f \in \mathrm{BMO}^\alpha_\rho(w)$ as we can see in (3.3). The integral in the second term is absolutely convergent (see estimate of Tf_2 in page 12). This fact follows in the same manner as in [7, p. 603], as does its well-definedness in the sense that equation (2.3) is independent of R. Moreover, we need to verify that (2.3) is well-defined for $f = 1$. Indeed, $\chi_{B(x_0,2R)} \in L^{s'}(\mathbb{R}^d)$, which ensures the finiteness of the first term and, for $x \in B(x_0, R)$

$$\begin{aligned}
\int_{B(x_0,2R)^c} |K(x,y)|dy &\le \sum_{j\ge 1} \left(\int_{2^jR \le |x_0-y| < 2^{j+1}R} |K(x,y)|^s dy \right)^{1/s} |B(x_0, 2^{j+1}R)|^{1/s'} \\
&\le C \sum_{j\ge 1} (2^jR)^\nu \exp\left(-c \left(1 + \frac{2^jR}{\rho(x)} \right)^m \right) \\
&\le CR^\nu \sum_{j \ge 1} (2^\nu)^j \exp\left(-c^* \left(1 + 2^j\right)^m \right) < \infty,
\end{aligned}$$

where $c^* = cC_0^{-m} 2^{-\frac{k_0 m}{k_0+1}}$ by (1.1). Here, the constant $C > 0$ is independent of $R > 0$ and $x \in \mathbb{R}^d$.

The following lemma will be useful to show that exponential SCZO of (ν, s, δ) type actually exhibits a smoothness condition on their kernels with exponential decay.

Lemma 2.7. *Let K be a kernel such that (1.6) and (1.7) hold for some $0 \le \nu < d$, $1 < s < \infty$, $0 < \delta \le 1$, and $c, m \ge 0$. Then, there exists a constant $C > 0$ such that for every $\delta' \in (0, \delta)$ and $R > 0$*

$$\left(\frac{1}{R^d} \int_{R < |x_0-y| \le 2R} |K(x,y) - K(x_0,y)|^s \, dy \right)^{\frac{1}{s}} \le \frac{C}{R^{d-\nu}} \left(\frac{r}{R} \right)^{\delta'} \exp\left(-c\sigma \left(1 + \frac{R}{2C_0\rho(x_0)} \right)^m \right),$$

where $|x - x_0| < r \le \rho(x_0)$, $r < R/2$ and $\sigma = 1 - \frac{\delta'}{\delta}$.

Proof. Let $\delta' \in (0, \delta)$. We define $\sigma \in (0,1)$ such that $(1-\sigma)\delta = \delta'$. Let $x_0, x \in \mathbb{R}^d$ and $r, R > 0$ such that $|x - x_0| < r \le \rho(x_0)$, $r < R/2$.

If we call

$$A = \left(\frac{1}{R^d} \int_{R < |x_0-y| \le 2R} |K(x,y) - K(x_0,y)|^s dy \right)^{\frac{1}{s}},$$

by (1.6) and (1.2) we get

$$\begin{aligned}
A^\sigma &\le \left(\frac{1}{R^d} \int_{R<|x_0-y|\le 2R} |K(x,y)|^s + |K(x_0,y)|^s \, dy \right)^{\sigma/s} \\
&\lesssim \frac{1}{R^{(d-\nu)\sigma}} \exp\left(-c\sigma \left(1 + \frac{R}{\rho(x)} \right)^m \right) + \frac{1}{R^{(d-\nu)\sigma}} \exp\left(-c\sigma \left(1 + \frac{R}{\rho(x_0)} \right)^m \right) \\
&\lesssim \frac{1}{R^{(d-\nu)\sigma}} \exp\left(-c\sigma \left(1 + \frac{R}{2C_0\rho(x_0)} \right)^m \right).
\end{aligned}$$

On the other hand, by (1.7) we have

$$A^{1-\sigma} \lesssim \frac{1}{R^{(d-\nu)(1-\sigma)}} \left(\frac{r}{R} \right)^{\delta(1-\sigma)} \sim \frac{1}{R^{(d-\nu)(1-\sigma)}} \left(\frac{r}{R} \right)^{\delta'}.$$

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Since $A = A^\sigma A^{1-\sigma}$, the result follows. □

The following inequality is a generalization of the classical Kolmogorov inequality for sublinear operators of weak-type $(1,1)$ (see, for instance, [17, Lemma 5.16]) to sublinear operators of weak-type (p,q) for any $p,q \geq 1$, and the proof can be obtained in a similar way (see [16, Theorem 3.3.1]). It will allow us to deduce sharper results when dealing with exponential SCZO of (s,δ) type.

Lemma 2.8. *Let S be a sublinear operator of weak-type (p,q) for some $p,q \geq 1$. Then, there exists $C > 0$ such that for any measurable set E with $0 < |E| < \infty$ and $0 < \gamma < q$,*

$$\int_E |Sf(x)|^\gamma dx \leq C|E|^{1-\frac{\gamma}{q}} \|f\|_p^\gamma,$$

for $f \in L^p(\mathbb{R}^d)$.

3. Proof of Theorem 1.7

Before presenting the proof of Theorem 1.7, we first state and prove some technical lemmas. Although we essentially follow the approach in [7], we outline the arguments here with the necessary modifications concerning the classes of weights, the doubling condition, the reverse–Hölder property, among other considerations.

Lemma 3.1. *Let $f \in \mathrm{BMO}_\rho^\alpha(w)$ with $0 \leq \alpha < 1$, $w \in D_{\kappa,c}^{\rho,m}$ for some $\kappa \geq 1$ and $c > 0$, and $B = B(x_0,r)$ with $r < \rho(x_0)$. Then, if $\kappa > 1$ or $\alpha > 0$, there exists a constant $C = C(c,m,d)$ such that*

$$|f_B| \leq C\|f\|_{\mathrm{BMO}_\rho^\alpha(w)} r^\alpha \frac{w(B)}{|B|} \left(\frac{\rho(x_0)}{r}\right)^{d(\kappa-1)+\alpha}. \tag{3.1}$$

When $\kappa = 1$ and $\alpha = 0$, there exists a constant $C = C(c,m,d)$ such that

$$|f_B| \leq C\|f\|_{\mathrm{BMO}_\rho^\alpha(w)} r^\alpha \frac{w(B)}{|B|} \left(1 + \log_2\left(\frac{\rho(x_0)}{r}\right)\right). \tag{3.2}$$

Proof. Let $f \in \mathrm{BMO}_\rho^\alpha(w)$ and fix $B = B(x_0,r)$ with $r < \rho(x_0)$. We choose $j_0 \in \mathbb{N}$ such that $2^{j_0-1}r < \rho(x_0) \leq 2^{j_0}r$, and define $B_j = 2^jB$ for every $j = 0,\ldots,j_0-1$. We have

$$\begin{aligned}
|f_B| &\leq \frac{1}{|B|}\int_B |f - f_B| + \sum_{j=1}^{j_0-1} |f_{B_{j-1}} - f_{B_j}| + |f_{B_{j_0-1}}| \\
&\leq \frac{1}{|B|}\int_B |f - f_B| + \sum_{j=1}^{j_0-1} \frac{2^d}{|B_j|}\int_{B_j} |f - f_{B_j}| + \frac{2^d}{|B_{j_0}|}\int_{B_{j_0}} |f| \\
&\leq \sum_{j=0}^{j_0-1} \frac{2^d}{|B_j|}\int_{B_j} |f - f_{B_j}| + \frac{2^d}{|B_{j_0}|}\int_{B_{j_0}} |f| \\
&\leq 2^d \|f\|_{\mathrm{BMO}_\rho^\alpha(w)} \sum_{j=0}^{j_0} \frac{w(B_j)}{|B_j|}|B_j|^{\frac{\alpha}{d}},
\end{aligned}$$

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where the last inequality was obtained from (1.4) and (1.5) since $2^{j_0-1}r \leq \rho(x_0) \leq 2^{j_0}r$. Applying the doubling property (2.2) of w we get that

$$\begin{aligned}
|f_B| &\leq 2^d \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} \frac{w(B)}{|B|}|B|^{\frac{\alpha}{d}} \sum_{j=0}^{j_0} 2^{(d\kappa+\alpha-d)j} \exp\left(c\left(1+\frac{2^j r}{\rho(x_0)}\right)^m\right)\\
&\leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)} \frac{w(B)}{|B|} r^\alpha 2^{(j_0+1)(d(\kappa-1)+\alpha)}\\
&\leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)} \frac{w(B)}{|B|} r^\alpha \left(\frac{\rho(x_0)}{r}\right)^{d(\kappa-1)+\alpha},
\end{aligned}$$

whenever $\kappa > 1$ or $\alpha > 0$, where C depends on the parameters of the doubling condition and on the dimension.

If $\kappa = 1$ and $\alpha = 0$, the sum obtained above equals $j_0 + 1$. Thus, from the choice of j_0, in this case we have

$$|f_B| \leq 2^d e^{c3^m} \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} \frac{w(B)}{|B|} r^\alpha (j_0+1) \leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)} \frac{w(B)}{|B|} r^\alpha \left(1+\log_2\left(\frac{\rho(x_0)}{r}\right)\right),$$

where C is as in the previous case, and depends on c, m and d. □

Lemma 3.2. *Let $f \in \mathrm{BMO}^\alpha_\rho(w)$ with $0 \leq \alpha < 1$, and a weight $w \in H^{\rho,m}_{\sigma',c_1} \cap D^{\rho,m}_{\kappa,c_3}$ for some $\sigma > 1$, $\kappa \geq 1$, $c_1, c_3 \geq 0$ and $m \geq 0$. Then, when $\kappa > 1$ or $\alpha > 0$, there exists $C > 0$ such that*

$$\begin{aligned}
&w(2^jB)^{1/\sigma'}\left(\int_{2^jB}|f-f_B|^\sigma w^{1-\sigma}\right)^{1/\sigma}\\
&\qquad\leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(B) r^\alpha 2^{j(\alpha+d\kappa)} \exp\left((c_1+c_3)\left(1+\frac{2^j r}{\rho(x_0)}\right)^m\right),
\end{aligned}$$

for every $j \in \mathbb{N}$ and every ball $B = B(x_0, r)$.

When $\kappa = 1$ and $\alpha = 0$, there exists $C > 0$ such that

$$\begin{aligned}
&w(2^jB)^{1/\sigma'}\left(\int_{2^jB}|f-f_B|^\sigma w^{1-\sigma}\right)^{1/\sigma}\\
&\qquad\leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(B) j 2^{jd} \exp\left((c_1+c_3)\left(1+\frac{2^j r}{\rho(x_0)}\right)^m\right),
\end{aligned}$$

for every $j \in \mathbb{N}$ and every ball $B = B(x_0, r)$.

Proof. We can write, for $j \in \mathbb{N}$,

$$\begin{aligned}
&w(2^jB)^{1/\sigma'}\left(\int_{2^jB}|f-f_B|^\sigma w^{1-\sigma}\right)^{1/\sigma}\\
&\leq w(2^jB)^{1/\sigma'}\left[\left(\int_{2^jB}|f-f_{2^jB}|^\sigma w^{1-\sigma}\right)^{1/\sigma} + \left(w^{1-\sigma}(2^jB)\right)^{1/\sigma}\sum_{i=1}^{j}|f_{2^iB}-f_{2^{i-1}B}|\right]\\
&\leq w(2^jB)\left(\frac{1}{w(2^jB)}\int_{2^jB}|f-f_{2^jB}|^\sigma w^{1-\sigma}\right)^{1/\sigma}\\
&\quad + w(2^jB)^{1/\sigma'}\left(w^{-\frac{1}{\sigma'-1}}(2^jB)\right)^{1/\sigma}\sum_{i=1}^{j}\frac{2^d}{|2^iB|}\int_{2^iB}|f-f_{2^iB}|\\
&:= I_1 + I_2.
\end{aligned}$$

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To estimate I_1, we can apply Lemma 2.6 with $q = p' = \sigma'$ since $w \in A^{\rho,\mathrm{loc}}_{\sigma'}$ (by (2.1)), together with the doubling condition of w, to get

$$I_1 \lesssim \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(2^jB)|2^jB|^{\alpha/d} \lesssim \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(B) r^\alpha 2^{j(d\kappa+\alpha)} \exp\left(c_3\left(1+\frac{2^jr}{\rho(x_0)}\right)^m\right).$$

For I_2, note that since $w \in H^{\rho,m}_{\sigma',c_1}$,

$$w(2^jB)^{1/\sigma'}\left(w^{-\frac{1}{\sigma'-1}}(2^jB)\right)^{1/\sigma} \leq C2^{jd}r^d \exp\left(c_1\left(1+\frac{2^jr}{\rho(x_0)}\right)^m\right).$$

On the other hand, since $f \in \mathrm{BMO}^\alpha_\rho(w)$ and $w \in D^{\rho,m}_{\kappa,c_3}$, if we first assume $\kappa > 1$ or $\alpha > 0$,

$$\begin{aligned}\sum_{i=1}^{j}\frac{1}{|2^iB|}\int_{2^iB}|f-f_{2^iB}| &\leq \|f\|_{\mathrm{BMO}^\alpha_\rho(w)}\sum_{i=1}^{j}|2^iB|^{\alpha/d-1}w(2^iB)\\ &\leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)} r^{\alpha-d}w(B)\sum_{i=1}^{j}2^{i(\alpha-d)}2^{id\kappa}\exp\left(c_3\left(1+\frac{2^ir}{\rho(x_0)}\right)^m\right)\\ &\leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)} r^{\alpha-d}w(B)\exp\left(c_3\left(1+\frac{2^jr}{\rho(x_0)}\right)^m\right)2^{j(d(\kappa-1)+\alpha)},\end{aligned}$$

where we have used that $d(\kappa-1)+\alpha > 0$ to bound the sum in this case.

Combining the above estimates, we get

$$I_1 + I_2 \leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(B) r^\alpha 2^{j(d\kappa+\alpha)} \exp\left((c_1+c_3)\left(1+\frac{2^jr}{\rho(x_0)}\right)^m\right).$$

If $\kappa = 1$ and $\alpha = 0$, we get instead

$$\sum_{i=1}^{j}\frac{1}{|2^iB|}\int_{2^iB}|f-f_{2^iB}| \leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)} j r^{-d} w(B)\exp\left(c_3\left(1+\frac{2^jr}{\rho(x_0)}\right)^m\right),$$

which combined with the estimate for I_1, leads to the result. In either case, the constant C is independent of j and B. ☐

Lemma 3.3. *Let $f \in \mathrm{BMO}^\alpha_\rho(w)$ with $0 \leq \alpha < 1$, and $w \in E^{\rho,m}_{s,c_1,c_2} \cap D^{\rho,m}_{\kappa,c_3}$ for some $s > 1$, $c_1, c_2, c_3 \geq 0$, $\kappa \geq 1$ and $m \geq 0$. Given a ball $B = B(x_0, r)$, consider the function*

$$g = \begin{cases} f\chi_{2B} & \text{if } r \geq \rho(x_0);\\ (f-f_B)\chi_{2B} & \text{if } r < \rho(x_0).\end{cases}$$

Then, there exists C such that for every $j \in \mathbb{N}$,

$$\left(\int_{2^jB}|g|^{s'}\right)^{\frac{1}{s'}} \leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)} 2^{j\left(d\left(\kappa-\frac{1}{s}\right)+\alpha\right)} w(B)|B|^{\frac{\alpha}{d}-\frac{1}{s}} \exp\left((c_2+c_3)\left(1+\frac{2^jr}{\rho(x_0)}\right)^m\right).$$

Proof. Fix $\eta > s'$ such that $w \in H^{\rho,m}_{s/\eta',c_1} \cap RH^{\rho,m}_{\eta,c_2}$. We set $\zeta = \frac{s-1}{s-\eta'} > 1$ and apply Hölder's inequality with this exponent to get

$$\left(\int_{2^jB}|g|^{s'}\right)^{\frac{1}{s'}} \leq \left(\int_{2^jB}|g|^{s'\zeta}w^{1-s'\zeta}\right)^{\frac{1}{s'\zeta}}\left(\int_{2^jB}w^{\left(s'-\frac{1}{\zeta}\right)\zeta'}\right)^{\frac{1}{s'\zeta'}}. \tag{3.3}$$

Since $w \in A^{\rho,\mathrm{loc}}_{s/\eta'}$ (by (2.1)), and noting that $s'\zeta = \left(\frac{s}{\eta'}\right)'$, we can use Lemma 2.6 in order to obtain

$$\left(\int_{2^jB}|g|^{s'\zeta}w^{1-s'\zeta}\right)^{\frac{1}{s'\zeta}} \leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(2^jB)^{\frac{1}{\left(\frac{s}{\eta'}\right)'}}|2^jB|^{\frac{\alpha}{d}} = C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(2^jB)^{\frac{1}{\left(\frac{s}{\eta'}\right)'}}|B|^{\frac{\alpha}{d}}2^{j\alpha},$$

for the corresponding cases of g.

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To estimate the second factor in (3.3), we observe that $\left(s'-\frac{1}{\zeta}\right)\zeta'=\eta$ and $s'\zeta'=\frac{s\eta}{\eta'}$, so we can use that $w\in RH^{\rho,m}_{\eta,c_2}$ to get

$$\left(\int_{2^jB} w^{\left(s'-\frac{1}{\zeta}\right)\zeta'}\right)^{\frac{1}{s'\zeta'}} = \left(\int_{2^jB} w^{\eta}\right)^{\frac{1}{\eta}\frac{\eta'}{s}} \lesssim \left(|2^jB|^{-\frac{1}{\eta'}} w(2^jB)\exp\left(c_2\left(1+\frac{2^jr}{\rho(x_0)}\right)^m\right)\right)^{\frac{\eta'}{s}}$$
$$\leq C2^{-j\frac{d}{s}}|B|^{-\frac{1}{s}}w(2^jB)^{\frac{\frac{1}{s}}{\eta'}}\exp\left(c_2\left(1+\frac{2^jr}{\rho(x_0)}\right)^m\right).$$

Combining both estimates above, and since $w\in D^{\rho,m}_{\kappa,c_3}$, from (2.2) we get

$$\left(\int_{2^jB}|g|^{s'}\right)^{\frac{1}{s'}} \leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)}2^{j\left(\alpha-\frac{d}{s}\right)}w(2^jB)|B|^{\frac{\alpha}{d}-\frac{1}{s}}\exp\left(c_2\left(1+\frac{2^jr}{\rho(x_0)}\right)^m\right)$$
$$\leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)}2^{j\left(d\left(\kappa-\frac{1}{s}\right)+\alpha\right)}w(B)|B|^{\frac{\alpha}{d}-\frac{1}{s}}\exp\left((c_2+c_3)\left(1+\frac{2^jr}{\rho(x_0)}\right)^m\right),$$

as claimed. □

Now, we are in a position to prove Theorem 1.7.

Proof of Theorem 1.7. To see that (i)⇒(ii), let $f\in\mathrm{BMO}^\alpha_\rho(w)$ and fix

$$w\in E^{\rho,m}_{s,c_1,c_2}\cap D^{\rho,m}_{\kappa,c_3}=\left(\bigcup_{\eta>s'}H^{\rho,m}_{s/\eta',c_1}\cap RH^{\rho,m}_{\eta,c_2}\right)\cap D^{\rho,m}_{\kappa,c_3},$$

with $(c_2+c_3)(4C_0)^m<c$. To verify condition (1.5), by (2.1) and Lemma 2.5, it is enough to consider critical balls. Let $B_\rho=B(x_0,\rho(x_0))$ with $x_0\in\mathbb{R}^d$.

According to the definition of Tf, we decompose the function as $f=f_1+f_2=f\chi_{2B_\rho}+f\chi_{(2B_\rho)^c}$. This allows us to bound

$$\int_{B_\rho}|Tf(x)|dx\leq\int_{B_\rho}|Tf_1(x)|dx+\int_{B_\rho}|Tf_2(x)|dx.$$

We begin by controlling the first term. From the hypothesis on w we can use Lemma 3.3 with $j=1$ and $r=\rho(x_0)$ to get

$$\left(\int_{2B_\rho}|f|^{s'}\right)^{\frac{1}{s'}}\leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)}|B_\rho|^{\frac{\alpha}{d}-\frac{1}{s}}w(B_\rho),$$

where the constant C depends on the parameters of the classes of weights.

Since T is of weak-type $\left(s',\frac{s'd}{d-s'\nu}\right)$ with $s',\frac{s'd}{d-s'\nu}\geq 1$, combining Lemma 2.8 with $\gamma=1$ and the previous inequality, we deduce

$$\int_{B_\rho}|Tf_1(x)|dx\leq C|B_\rho|^{1-\frac{1}{s'}+\frac{\nu}{d}}\left(\int_{2B_\rho}|f|^{s'}\right)^{\frac{1}{s'}}\leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)}|B_\rho|^{\frac{\alpha+\nu}{d}}w(B_\rho).$$

We next estimate the integral involving Tf_2, for which the kernel representation of T will be used. We fix $x\in B_\rho$ and make the standard decomposition into annuli,

$$|Tf_2(x)|\leq\sum_{j\in\mathbb{N}}\int_{2^{j+1}B_\rho\setminus 2^jB_\rho}|K(x,y)||f(y)|dy$$
$$\leq\sum_{j\in\mathbb{N}}\left(\int_{2^j\rho(x_0)\leq|y-x_0|<2^{j+1}\rho(x_0)}|K(x,y)|^sdy\right)^{\frac{1}{s}}\left(\int_{2^{j+1}B_\rho}|f|^{s'}\right)^{\frac{1}{s'}}.$$

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By the size condition (1.6) on the kernel with $R = 2^j\rho(x_0)$, $j \geq 1$, applying again Lemma 3.3 on each term, and using (1.2), we can see that for every $x \in B_\rho$,

$$
\begin{aligned}
|Tf_2(x)| &\lesssim \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(B_\rho)|B_\rho|^{\frac{\alpha}{d}-\frac{1}{s}} \\
&\quad \times \sum_{j\in\mathbb{N}} \frac{2^{(j+1)\left(d\left(\kappa-\frac{1}{s}\right)+\alpha\right)}}{(2^j\rho(x_0))^{\frac{d}{s'}-\nu}} \exp\left((c_2+c_3)(1+2^{j+1})^m - c\left(1+\frac{2^j\rho(x_0)}{\rho(x)}\right)^m\right) \\
&\lesssim \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(B_\rho)|B_\rho|^{\frac{\alpha+\nu}{d}-1} \\
&\quad \times \sum_{j\in\mathbb{N}} \exp\left((c_2+c_3)(1+2^{j+1})^m - c\left(1+\frac{1}{4C_0}2^{j+1}\right)^m\right) 2^{j(d(\kappa-1)+\alpha+\nu)} \\
&\lesssim \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(B_\rho)|B_\rho|^{\frac{\alpha+\nu}{d}-1} \\
&\quad \times \sum_{j\in\mathbb{N}} \exp\left(\left((c_2+c_3) - \frac{c}{(4C_0)^m}\right)(1+2^{j+1})^m\right) 2^{k(d(\kappa-1)+\alpha+\nu)} \\
&\lesssim \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(B_\rho)|B_\rho|^{\frac{\alpha+\nu}{d}-1},
\end{aligned}
$$

where the series converges since $(c_2 + c_3) < c(4C_0)^{-m}$. Finally, integrating on B_ρ, we have that Tf_2 satisfies the condition of averages (1.5) over critical balls. This finishes the proof of (1.5) for Tf.

Now, we shall see that condition (1.4) holds for Tf, for every $B = B(x_0, r)$ with $r < \rho(x_0)$ as we already observed it is sufficient to check it in this type of balls.

Fix $x_0 \in \mathbb{R}^d, r < \rho(x_0)$ and let $B = B(x_0, r)$. We split f in the following way,

$$f = (f - f_B)\chi_{2B} + (f - f_B)\chi_{(2B)^c} + f_B = f_1 + f_2 + f_3.$$

We choose $R \geq \max\{\rho(x_0), 2r\}$, which yields $\tilde{B} := B(x_0, R) \supset 2B$. Applying the definition given in (2.3) to the above decomposition of f, and adding and subtracting f_B in the integral over $\tilde{B}^c$,

$$
\begin{aligned}
Tf(x) &= T(f\chi_{\tilde{B}})(x) + \int_{\tilde{B}^c} K(x,y)f(y)dy \\
&= T((f-f_B)\chi_{2B}) + T((f-f_B)\chi_{\tilde{B}\setminus 2B})(x) + f_B T(\chi_{\tilde{B}})(x) \\
&+ \int_{\tilde{B}^c} K(x,y)(f-f_B) + f_B\int_{\tilde{B}^c} K(x,y)dy \\
&= T((f-f_B)\chi_{2B})(x) + \int_{(2B)^c} K(x,y)(f(y)-f_B)dy + f_B T1(x), \qquad \text{a.e. } x \in 2B.
\end{aligned}
$$

So, to estimate (1.4) for Tf, we write

$$
\begin{aligned}
\int_B |Tf(x) - (Tf)_B|dx \leq & \frac{1}{|B|}\int_B\int_B |Tf_1(x) - Tf_1(z)|dzdx \\
&+ \frac{1}{|B|}\int_B\int_B\int_{(2B)^c} |K(x,y) - K(z,y)||f(y) - f_B|dydzdx \\
&+ \int_B |Tf_3(x) - (Tf_3)_B|dx.
\end{aligned}
$$

For the first term, we can proceed as before, applying Lemma 3.3 to $f_1 = (f - f_B)\chi_{2B}$ and Lemma 2.8 to obtain the desired estimate

$$\frac{1}{|B|}\int_B\int_B |Tf_1(x) - Tf_1(z)|dzdx \leq 2\int_B |Tf_1(x)|dx \leq C\|f\|_{\mathrm{BMO}^\alpha_\rho(w)}|B|^{\frac{\alpha+\nu}{d}} w(B).$$

For the second term, given $x, z \in B$, we repeat the argument given on page 11. In this case, we apply it on the ball B instead of B_ρ and use Lemma 3.3 together with the smoothness condition

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obtained in Lemma 2.7 for $\delta' < \delta$ twice, with $R = 2^j r$, $j \geq 1$. In fact,

$$\begin{aligned}
&\int_{(2B)^c} |K(x,y) - K(z,y)||f(y) - f_B| dy \\
&\leq \sum_{j\in\mathbb{N}} \left(\int_{2^j r \leq |x_0 - y| < 2^{j+1} r} \left(|K(x,y) - K(x_0,y)| + |K(z,y) - K(x_0,y)| \right)^s dy \right)^{\frac{1}{s}} \left(\int_{2^{j+1}B} |f - f_B|^{s'} \right)^{\frac{1}{s'}} \\
&\lesssim \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(B) |B|^{\frac{\alpha+\nu}{d} - 1} \sum_{j\in\mathbb{N}} 2^{j(d(\kappa-1)+\alpha+\nu-\delta')} \\
&\quad \times \exp\left(-c\sigma \left(1 + \frac{2^j r}{2C_0 \rho(x_0)} \right)^m + (c_2 + c_3) \left(1 + \frac{2^{j+1} r}{\rho(x_0)} \right)^m \right) \\
&\lesssim \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(B) |B|^{\frac{\alpha+\nu}{d} - 1} \sum_{j\in\mathbb{N}} 2^{j(d(\kappa-1)+\alpha+\nu-\delta')} \exp\left(\left(c_2 + c_3 - \frac{c\sigma}{(4C_0)^m} \right) \left(1 + \frac{2^{j+1} r}{\rho(x_0)} \right)^m \right) \\
&\lesssim \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(B) |B|^{\frac{\alpha+\nu}{d} - 1}.
\end{aligned}$$

Choosing first $\sigma \in (0,1)$ such that $\sigma < 1 - \frac{d(\kappa-1)+\alpha}{\delta}$ and $(c_2 + c_3) < c\sigma(4C_0)^{-m}$, we also have that $d(\kappa - 1) + \alpha < (1 - \sigma)\delta = \delta'$ and the series converges.

Finally, to estimate the term with f_3 we use Lemma 3.1 and the $T1$ condition. In the case $\kappa > 1$ or $\alpha > 0$, we use (3.1) and (1.10) to get

$$\begin{aligned}
\int_B |Tf_3(x) - (Tf_3)_B| dx &= \int_B \left| f_B T1(x) - \frac{1}{|B|} \int_B |f_B T1(z)| dz \right| dx \\
&\leq |f_B| \int_B |T1(x) - (T1)_B| dx \\
&\lesssim \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} \frac{w(B)}{|B|} r^\alpha \left(\frac{\rho(x_0)}{r} \right)^{d(\kappa-1)+\alpha} \int_B |T1(x) - (T1)_B| dx \\
&\lesssim \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(B) |B|^{\frac{\alpha+\nu}{d}} \left(\frac{\rho(x_0)}{r} \right)^{d(\kappa-1)+\alpha} \left(\frac{r}{\rho(x_0)} \right)^{\alpha+d(\kappa-1)} \\
&\lesssim \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(B) |B|^{\frac{\alpha+\nu}{d}}.
\end{aligned}$$

If $\kappa = 1$ and $\alpha = 0$, we use instead (3.2) and (1.11) to have

$$\begin{aligned}
\int_B |Tf_3(x) - (Tf_3)_B| dx &\lesssim \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} |B|^{\frac{\alpha+\nu}{d}} w(B) \left(1 + \log_2 \left(\frac{\rho(x_0)}{r} \right) \right) \log^{-1} \left(\frac{\rho(x_0)}{r} \right) \\
&\lesssim \|f\|_{\mathrm{BMO}^\alpha_\rho(w)} w(B) |B|^{\frac{\alpha+\nu}{d}},
\end{aligned}$$

since $\frac{\rho(x_0)}{r} \geq 2$. The proof of the first implication is now complete.

To see that (ii)$\Rightarrow$(iii), let $w_{x_0}(x) = |x - x_0|^{d(\kappa-1)}$, which is the same type of weight considered in [7, Theorem 2(c)]. From that proof, we know that $w_{x_0} \in D_\kappa \cap A_{s/\eta'} \cap RH_\eta$ for every $\eta \geq 1$ and $s > \eta'\kappa$, being $\kappa \geq 1$. That is, w_{x_0} belongs to $E^{\rho,m}_{s,0,0} \cap D^{\rho,m}_{\kappa,0}$, which trivially satisfies $(c_2 + c_3)(4C_0)^m = 0 < c$, and thus the implication is proved.

Finally, the proof of (iii)$\Rightarrow$ (i) is analogous to the corresponding one given in [7, Theorem 2]. Here, we use that $w_{x_0} \in E^{\rho,m}_{s,0,0} \cap D^{\rho,m}_{\kappa,0}$ and the first part of the proof where the condition on the parameters is trivially satisfied. □

4. Applications

In this section we consider the generalized Schrödinger operator with measure μ,

$$\mathcal{L}_\mu = -\Delta + \mu,$$

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where μ is a nonnegative Radon measure on $\mathbb{R}^d$ and $d \geq 3$. This differential operator and some singular integrals associated with it were first investigated in detail by Z. Shen in [22]. Other operators related with $\mathcal{L}_\mu$ were later considered in [1, 13, 26], and the authors in [14].

Associated with $\mathcal{L}_\mu$, we will study several operators in order to establish weighted endpoint results by means of Theorem 1.7. In each case, we will briefly recall their boundedness properties on weighted and unweighted L^p spaces. In many cases, the results we shall obtain are not only new for the operators associated with $\mathcal{L}_\mu$, but also for $\mathcal{L}_V$ when V is a potential.

In order to achieve positive results, we will deal with a measure μ that satisfies the following conditions, as considered in [22]: there exist constants $\delta_\mu, C_\mu > 0$ and $D_\mu \geq 1$ such that

$$\mu(B(x,r)) \leq C_\mu \left(\frac{r}{R}\right)^{d-2+\delta_\mu} \mu(B(x,R)) \tag{4.1}$$

and

$$\mu(B(x,2r)) \leq D_\mu \left(\mu(B(x,r)) + r^{d-2}\right) \tag{4.2}$$

for all $x \in \mathbb{R}^d$ and $0 < r < R$.

When $d\mu(x) = V(x)dx$, with V a nonnegative function in the class RH_q for some $q > \frac{d}{2}$, the above conditions hold. In this case, it can be seen that the parameter $\delta_\mu =: \delta_V = 2 - \frac{q}{d} > 0$ (see [23, Lemma 1.2]).

From (4.1) it can be proved that (see [22, Remark 0.13])

$$\int_{B(x,R)} \frac{d\mu(y)}{|y-x|^{d-2}} \lesssim \frac{\mu(B(x,R))}{R^{d-2}}, \tag{4.3}$$

and if $\delta_\mu > 1$ then we also have

$$\int_{B(x,R)} \frac{d\mu(y)}{|y-x|^{d-1}} \lesssim \frac{\mu(B(x,R))}{R^{d-1}}, \tag{4.4}$$

for all $x \in \mathbb{R}^d$ and $R > 0$. Both conditions were already known for $d\mu(x) = V(x)dx$ (see [2, Lemma 2.6]).

For any nonnegative Radon measure μ on $\mathbb{R}^d$, $d \geq 3$, satisfying (4.1) and (4.2), the function given by

$$\rho_\mu(x) := \sup\left\{r > 0 : \frac{\mu(B(x,r))}{r^{d-2}} \leq 1\right\}, \quad x \in \mathbb{R}^d, \tag{4.5}$$

is a critical radius function.

A useful property that can be obtained immediately from the definition of ρ_μ is the following: for every $x \in \mathbb{R}^d$

$$\frac{\mu(B(x,\rho_\mu(x)))}{\rho_\mu(x)^{d-2}} \sim 1. \tag{4.6}$$

Since all the results on the previous sections were obtained in terms of the inequalities in (1.1), we may not need, in general, the definition (4.5) of ρ_μ.

In this setting, the natural metric is given by the Agmon distance, defined for any critical radius function ρ as

$$d_\rho(x,y) := \inf_\gamma \int_0^1 \rho(\gamma(t))^{-1}|\gamma'(t)|dt, \tag{4.7}$$

where the infimum is taken over every absolutely continuous function $\gamma : [0,1] \to \mathbb{R}^d$ with $\gamma(0) = x$ and $\gamma(1) = y$. When $\rho = \rho_\mu$, we will denote $d_\rho(x,y) = d_\mu(x,y)$ to emphasize the measure dependence.

In the following sections, for $\lambda \geq 0$ we will denote by $\Gamma_{\mu+\lambda}$ and Γ_λ the fundamental solutions of $-\Delta + \mu + \lambda$ and $-\Delta + \lambda$, respectively. In particular, Γ_0 will be the fundamental solution of $-\Delta$. Also, by $d_{\mu+\lambda}$ and d_λ we mean the Agmon distances related to the critical radius functions $\rho_{\mu+\lambda}$ and ρ_λ with respect to the measures $d\mu(x) + \lambda dx$ and λdx, respectively. Actually, from the definition (4.7), we have that $d_\lambda(x,y) = \sqrt{\omega_d \lambda}|x-y|$, where ω_d is the volume of the unit ball in $\mathbb{R}^d$ (see [1, p. 28]).

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4.1. **Riesz transforms.** Associated to $\mathcal{L}_\mu$ we define the singular integral operators $\mathcal{R}_\mu = \nabla(-\Delta+\mu)^{-\frac{1}{2}}$ and $\mathcal{R}_\mu^* = (-\Delta+\mu)^{-\frac{1}{2}}\nabla$ as in [22], which are the Riesz transforms in this setting.

The Riesz transform $\mathcal{R}_\mu$ can be expressed, for $f \in C_c^\infty(\mathbb{R}^d)$ and $x \notin \operatorname{supp}(f)$, as

$$\begin{aligned}\mathcal{R}_\mu f(x) &= \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\nabla(-\Delta+\mu+\lambda)^{-1}f(x)d\lambda \\ &= \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\int_{\mathbb{R}^d}\nabla_1\Gamma_{\mu+\lambda}(x,y)f(y)dy\,d\lambda \\ &= \int_{\mathbb{R}^d} K_\mu(x,y)f(y)dy,\end{aligned}$$

where K_μ is the singular kernel of $\mathcal{R}_\mu$ given by

$$K_\mu(x,y) = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\nabla_1\Gamma_{\mu+\lambda}(x,y)d\lambda. \tag{4.8}$$

The operator $\mathcal{R}_\mu^*$ also has an associated kernel, which is $K_\mu^*(x,y) = -K_\mu(y,x)$.

In [22, Theorem 0.20] it was established that $\mathcal{R}_\mu$ and $\mathcal{R}_\mu^*$ are Calderón-Zygmund operators when $\delta_\mu > 1$. So, in that case, $\mathcal{R}_\mu$ and $\mathcal{R}_\mu^*$ are bounded on $L^p(\mathbb{R}^d)$, for every $1<p<\infty$. Moreover, [14, Proposition 5.3] assures that, if $\delta_\mu > 1$, both $\mathcal{R}_\mu$ and $\mathcal{R}_\mu^*$ are exponential SCZO of $(0,\infty,\delta)$ type, with parameters $c = \frac{\epsilon}{2D_1}$ and $m = \frac{1}{k_0+1}$, where ϵ and D_1 depend on the Agmon distance d_μ. Although the parameter δ was not previously specified, from Lemma 4.3 it follows that $\delta = \min\{1, \delta_\mu - 1\}$ for $\mathcal{R}_\mu$, and by [22, Eq. (7.29)] there exists some $\delta \in (0,1)$ that gives the type for $\mathcal{R}_\mu^*$. This fact led us to recover the weighted L^p–boundedness result for $\mathcal{R}_\mu$ and $\mathcal{R}_\mu^*$ (see [14, Theorem 5.4]) previously obtained in [1, Theorem 4.1(i)], but gave us a better understanding of the kernel properties that not only these transforms have.

When $0 < \delta_\mu < 1$, $\mathcal{R}_\mu$ and $\mathcal{R}_\mu^*$ behave differently on L^p spaces, even in the potential case. It is well-known (see [23]) that, whenever $V \in RH_q$, $\frac{d}{2} < q < d$, $\mathcal{R}_V$ is bounded on $L^p(\mathbb{R}^d)$ for every $1 < p \le p_0$ and that $\mathcal{R}_V^*$ is bounded on $L^p(\mathbb{R}^d)$ for every $p_0' \le p < \infty$, being $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{d}$. This implies that they are not Calderón-Zygmund operators when $0 < \delta_V < 1$, where $\delta_V = 2 - \frac{d}{q}$. Analogous results were obtained for the generalized Riesz transforms $\mathcal{R}_\mu$ and $\mathcal{R}_\mu^*$ in [22] on the unweighted case, and in [1] and [14] with weights. In fact, in [1] it is proved that $\mathcal{R}_\mu$ is bounded on $L^p(w)$ for every $1 < p < \eta$ and $w^{-1/(p-1)} \in S^\mu_{p'/\eta',c}$ for some $c > 0$ and any $2 < \eta < (2-\delta_\mu)'$, and $\mathcal{R}_\mu^*$ is bounded on $L^p(w)$ for every $\eta' < p < \infty$ and $w \in S^\mu_{p/\eta',c}$ (for the definitions of these classes, see the mentioned article). In [14] we obtained by extrapolation a similar result but with weights in the $H^{\rho,m}_{p,c}$ classes, which are equivalent, in some sense, to the $S^\mu_{p,c}$ classes as stated in [1, Proposition 3.2]. We also proved ([14, Proposition 5.5]) that $\mathcal{R}_\mu^*$ is an exponential SCZO of $(0,s,\delta)$ type, with parameters $c = \frac{\epsilon}{4D_1}$ and $m = \frac{1}{k_0+1}$, provided that $1 < s < (2-\delta_\mu)'$ and for some $\delta \in (0,1)$ (see [22, Eq. (7.29)]).

First, note that, since $\mathcal{R}_\mu^*(1) = 0$, condition (i) of Theorem 1.7 is trivially verified. Therefore, the following result holds for $\mathcal{R}_\mu^*$, taking into account [14, Propositions 5.3 and 5.5], Theorem 1.7 and Corollary 1.8.

Theorem 4.1. *Let $m_0 = \frac{1}{k_0+1}$, $\delta \in (0,1)$ as above, $0 \le \alpha < \delta$ and $1 \le \kappa < \frac{\delta-\alpha}{d}+1$, and w a weight.*

(i) If $\delta_\mu > 1$, $\mathcal{R}_\mu^*$ is bounded on $\mathrm{BMO}^\alpha_\rho(w)$ if and only if $w \in E^{\rho,m_0}_{\infty,c_1,c_2} \cap D^{\rho,m_0}_{\kappa,c_3}$ with $c_1,c_2,c_3 \ge 0$ such that $(c_2+c_3) < \frac{\epsilon}{2D_1}\left(1 - \frac{d(\kappa-1)+\alpha}{\delta}\right)(4C_0)^{-m_0}$.

(ii) If $0 < \delta_\mu < 1$ and $1 < s < \frac{2-\delta_\mu}{1-\delta_\mu}$, $\mathcal{R}_\mu^*$ is bounded on $\mathrm{BMO}^\alpha_\rho(w)$ if and only if $w \in E^{\rho,m_0}_{s,c_1,c_2} \cap D^{\rho,m_0}_{\kappa,c_3}$ with $c_1,c_2,c_3 \ge 0$ such that $(c_2+c_3) < \frac{\epsilon}{4D_1}\left(1 - \frac{d(\kappa-1)+\alpha}{\delta}\right)(4C_0)^{-m_0}$.

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The behavior of $\mathcal{R}_\mu$ when $0 < \delta_\mu < 1$ says that we can only expect to obtain endpoint results on $\mathrm{BMO}_\rho^\alpha(w)$ when $\delta_\mu > 1$. In fact, we obtain $\mathrm{BMO}_\rho^\alpha(w)$ boundedness for R_μ provided $1 < \delta_\mu < d$, as well as the analogue for the particular case R_V. This result characterizes weighted boundedness for weights featuring exponential growth (cf. [7, Theorem 7] for weights with polynomial growth).

Theorem 4.2. *Let $m_0 = \frac{1}{k_0+1}$, $\delta = \min\{1, \delta_\mu - 1\}$, $0 \le \alpha < \delta$ and $1 \le \kappa < \frac{\delta-\alpha}{d} + 1$. Then, if $1 < \delta_\mu < d$, $\mathcal{R}_\mu$ is bounded on $\mathrm{BMO}_\rho^\alpha(w)$ if and only if $w \in E_{\infty,c_1,c_2}^{\rho,m_0} \cap D_{\kappa,c_3}^{\rho,m_0}$ with $c_1, c_2, c_3 \ge 0$ such that $(c_2 + c_3) < \frac{\epsilon}{4D_1}\left(1 - \frac{d(\kappa-1)+\alpha}{\delta}\right)(4C_0)^{-m_0}$.*

The proof of the previous theorem will be given at the end of this section, as several preliminary lemmas will be required. The first one was already obtained by Shen within the proof of [22, Theorem 7.18].

Lemma 4.3 ([22, (7.20) and (7.26)])**.** *Let $\delta_\mu > 1$. Then there exist constants $C, \epsilon > 0$ such that*

(i) for every $x \neq y$,

$$|K_\mu(x,y)| \le C\frac{e^{-\epsilon d_\mu(x,y)}}{|x-y|^d}. \tag{4.9}$$

(ii) When $|x-y| > 2|x-z|$,

$$|K_\mu(x,y) - K_\mu(z,y)| \le \frac{C}{|x-y|^d}\left(\frac{|x-z|}{|x-y|}\right)^{\delta_\mu - 1}. \tag{4.10}$$

In the next result we will denote by K_0 the kernel of the classical Riesz transform $\mathcal{R}_0 = \nabla(-\Delta)^{-\frac{1}{2}}$.

Lemma 4.4. *Let $1 < \delta_\mu < d$ and $0 < \delta \le \min\{1, \delta_\mu - 1\}$. Then there exists a constant $C > 0$ such that*

(i) For every $x \neq y$,

$$|K_\mu(x,y) - K_0(x,y)| \le \frac{C}{|x-y|^d}\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu}. \tag{4.11}$$

(ii) When $|x-y| \ge 2|x-z|$,

$$|[K_\mu(x,y) - K_0(x,y)] - [K_\mu(z,y) - K_0(z,y)]| \le C\frac{|x-z|^\delta}{|x-y|^{d+\delta}}\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu}. \tag{4.12}$$

Remark 4.5. We note that the restriction $\delta_\mu < d$ is asked only for the proof of Lemma 4.4(ii) and the proof of Theorem 4.2, independently. Although the properties (4.1) and (4.2) for μ do not seem to imply that δ_μ is smaller than the dimension, we observe that this is automatically satisfied in the potential case. Indeed, if $V \in RH_q$ for some $q > d$, $\delta_V = 2 - d/q < 2 < d$ since $d \ge 3$ and $q > 0$. Therefore, we recover [10, Lemma 4] when $d\mu(x) = V(x)dx$ with V as before since, in that case, the smoothness condition for K_V holds for any $0 < \delta < 1 - d/q = \min\{1, \delta_V - 1\}$.

Proof. Regarding (i), note that if $|x-y| > \rho(x)$, the result is true since both are Calderón-Zygmund kernels. If $|x-y| \le \rho(x)$, we use the following estimate (see proof of [22, Lemma 7.13]):

$$|K_\mu(x,y) - K_0(x,y)| \lesssim \frac{1}{|x-y|^{d-1}}\left(\int_{B(x,|x-y|/2)} \frac{d\mu(z)}{|z-x|^{d-1}} + \frac{1}{|x-y|}\left(\frac{|x-y|}{\rho(y)}\right)^{\delta_\mu}\right), \quad x \neq y.$$

By (4.4), (4.1) and (4.6), we obtain that the integral given above can be estimated as follows

$$\int_{B(x,|x-y|/2)} \frac{d\mu(z)}{|z-x|^{d-1}} \lesssim \frac{\mu(B(x,\rho(x)))}{|x-y|^{d-1}}\left(\frac{|x-y|}{2\rho(x)}\right)^{d-2+\delta_\mu} \sim \frac{1}{|x-y|}\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu}.$$

Then, we get

$$|K_\mu(x,y) - K_0(x,y)| \lesssim \frac{1}{|x-y|^d}\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu}.$$

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To prove item (ii), let $|x-y|\geq 2|x-z|$. Note that when $|x-y|\geq\rho(x)$, both operators have kernels that satisfy a Calderón-Zygmund smoothness estimate on the first variable. More precisely,

$$\begin{aligned}|K_\mu(x,y)-K_\mu(z,y)|+|K_0(x,y)-K_0(z,y)| &\leq \frac{C}{|x-y|^d}\left(\frac{|x-z|}{|x-y|}\right)^{\delta_\mu-1}+\frac{C}{|x-y|^d}\frac{|x-z|}{|x-y|}\\ &\leq C\frac{|x-z|^\delta}{|x-y|^{d+\delta}}\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu}\end{aligned}$$

for any $0<\delta\leq\min\{1,\delta_\mu-1\}$.

Now suppose $|x-y|<\rho(x)$. According to (4.8), the difference of the kernels can be written as

$$K_\mu(x,y)-K_0(x,y)=\frac{1}{\pi}\int_0^\infty\lambda^{-\frac{1}{2}}(\nabla_1\Gamma_{\mu+\lambda}(x,y)-\nabla_1\Gamma_\lambda(x,y))d\lambda.$$

We will rewrite the integrand above, so we recall that $u(\cdot,y)=\Gamma_{\mu+\lambda}(\cdot,y)-\Gamma_\lambda(\cdot,y)$ (as a function of the first variable) satisfies the equation $(-\Delta+\lambda)u=-\mu\Gamma_{\mu+\lambda}$. Then, we get

$$\Gamma_{\mu+\lambda}(x,y)-\Gamma_\lambda(x,y)=-\int_{\mathbb{R}^d}\Gamma_\lambda(x,v)\Gamma_{\mu+\lambda}(v,y)d\mu(v).$$

Hence,

$$K_\mu(x,y)-K_0(x,y)=-\frac{1}{\pi}\int_0^\infty\lambda^{-\frac{1}{2}}\left(\int_{\mathbb{R}^d}\nabla_1\Gamma_\lambda(x,v)\Gamma_{\mu+\lambda}(v,y)d\mu(v)d\lambda\right).$$

Consequently, the difference of interest is

$$\begin{aligned}&[K_\mu(x,y)-K_0(x,y)]-[K_\mu(z,y)-K_0(z,y)]\\ &\qquad=-\frac{1}{\pi}\int_0^\infty\lambda^{-\frac{1}{2}}\int_{\mathbb{R}^d}(\nabla_1\Gamma_\lambda(x,v)-\nabla_1\Gamma_\lambda(z,v))\Gamma_{\mu+\lambda}(v,y)d\mu(v)d\lambda. \qquad (4.13)\end{aligned}$$

We will deal first with the absolute value of the inner integral before performing the integration on λ. As in the proof of [10, Lemma 4], we consider four regions covering $\mathbb{R}^d$,

$$\begin{aligned}E_1&=\left\{v\in\mathbb{R}^d:|v-x|<\tfrac{3}{2}|x-z|\right\},\\ E_2&=\left\{v\in\mathbb{R}^d:\tfrac{3}{2}|x-z|\leq|v-x|<\tfrac{1}{2}|x-y|\right\},\\ E_3&=\left\{v\in\mathbb{R}^d:\tfrac{1}{2}|x-y|\leq|v-x|<2|x-y|\right\},\\ E_4&=\left\{v\in\mathbb{R}^d:|v-x|\geq 2|x-y|\right\},\end{aligned}$$

and we denote

$$I_j=\int_{E_j}|\nabla_1\Gamma_\lambda(x,v)-\nabla_1\Gamma_\lambda(z,v)||\Gamma_{\mu+\lambda}(v,y)|d\mu(v),\quad j=1,2,3,4.$$

Provided we can show that for some $N>0$ and any $k\geq N$,

$$I_j\lesssim\frac{C_{k,d}|x-z|^\delta}{(1+\lambda^{1/2}|x-y|)^{k\epsilon_2}|x-y|^{d+\delta-1}}\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu},\quad\forall\ 0<\delta<\min\{1,\delta_\mu-1\},\ j=1,2,3,4,\qquad (4.14)$$

integrating in λ the expression (4.13), and choosing $k>\max\{N,\frac{1}{\epsilon_2}\}$ we obtain (4.12). Therefore, let us prove (4.14).

For I_1, we bound by the sum of the gradients and estimate each integral separately. That is

$$\int_{|v-x|<\frac{3}{2}|x-z|}|\nabla_1\Gamma_\lambda(x,v)||\Gamma_{\mu+\lambda}(v,y)|d\mu(v)+\int_{|v-x|<\frac{3}{2}|x-z|}|\nabla_1\Gamma_\lambda(z,v)||\Gamma_{\mu+\lambda}(v,y)|d\mu(v):=I_{1,1}+I_{1,2}.$$

On each one, we will use estimates for the gradient of the fundamental solution Γ_λ, for $\lambda\geq 0$. It is known, and can be found in [23, Eq. (4.4)], that for any $k>0$, there exists $C_k>0$ such that

$$|\nabla_1\Gamma_\lambda(x,v)|\leq\frac{C_k}{(1+\lambda^{1/2}|x-v|)^k}\frac{1}{|x-v|^{d-1}},\quad x\neq v,\qquad (4.15)$$

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and an analogous bound for $|\nabla_1\Gamma_\lambda(z,v)|$.

To estimate $\Gamma_{\mu+\lambda}(v,y)$, we use [22, Eq. (3.12)] along with the inequality (see [1, Eq. (21)])

$$d_{\mu+\lambda}(v,y) \geq \frac{1}{2}d_\mu(v,y) + \frac{\sqrt{\omega_d}}{2}\sqrt{\lambda}|v-y|. \tag{4.16}$$

Thus, for any $k>0$, there exist positive constants C, ϵ_1, and ϵ_2, independent of v and y, such that the following holds

$$0 \leq \Gamma_{\mu+\lambda}(v,y) \leq \frac{Ce^{-\epsilon_1 d_{\mu+\lambda}(v,y)}}{|v-y|^{d-2}} \leq \frac{C_{k,d}\, e^{-\epsilon_2 d_\mu(v,y)}}{|v-y|^{d-2}(1+\lambda^{1/2}|v-y|)^{k\epsilon_2}}, \quad v \neq y, \tag{4.17}$$

Since $|x-y| \geq 2|x-z|$, for $v \in E_1$ we have $|v-y| \geq \frac{1}{4}|x-y|$. Then,

$$\Gamma_{\mu+\lambda}(v,y) \leq \frac{C_{k,d}}{|x-y|^{d-2}(1+\lambda^{1/2}|x-y|)^{k\epsilon_2}}, \quad x \neq y. \tag{4.18}$$

By (4.15) and (4.18), and applying the properties (4.4), (4.1) and (4.6) for the measure μ, we obtain

$$\begin{aligned}
I_{1,1} &\leq \frac{C_{k,d}}{(1+\lambda^{1/2}|x-y|)^{k\epsilon_2}|x-y|^{d-2}} \int_{B(x,\frac{3}{2}|x-z|)} \frac{d\mu(v)}{|x-v|^{d-1}} \\
&\lesssim \frac{C_{k,d}}{(1+\lambda^{1/2}|x-y|)^{k\epsilon_2}|x-y|^{d-2}} \frac{\mu\left(B(x,\frac{3}{2}|x-z|)\right)}{|x-z|^{d-1}} \\
&\lesssim \frac{C_{k,d}}{(1+\lambda^{1/2}|x-y|)^{k\epsilon_2}|x-y|^{d-2}} \frac{\mu(B(x,\rho(x)))}{\rho(x)^{d-2}} \frac{|x-z|^{\delta_\mu-1}}{\rho(x)^{\delta_\mu}} \\
&\lesssim \frac{C_{k,d}|x-z|^\delta}{(1+\lambda^{1/2}|x-y|)^{k\epsilon_2}|x-y|^{d-1+\delta}} \left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu},
\end{aligned}$$

for $\delta = \delta_\mu - 1$. Since $|x-y| < \rho(x)$, the above inequality also holds for any $0<\delta<\delta_\mu-1$.

For $I_{1,2}$ we use again (4.18), (4.15) with z instead of x, and take into account that $B(x,\frac{3}{2}|x-z|) \subseteq B(x,\frac{5}{2}|x-z|)$ so the argument follows in the same way. Combining both bounds, we have

$$I_1 \lesssim \frac{C_{k,d}|x-z|^\delta}{(1+\lambda^{1/2}|x-y|)^{k\epsilon_2}|x-y|^{d-1+\delta}} \left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu}, \quad 0<\delta\leq\delta_\mu-1. \tag{4.19}$$

Next, to take care of the integrals in the other three regions, we apply the Mean Value Theorem and use an estimate for $\nabla_1^2\Gamma_\lambda$, which is also known (see [23, Eq. (4.4)]). Then we have that for every $k>0$, there exists $C_k>0$ such that

$$\begin{aligned}
|\nabla_1\Gamma_\lambda(x,v) - \nabla_1\Gamma_\lambda(z,v)| &= \left|\nabla_1^2\Gamma_\lambda(\xi,v)\right| |x-z| \\
&\leq \frac{C_k|x-z|}{(1+\lambda^{1/2}|\xi-v|)^k|\xi-v|^d} \leq \frac{C_k|x-z|}{(1+\lambda^{1/2}|x-v|)^k|x-v|^d},
\end{aligned} \tag{4.20}$$

where $\xi = (1-\theta)x + \theta z$, for some $\theta \in (0,1)$.

Then, using the above inequality and the estimate (4.17) for $\Gamma_{\mu+\lambda}$, for every $j=2,3,4$, we get

$$I_j \leq \int_{E_j} \frac{C_{k,d}|x-z|}{(1+\lambda^{1/2}|x-v|)^k|x-v|^d} \frac{e^{-\epsilon_2 d_\mu(v,y)}}{|v-y|^{d-2}(1+\lambda^{1/2}|v-y|)^{k\epsilon_2}} d\mu(v).$$

For $j=2$ we use that $|v-y| > \frac{1}{2}|x-y|$ for every $v \in E_2$ to get

$$\begin{aligned}
I_2 &\lesssim \frac{C_{k,d}|x-z|^\delta}{(1+\lambda^{1/2}|x-y|)^{k\epsilon_2}|x-y|^{d-2}} \int_{\frac{3}{2}|x-z|\leq|v-x|<\frac{1}{2}|x-y|} \frac{|x-z|^{1-\delta}}{|x-v|^d} d\mu(v) \\
&\lesssim \frac{C_{k,d}|x-z|^\delta}{(1+\lambda^{1/2}|x-y|)^{k\epsilon_2}|x-y|^{d-2}} \int_{B(x,|x-y|)} \frac{d\mu(v)}{|x-v|^{d-(1-\delta)}}.
\end{aligned}$$

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Decomposing the last integral into annulli, recalling that we are in the case $|x-y|<\rho(x)$ and using (4.1), it becomes

$$\begin{aligned}\int_{B(x,|x-y|)} \frac{d\mu(v)}{|x-v|^{d-(1-\delta)}} &\leq \sum_{j=0}^{\infty} \int_{|v-x|<2^{-j}|x-y|} \left(2^{-j-1}|x-y|\right)^{-d+(1-\delta)} d\mu(v) \\ &\lesssim \sum_{j=0}^{\infty} \frac{\mu(B(x,2^{-j}|x-y|))}{(2^{-j}|x-y|)^{d-(1-\delta)}} \\ &\lesssim \sum_{j=0}^{\infty} \frac{\mu(B(x,\rho(x)))}{(2^{-j}|x-y|)^{d-(1-\delta)}} \left(\frac{2^{-j}|x-y|}{\rho(x)}\right)^{d-2+\delta_\mu} \\ &\lesssim \frac{1}{|x-y|^{1+\delta}} \left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu} \sum_{j=0}^{\infty} 2^{-j(\delta_\mu-(1+\delta))} \\ &\lesssim \frac{1}{|x-y|^{1+\delta}} \left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu},\end{aligned}$$

provided that $\delta<\delta_\mu-1$. Therefore,

$$I_2 \lesssim \frac{C_{k,d}|x-z|^\delta}{(1+\lambda^{1/2}|x-y|)^{k\epsilon_1}|x-y|^{d-1+\delta}} \left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu}, \quad 0<\delta<\delta_\mu-1. \tag{4.21}$$

For $v\in E_3$, $\frac{1}{2}|x-y|\leq|v-x|\leq 2|x-y|$ which also yields $|v-y|<3|x-y|$. Then, using again (4.20), (4.17) and (4.3), we have

$$I_3 \lesssim \frac{C_{k,d}|x-z|}{(1+\lambda^{1/2}|x-y|)^{k\epsilon_2}|x-y|^d} \frac{\mu(B(x,4|x-y|))}{|x-y|^{d-2}}.$$

Applying (4.1), (4.2) and (4.6), we get the estimate (4.14) for I_3 with $\delta=1$ and, therefore,

$$I_3 \lesssim \frac{C_{k,d}|x-z|^\delta}{(1+\lambda^{1/2}|x-y|)^{k\epsilon_2}|x-y|^{d+\delta}} \left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu}, \quad 0<\delta\leq 1. \tag{4.22}$$

Finally, to deal with E_4 we use again (4.20) and (4.17), and the fact that $\rho(x)\sim\rho(y)$ by (1.2) (we are considering the case $|x-y|<\rho(x)$). We also note that for $v\in E_4$, $|v-x|\sim|v-y|$.

Since we are integrating in the global part, we will take advantage of the exponential decay by considering the lower bound for the Agmon distance d_μ given in [14, Lemma 5.1], i.e., for some $D_1, D_0>1$, there exists $C>0$ depending on them and k_0 such that

$$d_\mu(v,u) \geq D_1^{-1}\left(1+\frac{|v-u|}{\rho(u)}\right)^{\frac{1}{k_0+1}} + D_0^{-1}\left(1+\frac{\rho(u)}{|v-u|}\right)^{-1} - C, \quad v,u\in\mathbb{R}^d.$$

This yields, for every $k>0$, there exists C_k such that

$$e^{-\epsilon_2 d_\mu(v,u)} \leq C_k \left(1+\frac{|v-u|}{\rho(u)}\right)^{\frac{-k\epsilon_2}{k_0+1}}.$$

Therefore, from all of the above considerations,

$$I_4 \lesssim \frac{C_k|x-z|}{(1+\lambda^{1/2}|x-y|)^{k\epsilon_2}} \int_{|v-x|\geq 2|x-y|} \frac{\left(1+\frac{|v-x|}{\rho(x)}\right)^{\frac{-k\epsilon_2}{k_0+1}}}{|v-x|^{2d-2}} d\mu(v). \tag{4.23}$$

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We shall split the integral above inside the critical ball $B(x,\rho(x))$ and outside of it. For the first part, let $j_0\in\mathbb{N}$ such that $2^{j_0-1}|x-y|\le\rho(x)<2^{j_0}|x-y|$. Then, by splitting into dyadic annuli

$$\int_{2|x-y|\le|v-x|<\rho(x)}\frac{\left(1+\frac{|v-x|}{\rho(x)}\right)^{\frac{-k\epsilon_2}{k_0+1}}}{|v-x|^{2d-2}}d\mu(v)\lesssim\frac{1}{|x-y|^d}\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu}\sum_{j=1}^{j_0-1}2^{j(\delta_\mu-d)}\lesssim\frac{1}{|x-y|^d}\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu},\tag{4.24}$$

where we have used (4.1), and that $\delta_\mu-d<0$.

When $|v-x|>\rho(x)$, by again splitting into dyadic annuli and choosing $k>\max\left\{\frac{(k_0+1)(\log_2(D_\mu)-d)}{\epsilon_2},0\right\}$, we obtain

$$\begin{aligned}\int_{|v-x|>\rho(x)}\frac{\left(1+\frac{|v-x|}{\rho(x)}\right)^{-\frac{k\epsilon_2}{k_0+1}}}{|v-x|^{2d-2}}d\mu(v)&\lesssim\rho(x)^{2-2d}\sum_{j\in\mathbb{N}}2^{-j\left(2d-2+\frac{k\epsilon_2}{k_0+1}\right)}\left(D_\mu^{j+1}2^{(d-2)(j+1)}\rho(x)^{d-2}\right)\\&\lesssim\rho(x)^{-d}\sum_{j\in\mathbb{N}}\left(D_\mu2^{-d-\frac{k\epsilon_2}{k_0+1}}\right)^j\lesssim\rho(x)^{-d}\lesssim\frac{1}{|x-y|^d}\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu},\end{aligned}$$

where we have used (4.2), (4.6), and the fact that $\delta_\mu-d<0$ and $|x-y|<\rho(x)$.

Thus, from (4.23) and (4.24) we obtain

$$I_4\le\frac{C_k|x-z|^\delta}{(1+\lambda^{1/2}|x-y|)^{k\epsilon_1}}\frac{1}{|x-y|^{d+\delta}}\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_\mu},\quad 0<\delta\le1.\tag{4.25}$$

Therefore, in (4.19), (4.21), (4.22) and (4.25) we have proved (4.14) for each $j=1,2,3,4$. □

Proof of Theorem 4.2. By Corollary 1.8 and Remark 1.9, we need to verify that for $1<\delta_\mu<d$, $x_0\in\mathbb{R}^d$ and $0<r\le\frac{1}{2}\rho(x_0)$, there exists a constant C, independent of $B=B(x_0,r)$, such that

$$\frac{1}{|B|}\int_B|\mathcal{R}_\mu1(y)-(\mathcal{R}_\mu1)_B|\,dy\le C\left(\frac{r}{\rho(x_0)}\right)^\beta,\quad 0<\beta<\min\{1,\delta_\mu-1\}.$$

The proof follows the general strategy of [21, Proposition 4.12], with several modifications. In particular, we provide a self-contained argument, as some of the estimates used there do not apply directly in our setting.

Let $y,z\in B$. By (1.2) we have $\rho(y)\sim\rho(x_0)\sim\rho(z)$. Since

$$\mathcal{R}_\mu1(x)=\lim_{\epsilon\to0^+}\int_{|x-y|>\epsilon}K_\mu(x,y)dy\qquad a.e.\quad x\in\mathbb{R}^d$$

we have

$$\begin{aligned}|\mathcal{R}_\mu1(y)-\mathcal{R}_\mu1(z)|\le\lim_{\epsilon\to0^+}&\left|\int_{\epsilon<|x-y|<4\rho(x_0)}K_\mu(y,x)dx-\int_{\epsilon<|z-x|<4\rho(x_0)}K_\mu(z,x)dx\right|\\&+\left|\int_{|x-y|>4\rho(x_0)}K_\mu(y,x)dx-\int_{|z-x|>4\rho(x_0)}K_\mu(z,x)dx\right|\\&=:\lim_{\epsilon\to0^+}A_\epsilon+\tilde{A}.\end{aligned}$$

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To analyze A_ϵ, we consider $0 < \epsilon < 4\rho(x_0) - 2r$. Since $\int_{r_1<|x-y|<r_2} K_0(x,y)dy = 0$ for every $0 < r_1 < r_2$, we can write

$$\begin{aligned}
A_\epsilon &= \left| \int_{\epsilon<|x-y|\le 4\rho(x_0)} (K_\mu(y,x) - K_0(y,x))dx - \int_{\epsilon<|z-x|<4\rho(x_0)} (K_\mu(z,x) - K_0(z,x))dx \right| \\
&\le \int_{\mathbb{R}^d} \left| (K_\mu(y,x) - K_0(y,x))(\chi_{\epsilon<|x-y|\le 4\rho(x_0)}(x) - \chi_{\epsilon<|x-z|\le 4\rho(x_0)}(x)) \right| dx \\
&+ \int_{\mathbb{R}^d} \left| [(K_\mu(y,x) - K_0(y,x)) - (K_\mu(z,x) - K_0(z,x))]\chi_{\epsilon<|x-z|\le 4\rho(x_0)}(x) \right| dx \\
&:= A_{\epsilon,1} + A_{\epsilon,2}.
\end{aligned}$$

The term $A_{\epsilon,1}$ is nonzero only in the following 4 cases:

(i) $\epsilon < |x-y| \le 4\rho(x_0)$ and $|x-z| \le \epsilon$;
(ii) $\epsilon < |x-y| \le 4\rho(x_0)$ and $|x-z| > 4\rho(x_0)$;
(iii) $\epsilon < |x-z| \le 4\rho(x_0)$ and $|x-y| \le \epsilon$;
(iv) $\epsilon < |x-z| \le 4\rho(x_0)$ and $|x-y| > 4\rho(x_0)$.

In the first case we have $\epsilon < |x-y| \le |x-z| + |z-y| < \epsilon + 2r$. Then, using the estimate (4.11), integrating and using the Mean Value Theorem we get

$$A_{\epsilon,1} \lesssim \frac{1}{\rho(x_0)^{\delta_\mu}} \int_\epsilon^{\epsilon+2r} t^{\delta_\mu-1}dt \sim \frac{1}{\rho(x_0)^{\delta_\mu}} \frac{1}{\delta_\mu} \left((\epsilon+2r)^{\delta_\mu} - \epsilon^{\delta_\mu} \right) \lesssim \frac{(4\rho(x_0))^{\delta_\mu-1}}{\rho(x_0)^{\delta_\mu}} 2r \sim \frac{r}{\rho(x_0)}. \tag{4.26}$$

In the second case, observe that $4\rho(x_0) < |x-z| < |x-y| + 2r$. Thus, $4\rho(x_0) - 2r < |x-y| \le 4\rho(x_0)$. Then, proceeding as in the previous case,

$$A_{\epsilon,1} \lesssim \int_{4\rho(x_0)-2r<|x-y|<4\rho(x_0)} \frac{1}{|x-y|^d} \left(\frac{|x-y|}{\rho(x)} \right)^{\delta_\mu} dx \lesssim \frac{r}{\rho(x_0)}.$$

With respect to the third and fourth cases, although the conditions seem to be symmetric, the kernels are still evaluated in y instead of z.

In the third case, without loss of generality, as we will take $\epsilon \to 0^+$, we may assume $\epsilon < 4r$. Thus, by (4.11), we deduce

$$A_{\epsilon,1} \lesssim \frac{1}{\rho(x_0)^{\delta_\mu}} \int_{|x-y|\le\epsilon} |x-y|^{\delta_\mu-d}dx \lesssim \frac{1}{\rho(x_0)^{\delta_\mu}} \epsilon^{\delta_\mu} \lesssim \left(\frac{r}{\rho(x_0)} \right)^{\delta_\mu} \lesssim \frac{r}{\rho(x_0)},$$

where in the last inequality we have used that $r < \rho(x_0)$ and $\delta_\mu > 1$.

To estimate in the fourth case, note that $4\rho(x_0) < |x-y| < 4\rho(x_0) + 2r$, which gives, proceeding similarly to (4.26), that

$$A_{\epsilon,1} \lesssim \frac{1}{\rho(x_0)^{\delta_\mu}} \left((4\rho(x_0)+2r)^{\delta_\mu} - (4\rho(x_0))^{\delta_\mu} \right) \lesssim \frac{1}{\rho(x_0)^{\delta_\mu}} (5\rho(x_0))^{\delta_\mu-1} 2r \lesssim \frac{r}{\rho(x_0)}.$$

In conclusion, since $r < \rho(x_0)$,

$$A_{\epsilon,1} \le C \left(\frac{r}{\rho(x_0)} \right)^\delta, \quad 0 < \delta \le 1. \tag{4.27}$$

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Now we consider $A_{\epsilon,2}$, and we split it as follows,

$$\begin{aligned}
A_{\epsilon,2} &= \int_{\mathbb{R}^d} \left|[(K_\mu(y,x)-K_0(y,x))-(K_\mu(z,x)-K_0(z,x))]\chi_{\epsilon<|x-z|\le 4\rho(x_0)}(x)\right| dx \\
&\le \int_{|z-x|\ge 2|z-y|} \left|[(K_\mu(z,x)-K_0(z,x))-(K_\mu(y,x)-K_0(y,x))]\chi_{\epsilon<|x-z|\le 4\rho(x_0)}(x)\right| dx \\
&+ \int_{|z-x|<2|z-y|} |[(K_\mu(z,x)-K_0(z,x))-(K_\mu(y,x)-K_0(y,x))]|\, dx \\
&= A_{\epsilon,2,1}+A_{\epsilon,2,2}.
\end{aligned}$$

By (4.12), for any $0<\delta<\min\{1,\delta_\mu-1\}$,

$$A_{\epsilon,2,1} \lesssim \frac{|y-z|^\delta}{\rho(x_0)^{\delta_\mu}} \int_{|x-z|<4\rho(x_0)} |x-z|^{\delta_\mu-d-\delta} dx \lesssim \frac{r^\delta}{\rho(x_0)^{\delta_\mu}} (4\rho(x_0))^{\delta_\mu-\delta} \lesssim \left(\frac{r}{\rho(x_0)}\right)^\delta. \tag{4.28}$$

On the other hand, by (4.11) we get

$$\begin{aligned}
A_{\epsilon,2,2} &\lesssim \int_{|x-z|<2|y-z|} \frac{1}{|x-z|^d}\left(\frac{|x-z|}{\rho(z)}\right)^{\delta_\mu} dx + \int_{|x-y|<3|y-z|} \frac{1}{|x-y|^d}\left(\frac{|x-y|}{\rho(y)}\right)^{\delta_\mu} dx \\
&\lesssim \frac{1}{\rho(x_0)^{\delta_\mu}}|y-z|^{\delta_\mu} \lesssim \left(\frac{r}{\rho(x_0)}\right)^\delta \qquad , \qquad\qquad (4.29)
\end{aligned}$$

for any $\delta \le \delta_\mu$. Hence, combining the estimates obtained in (4.27), (4.28) and (4.29) we have that for all $\epsilon>0$ sufficiently small, and any $0<\delta<\min\{1,\delta_\mu-1\}$,

$$A_\epsilon \le C\left(\frac{r}{\rho(x_0)}\right)^\delta. \tag{4.30}$$

Let us now estimate $\tilde{A}$, by writing

$$\begin{aligned}
\tilde{A} &= \left| \int_{|x-y|>4\rho(x_0)} K_\mu(y,x)dx - \int_{|z-x|>4\rho(x_0)} K_\mu(z,x)dx \right| \\
&\le \int_{|x-y|>4\rho(x_0)} |K_\mu(y,x)-K_\mu(z,x)|dx + \int_{\mathbb{R}^d} |K_\mu(z,x)||\chi_{|x-z|>4\rho(x_0)}(x)-\chi_{|x-y|>4\rho(x_0)}(x)|dx \\
&=: \tilde{A}_1+\tilde{A}_2.
\end{aligned}$$

In the integral of $\tilde{A}_1$, we have $|x-y|>4\rho(x_0)\ge 8r>2|y-z|$. Therefore, from the smoothness of the Riesz kernel K_μ given in (4.10), and recalling that $y,z\in B$ and $\delta_\mu>1$, we have

$$\tilde{A}_1 \le C|y-z|^{\delta_\mu-1}\int_{|x-y|>4\rho(x_0)} \frac{1}{|x-y|^{d+\delta_\mu-1}}dx \le C\left(\frac{r}{\rho(x_0)}\right)^{\delta_\mu-1}.$$

For $\tilde{A}_2$ we use the size condition (4.9) to have

$$\tilde{A}_2 \lesssim \int_{\mathbb{R}^d} \frac{1}{|z-x|^d}\left|\chi_{|x-z|>4\rho(x_0)}(x)-\chi_{|x-y|>4\rho(x_0)}(x)\right| dx.$$

When $|x-z|>4\rho(x_0)\ge|x-y|$, we get $4\rho(x_0)<|z-x|<4\rho(x_0)+2r$. Thus,

$$\tilde{A}_2 \lesssim \int_{4\rho(x_0)<|z-x|<4\rho(x_0)+2r} \frac{1}{|z-x|^d}dx \sim \int_{4\rho(x_0)}^{4\rho(x_0)+2r} \frac{1}{t}dt \lesssim \frac{2r}{4\rho(x_0)} \sim \frac{r}{\rho(x_0)}.$$

In the other case, when $|x-y|>4\rho(x_0)\ge|x-z|$, we get $2\rho(x_0)\le 4\rho(x_0)-2r<|z-x|\le 4\rho(x_0)$. Proceeding as above, we get $\tilde{A}_2 \le C\frac{r}{\rho(x_0)}$. Hence

$$\tilde{A} \le C\left(\frac{r}{\rho(x_0)}\right)^\delta, \quad 0<\delta<\min\{1,\delta_\mu-1\}. \tag{4.31}$$

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Then, the estimates (4.30) and (4.31) imply (1.10) as we wanted to prove. $\square$

4.2. **Laplace transform-type multipliers.** Given $\phi \in L^\infty(0,\infty)$, we define

$$m_\phi(\lambda) = \lambda \int_0^\infty e^{-\lambda t}\phi(t)\,dt, \quad \lambda > 0.$$

By means of the Spectral Theorem, we can define the Laplace transform-type multipliers $m_\phi(\mathcal{L}_\mu)$. If we consider the heat semigroup associated with $\mathcal{L}_\mu$,

$$\mathcal{W}_t f(x) := e^{-t\mathcal{L}_\mu} f(x) = \int_{\mathbb{R}^d} \mathcal{W}_t(x,y) f(y)dy, \quad f \in L^2(\mathbb{R}^d),\ x \in \mathbb{R}^d,\ t>0, \tag{4.32}$$

we can write

$$m_\phi(\mathcal{L}_\mu)f(x) = \int_0^\infty \phi(t)\mathcal{L}_\mu e^{-t\mathcal{L}_\mu} f(x)dt = -\int_0^\infty \phi(t)\partial_t \mathcal{W}_t f(x)dt, \qquad x \in \mathbb{R}^d.$$

Its kernel can be expressed as

$$\mathcal{M}_\phi(x,y) = -\int_0^\infty \phi(t)\partial_t \mathcal{W}_t(x,y)dt, \quad x,y \in \mathbb{R}^d.$$

From classical theory, it can be deduced that $m_\phi(\mathcal{L}_\mu)$ is bounded on $L^2(\mathbb{R}^d)$ for every $\phi \in L^\infty(0,\infty)$ (see [24, Corollary 3, p. 121]). In the case of the Schrödinger operator $\mathcal{L}_V$, with $V \in RH_q$, $q > d/2$, it is known that $m_\phi(\mathcal{L}_V)$ can be extended to a bounded operator on $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$, and also from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ (see [3, Theorem 2]).

In the particular case of $\phi(\lambda) = \lambda^{i\gamma}$, for $\lambda > 0$ and $\gamma \in \mathbb{R}$, we get the imaginary powers $\mathcal{L}_V^{i\gamma}$. In [5] it was proved that $\mathcal{L}_V^{i\gamma}$ is bounded on $L^p(w)$ for all $\gamma > 0$, $1 < p < \infty$, and $w \in A_p^\rho$. Recently, it was proved in [14] that the operator $m_\phi(\mathcal{L}_\mu)$ is an exponential SCZO of $(0,\infty,\delta)$ type, for $0 < \delta < \min\{1,\delta_\mu\}$, with parameters $c = c_0$ (the constant from Lemma 4.6(i) given below) and $m = \frac{1}{k_0+1}$. Consequently, it is bounded on $L^p(w)$ for all $1 < p < \infty$ and every $w \in H_{p,c^*}^{\rho,m^*}$, where $m^* = \frac{1}{(k_0+1)^2}$ and $c^* < c_0(2^{3k_0+7}C_0^{k_0+3})^{-m^*}$ (see [14, Theorem 5.9]). The proofs of these results rely on estimates for the kernel $\mathcal{M}_\phi(x,y)$, which are derived from estimates for $\mathcal{W}_t(x,y)$ and $\partial_t\mathcal{W}_t(x,y)$. These estimates for the heat kernels are provided here, as they will be employed in subsequent examples.

Lemma 4.6 ([26, Theorem 1.1, Lemma 3.7])**.** *Let $x,y \in \mathbb{R}^d$ and $t > 0$.*

(i) There exist positive constants C, c, and c_0 such that

$$0 \le \mathcal{W}_t(x,y) \le \frac{C}{t^{\frac{d}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) \exp\left(-c_0\left(1+\frac{\max\{|x-y|,\sqrt{t}\}}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right).$$

(ii) For every $0 < \beta < \min\{1,\delta_\mu\}$ and every $N > 0$, there exist positive constants c and C_N such that, for all $|h| \le \sqrt{t}$,

$$|\mathcal{W}_t(x+h,y) - \mathcal{W}_t(x,y)| \le \frac{C_N}{t^{\frac{d}{2}}}\left(\frac{|h|}{\sqrt{t}}\right)^\beta \exp\left(-c\frac{|x-y|^2}{t}\right)\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

Lemma 4.7 ([26, Lemma 3.8])**.** *Let $x,y \in \mathbb{R}^d$ and $t > 0$.*

(i) There exist positive constants C and c such that

$$|t\partial_t\mathcal{W}_t(x,y)| \le \frac{C}{t^{\frac{d}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) \exp\left(-c_0\left(1+\frac{\max\{|x-y|,\sqrt{t}/2\}}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right),$$

where c_0 is the constant from Lemma 4.6(i).

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(ii) *For every $0 < \beta < \min\{1, \delta_\mu\}$ and $N > 0$, there exist positive constants c and C_N such that, for all $|h| \leq \sqrt{t}$,*

$$|t\partial_t \mathcal{W}_t(x+h, y) - t\partial_t \mathcal{W}_t(x, y)| \leq \frac{C_N}{t^{\frac{d}{2}}} \left(\frac{|h|}{\sqrt{t}}\right)^{\beta} \exp\left(-c\frac{|x-y|^2}{t}\right) \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

(iii) *For every $N > 0$, there exists a constant C_N such that*

$$\left|\int_{\mathbb{R}^d} t\partial_t \mathcal{W}_t(x, y) dy\right| \leq C_N \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_\mu} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N}.$$

To establish boundedness results on $\mathrm{BMO}^\alpha_\rho(w)$ for $m_\phi(\mathcal{L}_\mu)$, extending those already known for $m_\phi(\mathcal{L}_V)$ given in [21, Theorem 1.3], we will give some estimates comparing the kernel $\mathcal{W}_t(x, y)$ with the classical heat semigroup kernel $\{W_t\}_{t>0} = \{e^{-t\Delta}\}_{t>0}$.

We note first that we can express, in the weak sense, that

$$\int_{\mathbb{R}^d} \partial_t \mathcal{W}_t(x, y) dy = -e^{-t\mathcal{L}_\mu}\mathcal{L}_\mu 1(x) = -\int_{\mathbb{R}^d} \mathcal{W}_t(x, y)\mathcal{L}_\mu 1(y) dy = -\int_{\mathbb{R}^d} \mathcal{W}_t(x, y) d\mu(y). \tag{4.33}$$

We also have the following inequality given in [26, Eq. (2.2)]: for every $x \in \mathbb{R}^d$ and $t > 0$,

$$\int_{\mathbb{R}^d} t^{-d/2} \exp\left(-\frac{|x-y|^2}{t}\right) d\mu(y) \leq \begin{cases} \frac{C}{t}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_\mu}, & t < \rho(x)^2 \\ \frac{C}{t}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\log_2(D_\mu)}, & t \geq \rho(x)^2. \end{cases} \tag{4.34}$$

Proposition 4.8 ([26, Lemma 3.6]). *Let $x, y, z \in \mathbb{R}^d$ and $t > 0$.*

(i) *There exist positive constants C and c such that*

$$|\mathcal{W}_t(x, y) - W_t(x, y)| \leq \begin{cases} \frac{C}{t^{d/2}}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_\mu} \exp\left(-c\frac{|x-y|^2}{t}\right), & \text{if } \sqrt{t} < \rho(x), \\ \frac{C}{t^{d/2}}\left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta_\mu} \exp\left(-c\frac{|x-y|^2}{t}\right), & \text{if } \sqrt{t} < \rho(y), \\ \frac{C}{t^{d/2}} \exp\left(-\frac{|x-y|^2}{4t}\right), & \text{otherwise.} \end{cases}$$

(ii) *For every $0 < \beta < \min\{1, \delta_\mu\}$ and every positive constant C, there exist positive constants C' and c such that*

$$|(\mathcal{W}_t(x, y) - W_t(x, y)) - (\mathcal{W}_t(z, y) - W_t(z, y))| \leq \frac{C'}{t^{\frac{d}{2}}}\left(\frac{|x-z|}{\rho(y)}\right)^{\beta} \exp\left(-c\frac{|y-z|^2}{t}\right),$$

whenever $|x - z| \leq |x - y|/4$ and $|x - z| \leq C\rho(x)$.

Theorem 4.9. *Let $m = \frac{1}{k_0+1}$ and $\delta = \min\{1, \delta_\mu\}$. Then, for all $0 \leq \alpha < \delta$, the operator $m_\phi(\mathcal{L})$ is bounded on $\mathrm{BMO}^\alpha_\rho(w)$ provided that $w \in E^{\rho,m}_{\infty,c_1,c_2} \cap D^{\rho,m}_{\kappa,c_3}$ with $c_1, c_2, c_3 \geq 0$ such that $(c_2 + c_3) < c_0\left(1 - \frac{d(\kappa-1)+\alpha}{\delta}\right)(4C_0)^{-m}$ and $1 \leq \kappa < \frac{\delta-\alpha}{d} + 1$.*

Proof. By Corollary 1.8 and Remark 1.9 it suffices to show that for every ball $B = B(x_0, r)$ with $x_0 \in \mathbb{R}^d$ and $0 < r \leq \frac{1}{2}\rho(x_0)$, and for every $0 < \beta < \min\{1, \delta_\mu\}$, the following holds:

$$\frac{1}{|B|}\int_B |m_\phi(\mathcal{L}_\mu)1(y) - (m_\phi(\mathcal{L}_\mu)1)_B| dy \leq C\left(\frac{r}{\rho(x_0)}\right)^{\beta}. \tag{4.35}$$

Consider $y, z \in B$. As $\phi \in L^\infty(0, \infty)$,

$$|m_\phi(\mathcal{L}_\mu)1(y) - (m_\phi(\mathcal{L}_\mu)1)(z)| \leq C\int_0^\infty \left|\int_{\mathbb{R}^d} (\partial_t \mathcal{W}_t(y, x) - \partial_t \mathcal{W}_t(z, x)) dx\right| dt.$$

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We split the integral over t into three regions. From 0 to $4r^2$, we control by the sum and we analyze only the first term, since the other is handled in an analogous manner, keeping in mind that since $y, z \in B$, then $\rho(y) \sim \rho(x_0) \sim \rho(z)$. Applying Lemma 4.7(iii),

$$\int_0^{4r^2} \left| \int_{\mathbb{R}^d} \partial_t \mathcal{W}_t(y,x) dx \right| dt \lesssim \int_0^{4r^2} \left(\frac{\sqrt{t}}{\rho(y)} \right)^{\delta_\mu} \frac{dt}{t} \lesssim \frac{1}{\rho(x_0)^{\delta_\mu}} \int_0^{4r^2} t^{\delta_\mu/2-1} dt \lesssim \left(\frac{r}{\rho(x_0)} \right)^{\delta_\mu}. \tag{4.36}$$

Now, let us examine the integral over t from $\rho(x_0)^2$ to ∞. Here, we apply Lemma 4.7(ii) since $|y-z| \leq 2r \leq \rho(x_0) \leq \sqrt{t}$. It follows that, for every $0 < \beta < \min\{1, \delta_\mu\}$,

$$\begin{aligned} \int_{\rho(x_0)^2}^{\infty} \int_{\mathbb{R}^d} |\partial_t \mathcal{W}_t(y,x) - \partial_t \mathcal{W}_t(z,x)| \, dx dt &\lesssim \int_{\rho(x_0)^2}^{\infty} \int_{\mathbb{R}^d} \left(\frac{|z-y|}{\sqrt{t}} \right)^{\beta} t^{-d/2-1} \exp\left(-c \frac{|x-y|^2}{t} \right) dx dt \\ &\lesssim (2r)^{\beta} \int_{\rho(x_0)^2}^{\infty} t^{-\beta/2-1} dt \lesssim \left(\frac{r}{\rho(x_0)} \right)^{\beta}, \end{aligned} \tag{4.37}$$

since $\int_{\mathbb{R}^d} e^{-c\frac{|x-y|^2}{t}} dx = (c\pi t)^{d/2}$.

Finally, for $4r^2 \leq t \leq \rho(x_0)^2$, by (4.33), Lemma 4.6(ii), the fact that $|y-z| \leq 2r \leq \sqrt{t}$ in the integration domain and $t \leq \rho(x_0)^2 < C\rho(y)^2$, together with inequality (4.34), we have for every $0 < \beta < \min\{1, \delta_\mu\}$

$$\begin{aligned} \int_{4r^2}^{\rho(x_0)^2} \left| \int_{\mathbb{R}^d} \partial_t \mathcal{W}_t(y,x) - \partial_t \mathcal{W}_t(z,x) dx \right| dt &\leq \int_{4r^2}^{\rho(x_0)^2} \int_{\mathbb{R}^d} |\mathcal{W}_t(y,x) - \mathcal{W}_t(z,x))| \, d\mu(x) dt \\ &\lesssim \int_{4r^2}^{\rho(x_0)^2} \left(\frac{|z-y|}{\sqrt{t}} \right)^{\beta} \int_{\mathbb{R}^d} t^{-d/2} \exp\left(-c \frac{|y-x|^2}{t} \right) d\mu(x) dt \\ &\lesssim r^{\beta} \rho(x_0)^{-\delta_\mu} \int_0^{\rho(x_0)^2} t^{-1+\frac{\delta_\mu - \beta}{2}} dt \\ &\lesssim r^{\beta} \rho(x_0)^{-\delta_\mu} \rho(x_0)^{\delta_\mu - \beta} \sim \left(\frac{r}{\rho(x_0)} \right)^{\beta}, \end{aligned} \tag{4.38}$$

for all $0 < \beta < \min\{1, \delta_\mu\}$.

Therefore, by combining estimates (4.36), (4.37), and (4.38), we obtain (4.35) as desired. The boundedness property of $m_\phi(\mathcal{L}_\mu)$ follows from Corollary 1.8. $\square$

4.3. **Maximal operators for the heat-diffusion semigroup associated with $e^{-t\mathcal{L}_\mu}$.** Let $\{\mathcal{W}_t\}_{t>0}$ be the heat semigroup associated with $\mathcal{L}_\mu$, given for $f \in L^2(\mathbb{R}^d)$ by

$$\mathcal{W}_t f(x) = e^{-t\mathcal{L}_\mu} f(x) = \int_{\mathbb{R}^n} \mathcal{W}_t(x,y) f(y) \, dy, \quad x \in \mathbb{R}^d, \ t > 0,$$

where $\mathcal{W}_t(x,y)$ is the corresponding heat kernel, for $x, y \in \mathbb{R}^d$ and $t > 0$.

We consider the maximal operator of the heat semigroup

$$\mathcal{W}^* f(x) = \sup_{t>0} \left| e^{-t\mathcal{L}_\mu} \right| = \sup_{t>0} |\mathcal{W}_t f(x)|, \quad f \in L^2(\mathbb{R}^d).$$

We interpret the operator $\mathcal{W}^*$ in a vector-valued sense, as in [25], and take into account Remark 1.10 in the Banach space $\mathbb{B} = L^\infty((0,\infty), dt)$. Then, we can write $\mathcal{W}^* f(x) = \|\boldsymbol{W} f(x)\|_{\mathbb{B}}$, where we denote $\boldsymbol{W} f(x) = \{\mathcal{W}_t f(x)\}_{t>0}$, for every $x \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$.

$\boldsymbol{W}$ is bounded from $L^2(\mathbb{R}^d)$ into $L^2_{\mathbb{B}}(\mathbb{R}^d)$ and, from the kernel conditions to be proven below, it follows that it is a vector-valued Calderón–Zygmund operator; consequently, it is bounded from $L^p(\mathbb{R}^d)$ into $L^p_{\mathbb{B}}(\mathbb{R}^d)$ for all $1 < p < \infty$. Furthermore, we shall prove that conditions (1.8) and (1.9) are satisfied in $\mathbb{B}$, with parameters $c = c_0$ and $m = \frac{1}{k_0+1}$. This implies that $\mathcal{W}^*$ is bounded on $L^p(w)$ for $w \in H^{\rho,m^*}_{p,c^*}$, where $m^* = \frac{1}{(k_0+1)^2}$ and $c^* < c_0(2^{3k_0+7} C_0^{k_0+3})^{-m^*}$ (see [14, Theorem 4.1 and Proposition 4.2]).

Proposition 4.10. *Let $x_0, x, y \in \mathbb{R}^d$ and $t > 0$. Then,*

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(i) for every $x \neq y$,

$$\|\mathcal{W}_t(x,y)\|_{\mathbb{B}} \lesssim \frac{1}{|x-y|^d} \exp\left(-c_0\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right);$$

(ii) for every $0<\delta<\min\{1,\delta_\mu\}$, there exists C_δ such that

$$\|\mathcal{W}_t(x,y)-\mathcal{W}_t(x_0,y)\|_{\mathbb{B}} \leq C_\delta \frac{|x-x_0|^\delta}{|x-y|^{d+\delta}},$$

provided that $|x-y|>2|x-x_0|$.

Proof. By Lemma 4.6(i), we obtain (i) when $t>|x-y|^2$. For $t \leq |x-y|^2$, observe that for each N, there exists C_N such that

$$\exp\left(-c\frac{|x-y|^2}{t}\right) \leq C_N\left(\frac{t}{|x-y|^2}\right)^N. \tag{4.39}$$

Then, if $N=d/2$ is taken above, it follows that

$$0 \leq \mathcal{W}_t(x,y) \lesssim \frac{1}{|x-y|^d}\exp\left(-c_0\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right).$$

For the smoothness condition (ii), we consider $|x-y|>2|x-x_0|$, that implies $|x-y|\sim|x_0-y|$. For $0<\delta'<\min\{1,\delta_\mu\}$ and $|x-x_0|<\sqrt{t}$, by Lemma 4.6(i) and taking $N=\frac{d}{2}+\frac{\delta'}{2}$ in (4.39)

$$|\mathcal{W}_t(x,y)-\mathcal{W}_t(x_0,y)| \leq C\left(\frac{|x-x_0|}{\sqrt{t}}\right)^{\delta'} t^{-d/2}\left(\frac{|x-y|^2}{4t}\right)^{-\frac{d}{2}-\frac{\delta'}{2}} \leq C_{\delta'}\frac{1}{|x-y|^d}\left(\frac{|x-x_0|}{|x-y|}\right)^{\delta'}.$$

For $|x-x_0|>\sqrt{t}$, we use Lemma 4.6(i) and $N=\frac{d}{2}+\frac{\delta'}{2}$ in (4.39) to obtain

$$|\mathcal{W}_t(x,y)| \leq Ct^{-d/2}\exp\left(-\frac{|x-y|^2}{4t}\right) \leq C_{\delta'}\frac{1}{|x-y|^d}\left(\frac{|x-x_0|}{|x-y|}\right)^{\delta'},$$

and the same estimate follows for $\mathcal{W}_t(x_0,y)$. This completes the proof. $\square$

Theorem 4.11. *Let $m=\frac{1}{k_0+1}$, c_0 as in Lemma 4.6(i), $\delta=\min\{1,\delta_\mu\}$, with $\delta_\mu<d$ and $0\leq\alpha<\delta$. Then $\mathcal{W}^*$ is bounded on $\mathrm{BMO}^\alpha_\rho(w)$, for $w\in E^{\rho,m}_{\infty,c_1,c_2}\cap D^{\rho,m}_{\kappa,c_3}$, where $1\leq\kappa<\frac{\delta-\alpha}{d}+1$ and $c_1,c_2,c_3\geq 0$ such that $(c_2+c_3)<c_0\left(1-\frac{d(\kappa-1)+\alpha}{\delta}\right)(4C_0)^{-m}$.*

Remark 4.12. The condition $\delta_\mu<d$ in Theorem 4.11 differs from the case $d\mu(x)=V(x)dx$. In the latter, $\delta_V=2-d/q$ (where $V\in RH_q$), and $\delta_V<d$ is always satisfied.

Proof. By Corollary 1.8 and Remarks 1.9 and 1.10 it is enough to prove that there exists a constant C such that

$$\frac{1}{|B|}\int_B \|\mathcal{W}_t1(y)-(\mathcal{W}_t1)_B\|_{\mathbb{B}}\,dy \leq C\left(\frac{r}{\rho(x_0)}\right)^\beta,$$

for $B=B(x_0,r)$ with $0<r\leq\frac{1}{2}\rho(x_0)$ and every $0<\beta<\min\{1,\delta_\mu\}$. To this end, since

$$\|\mathcal{W}_t1(y)-(\mathcal{W}_t1)_B\|_{\mathbb{B}} \leq \frac{1}{|B|}\int_B \|\mathcal{W}_t1(y)-\mathcal{W}_t1(z)\|_{\mathbb{B}}dz,$$

we shall estimate the integrand $\|\mathcal{W}_t1(y)-\mathcal{W}_t1(z)\|_{\mathbb{B}}$.

Let $t>0$ and let $y,z\in B$. Given that $\rho(y)\sim\rho(x_0)\sim\rho(z)$, if $\sqrt{t}<2r$, then $\sqrt{t}<\rho(x_0)\lesssim\rho(z)$ and $\sqrt{t}\lesssim\rho(y)$. Since $W_t1(x)\equiv 1$, by applying Proposition 4.8(i), and changing variables, it follows that

$$\begin{aligned}|\mathcal{W}_t1(y)-\mathcal{W}_t1(z)| &\lesssim \frac{1}{t^{d/2}}\left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta_\mu}\int_{\mathbb{R}^d}\exp\left(-c\frac{|y-x|^2}{t}\right)dx+\left(\frac{\sqrt{t}}{\rho(z)}\right)^{\delta_\mu}\int_{\mathbb{R}^d}\exp\left(-c\frac{|z-x|^2}{t}\right)dx\\ &\lesssim \left(\frac{\sqrt{t}}{\rho(x_0)}\right)^{\delta_\mu} \lesssim \left(\frac{r}{\rho(x_0)}\right)^{\delta_\mu}.\end{aligned}$$

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For $\sqrt{t} > \rho(x_0) \geq 2r$, it follows that $|y-z| < 2r < \sqrt{t}$. Therefore, by Lemma 4.6(i), for every $0 < \beta < \min\{1, \delta_\mu\}$, we have

$$|\mathcal{W}_t 1(y) - \mathcal{W}_t 1(z)| \leq \int_{\mathbb{R}^d} |\mathcal{W}_t(y,x) - \mathcal{W}_t(z,x)| dx \lesssim \left(\frac{|y-z|}{\sqrt{t}}\right)^\beta t^{-d/2} \int_{\mathbb{R}^d} \exp\left(-c\frac{|y-x|^2}{t}\right) dx$$
$$\lesssim \left(\frac{r}{\sqrt{t}}\right)^\beta \lesssim \left(\frac{r}{\rho(x_0)}\right)^\beta. \tag{4.40}$$

Finally, if $2r < \sqrt{t} < \rho(x_0)$ we can write

$$\begin{aligned}|\mathcal{W}_t 1(y) - \mathcal{W}_t 1(z)| \leq &\int_{4|y-z|<|x-y|\leq a\rho(y)} |(\mathcal{W}_t(y,x) - W_t(y,x)) - (\mathcal{W}_t(z,x) - W_t(z,x))|\, dx \\ &+ \int_{|x-y|\leq 4|y-z|} |(\mathcal{W}_t(y,x) - W_t(y,x)) - (\mathcal{W}_t(z,x) - W_t(z,x))|\, dx \\ &+ \int_{|x-y|> a\rho(y)} |(\mathcal{W}_t(y,x) - W_t(y,x)) - (\mathcal{W}_t(z,x) - W_t(z,x))|\, dx,\end{aligned} \tag{4.41}$$

where $a = 2^{k_0+1}C_0$.

For the first integral, we apply Proposition 4.8(ii) taking into account that $|x-z| \sim |x-y|$ and $\rho(x) \sim \rho(y)$ in the integration region to have

$$\begin{aligned}&\int_{4|y-z|<|x-y|<a\rho(y)} |(\mathcal{W}_t(y,x) - W_t(y,x)) - (\mathcal{W}_t(z,x) - W_t(z,x))| dx \\ &\lesssim \left(\frac{|y-z|}{\rho(y)}\right)^\beta t^{-d/2} \int_{4|y-z|<|x-y|<a\rho(y)} \exp\left(-c\frac{|y-x|^2}{t}\right) dx \lesssim \left(\frac{r}{\rho(x_0)}\right)^\beta.\end{aligned} \tag{4.42}$$

Regarding the second integral, since x is close to both y and z, by Proposition 4.8(i) and $\sqrt{t} < \rho(x_0) \sim \rho(y) \sim \rho(z)$ we have

$$\begin{aligned}&\int_{|x-y|<4|y-z|} |(\mathcal{W}_t(y,x) - W_t(y,x)) - (\mathcal{W}_t(z,x) - W_t(z,x))| dx \\ &\leq \int_{|x-y|<4|y-z|} |\mathcal{W}_t(y,x) - W_t(y,x)| dx + \int_{|x-z|<5|y-z|} |\mathcal{W}_t(z,x) - W_t(z,x)| dx \\ &\lesssim \frac{1}{t^{d/2}}\left(\frac{\sqrt{t}}{\rho(x_0)}\right)^{\delta_\mu} \left[\int_{|x-y|<4|y-z|} \exp\left(-c\frac{|x-y|^2}{t}\right) dx + \int_{|x-z|<5|y-z|} \exp\left(-c\frac{|x-z|^2}{t}\right) dx\right] \\ &\lesssim \frac{1}{t^{d/2}}\left(\frac{\sqrt{t}}{\rho(x_0)}\right)^{\delta_\mu} \int_0^{5|y-z|} e^{-c\frac{u^2}{t}} u^{d-1} du \lesssim \left(\frac{\sqrt{t}}{\rho(x_0)}\right)^{\delta_\mu} \left(\frac{|y-z|}{\sqrt{t}}\right)^d \lesssim \left(\frac{r}{\rho(x_0)}\right)^{\delta_\mu},\end{aligned} \tag{4.43}$$

where we have used that $|y-z| < 2r < \sqrt{t}$ and $\delta_\mu < d$.

Finally, for the third term in (4.41), we use Lemma 4.6(i) for $\mathcal{W}_t$, and the smoothness condition for W_t. Also, since $|y-z| < \rho(x_0)$ and by the choice of a, it follows that $|x-y| > 2|y-z|$. Therefore, for every $0 < \beta < \min\{1, \delta_\mu\}$, we have

$$\begin{aligned}&\int_{|x-y|>a\rho(y)} |(\mathcal{W}_t(y,x) - W_t(y,x)) - (\mathcal{W}_t(z,x) - W_t(z,x))| dx \\ &\qquad \leq \int_{|x-y|>a\rho(y)} |\mathcal{W}_t(y,x) - \mathcal{W}_t(z,x)| dx + \int_{|x-y|>a\rho(y)} |W_t(y,x) - W_t(z,x)| dx \\ &\qquad \lesssim |y-z|^\beta \int_{|x-y|>a\rho(y)} \frac{1}{|x-y|^{d+\beta}} dx \lesssim r^\beta \int_{a\rho(y)}^\infty \frac{1}{s^{d+\beta}} s^{d-1} ds \lesssim \left(\frac{r}{\rho(x_0)}\right)^\beta.\end{aligned} \tag{4.44}$$

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By combining (4.42), (4.43), and (4.44) we obtain

$$\|\mathcal{W}_t 1(y) - \mathcal{W}_t 1(z)\|_{\mathbb{B}} \le C\left(\frac{r}{\rho(x_0)}\right)^{\beta}, \quad 0<\beta<\min\{1,\delta_\mu\}, \tag{4.45}$$

and finish the proof. □

4.4. **Maximal operators for the generalized Poisson operators $\mathcal{P}_t^\sigma$.** For $0<\sigma<1$ and $f\in L^2(\mathbb{R}^d)$, the generalized Poisson operators $\mathcal{P}_t^\sigma$ are defined through the subordination formula

$$\mathcal{P}_t^\sigma f(x) = \frac{t^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-\frac{t^2}{4r}}\mathcal{W}_r f(x)\frac{dr}{r^{1+\sigma}} = \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-r}\mathcal{W}_{\frac{t^2}{4r}} f(x)\frac{dr}{r^{1-\sigma}}, \tag{4.46}$$

for $x\in\mathbb{R}^d$ and $t>0$, where $\mathcal{W}_{\frac{t^2}{4r}} f$ is as in (4.32). Thus,

$$\mathcal{P}_t^\sigma f(x) = \int_{\mathbb{R}^d}\mathcal{P}_t^\sigma(x,y)f(y)dy, \quad x\in\mathbb{R}^d,$$

where the expression for the kernel is given by

$$\mathcal{P}_t^\sigma(x,y) = \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-r}\mathcal{W}_{\frac{t^2}{4r}}(x,y)\frac{dr}{r^{1-\sigma}}, \quad x,y\in\mathbb{R}^d. \tag{4.47}$$

We shall consider the maximal operator associated with $\mathcal{P}_t^\sigma$, for $f\in L^2(\mathbb{R}^d)$, as

$$\mathcal{P}^{\sigma,*}f(x) := \sup_{t>0}|\mathcal{P}_t^\sigma f(x)| = \|\mathcal{P}_t^\sigma f(x)\|_{\mathbb{B}}, \tag{4.48}$$

in the Banach space $\mathbb{B}=L^\infty((0,\infty),dt)$ as before. Moreover, the inequality $\mathcal{P}^{\sigma,*}f(x)\le\mathcal{W}^*f(x)$ for all $x\in\mathbb{R}^d$ and $f\in L^2(\mathbb{R}^d)$ derived from (4.46) yields the boundedness from $L^2(\mathbb{R}^d)$ into $L^2_{\mathbb{B}}(\mathbb{R}^d)$. Additionally, we have the same result for $\mathcal{P}^{\sigma,*}$ as for the maximal operator $\mathcal{W}^*$ on weighted Lebesgue spaces, that is, $\mathcal{P}^{\sigma,*}$ is bounded on $L^p(w)$ for $w\in H^{\rho,m^*}_{p,c^*}$, where $m^*=\frac{1}{(k_0+1)^2}$ and $c^*<c_0(2^{3k_0+7}C_0^{k_0+3})^{-m^*}$.

In $\mathrm{BMO}^\alpha_\rho(\mathbb{R}^d)$ spaces, these maximal operators $\mathcal{P}^{\sigma,*}$ were studied in [21] for the particular case where $d\mu(x)=V(x)dx$. To obtain a weighted version of these results, for a general measure μ, we shall prove first that size and smoothness conditions with norm $\mathbb{B}$ are satisfied.

Proposition 4.13. *Let $x_0,x,y\in\mathbb{R}^d$, $t>0$ and $0<\sigma<1$.*

(i) There exists a constant C such that

$$\|\mathcal{P}_t^\sigma(x,y)\|_{\mathbb{B}} \le C\frac{1}{|x-y|^d}\exp\left(-c_0\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right),$$

where c_0 is the constant from Lemma 4.6(i).

(ii) For every $0<\delta<\min\{1,\delta_\mu\}$, there exists a constant C such that

$$\|\mathcal{P}_t^\sigma(x,y)-\mathcal{P}_t^\sigma(x_0,y)\|_{\mathbb{B}} \le C\frac{|x-x_0|^\delta}{|x-y|^{d+\delta}},$$

whenever $|x-y|>2|x-x_0|$.

Proof. The proof of item (i) is a consequence of definitions (4.48) and (4.47), Proposition 4.10 and the definition of the Gamma function $\Gamma(\sigma)=\int_0^\infty r^{\sigma-1}e^{-r}dr$.

To prove the smoothness condition (ii), we use Proposition 4.10(ii) to obtain

$$\|\mathcal{P}_t^\sigma(x,y)-\mathcal{P}_t^\sigma(x_0,y)\|_{\mathbb{B}} \le \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-r}\left\|\mathcal{W}_{\frac{t^2}{4r}}(x,y)-\mathcal{W}_{\frac{t^2}{4r}}(x_0,y)\right\|_{\mathbb{B}}\frac{dr}{r^{1-\sigma}} \lesssim \frac{|x-x_0|^\delta}{|x-y|^{d+\delta}},$$

whenever $|x-y|>2|x-x_0|$ and $0<\delta<\min\{1,\delta_\mu\}$. □

Theorem 4.14. *Let $0<\sigma<1$, $m=\frac{1}{k_0+1}$, c_0 be the constant from Lemma 4.6(i), and $\delta=\min\{1,\delta_\mu\}$. Then, for every $0\le\alpha<\delta$, the operator $\mathcal{P}^{\sigma,*}$ is bounded on $\mathrm{BMO}^\alpha_\rho(w)$ provided that $w\in E^{\rho,m}_{\infty,c_1,c_2}\cap D^{\rho,m}_{\kappa,c_3}$ with $c_1,c_2,c_3\ge0$ such that $1\le\kappa<\frac{\delta-\alpha}{d}+1$ and $(c_2+c_3)<c_0\left(1-\frac{d(\kappa-1)+\alpha}{\delta}\right)(4C_0)^{-m}$.*

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Proof. Fix $0 < \sigma < 1$. By Corollary 1.8 and Remark 1.10, and in view of the kernel conditions given in Proposition 4.13, it is sufficient to prove that there exists a constant C such that

$$\frac{1}{|B|}\int_B \|\mathcal{P}_t^\sigma 1(y) - (\mathcal{P}_t^\sigma 1)_B\|_{\mathbb{B}} dy \le C\left(\frac{r}{\rho(x_0)}\right)^\beta, \tag{4.49}$$

for every ball $B = B(x_0, r)$ with $x_0 \in \mathbb{R}^d$, $0 < r \le \frac{1}{2}\rho(x_0)$, and $0 \le \beta < \min\{1, \delta_\mu\}$.

To show this, we shall estimate $\|\mathcal{P}_t^\sigma 1(y) - \mathcal{P}_t^\sigma 1(z)\|_{\mathbb{B}}$ for all $y, z \in B = B(x_0, r)$. According to the kernel representation of the generalized Poisson operator (4.47), and using the estimate (4.45), it follows that for every $0 < \beta < \min\{1, \delta_\mu\}$:

$$\begin{aligned}\|\mathcal{P}_t^\sigma 1(y) - \mathcal{P}_t^\sigma 1(z)\|_{\mathbb{B}} &= \left\|\int_{\mathbb{R}^d}\frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-s}\left(\mathcal{W}_{\frac{t^2}{4s}}(y,x) - \mathcal{W}_{\frac{t^2}{4s}}(z,x)\right)\frac{ds}{s^{1-\sigma}}dx\right\|_{\mathbb{B}} \\ &\le \frac{1}{\Gamma(\sigma)}\int_0^\infty s^{\sigma-1}e^{-s}\|\mathcal{W}_t 1(y) - \mathcal{W}_t 1(z)\|_{\mathbb{B}} ds \lesssim \left(\frac{r}{\rho(x_0)}\right)^\beta.\end{aligned}$$

By averaging over $y, z \in B$, we obtain the desired result. $\square$

4.5. **Littlewood-Paley function for the heat semigroup.** The Littlewood–Paley g-function associated with the family $\{\mathcal{W}_t\}_{t>0}$ is defined by

$$g_{\mathcal{W}}(f)(x) = \left(\int_0^\infty |t\partial_t \mathcal{W}_t f(x)|^2 \,\frac{dt}{t}\right)^{1/2} = \|t\partial_t\mathcal{W}_t f(x)\|_{\mathbb{F}},$$

where $\mathbb{F} = L^2\left((0,\infty), \frac{dt}{t}\right)$.

For $d\mu(x) = V(x)dx$ with $V \in RH_q$, the boundedness of $g_{\mathcal{W}}$ on $L^p(w)$ for weights $w \in A_p^\rho$ was established in [11] and unweighted estimates in the $\mathrm{BMO}_\rho^\alpha(\mathbb{R}^d)$ space can be found in [21]. To extend this to general measures, we adopt the vector-valued framework using the Banach space $\mathbb{F}$. Thus, since $\{t\partial_t\mathcal{W}_t\}_{t>0}$ is bounded from L^2 to $L^2_{\mathbb{F}}$ it is enough to verify that the kernel satisfies the size and smoothness conditions in the $\mathbb{F}$ norm to obtain that it is a vector-valued Calderón–Zygmund operator and bounded from L^p to $L^p_{\mathbb{F}}$ for all $1 < p < \infty$.

Moreover, as a consequence of Proposition 4.15 we shall prove that (1.8) and (1.9) are satisfied in $\mathbb{F}$, with parameters $c = c_0$ and $m = \frac{1}{k_0+1}$. This implies that $g_{\mathcal{W}}$ is bounded on $L^p(w)$ for $w \in H_{p,c^*}^{\rho,m^*}$, where $m^* = \frac{1}{(k_0+1)^2}$ and $c^* < c_0(2^{3k_0+7}C_0^{k_0+3})^{-m^*}$ (see [14, Theorem 4.1 and Proposition 4.2]).

Proposition 4.15. *Let $x_0, x, y \in \mathbb{R}^d$ and $t > 0$.*

(i) There exists a constant C such that

$$\|t\partial_t\mathcal{W}_t(x,y)\|_{\mathbb{F}} \le \frac{C}{|x-y|^d}\exp\left(-c_0\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right),$$

where c_0 is the constant from Lemma 4.6(i).

(ii) For every $0 < \delta < \min\{1, \delta_\mu\}$ there exists a constant C such that

$$\|t\partial_t\mathcal{W}_t(x,y) - t\partial_t\mathcal{W}_t(x_0,y)\|_{\mathbb{F}} \le C\frac{|x-x_0|^\delta}{|x-y|^{d+\delta}},$$

whenever $|x-y| > 2|x-x_0|$.

Proof. The proof is analogous to [14, Proposition 5.8]. For condition (i) we use Lemma 4.7(i). Then, there exist C and c such that

$$\begin{aligned}\|t\partial_t\mathcal{W}_t(x,y)\|_{\mathbb{F}}^2 &\lesssim \exp\left(-2c_0\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right)\int_0^\infty t^{-d-1}\exp\left(-2c\frac{|x-y|^2}{t}\right)dt \\ &\lesssim \frac{1}{|x-y|^{2d}}\exp\left(-2c_0\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right).\end{aligned}$$

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For condition (ii), consider $|x-y|>2|x-x_0|$. From Lemma 4.7(ii) it follows, on one hand, that

$$\int_{|x-x_0|^2}^{\infty} |t\partial_t\mathcal{W}_t(x,y)-t\partial_t\mathcal{W}_t(x_0,y)|^2\frac{dt}{t}\lesssim \frac{1}{|x-y|^{2d}}\left(\frac{|x-x_0|}{|x-y|}\right)^{2\delta},$$

for every $0<\delta<\min\{1,\delta_\mu\}$ and some constant C.

On the other hand, keeping in mind that $|x_0-y|>\frac{1}{2}|x-y|$, from Lemma 4.7(i) we have

$$\begin{aligned}\int_0^{|x-x_0|^2} |t\partial_t\mathcal{W}_t(x,y)-t\partial_t\mathcal{W}_t(x_0,y)|^2\frac{dt}{t}&\le \int_0^{|x-x_0|^2}|t\partial_t\mathcal{W}_t(x,y)|^2\frac{dt}{t}+\int_0^{|x-x_0|^2}|t\partial_t\mathcal{W}_t(x_0,y)|^2\frac{dt}{t}\\ &\lesssim \left(\frac{|x-x_0|}{|x-y|}\right)^{2\delta}\int_0^\infty t^{-d-1}\exp\left(-c\frac{|x-y|^2}{t}\right)dt\\ &\lesssim \frac{1}{|x-y|^{2d}}\left(\frac{|x-x_0|}{|x-y|}\right)^{2\delta},\end{aligned}$$

for every $0<\delta<\min\{1,\delta_\mu\}$. □

Theorem 4.16. *Let $m=\frac{1}{k_0+1}$, c_0 be the constant from Lemma 4.6(i), and $\delta=\min\{1,\delta_\mu\}$. Then, for every $0\le\alpha<\delta$, the operator $g_{\mathcal{W}}$ is bounded on $\mathrm{BMO}^\alpha_\rho(w)$ provided that $w\in E^{\rho,m}_{\infty,c_1,c_2}\cap D^{\rho,m}_{\kappa,c_3}$ with $1\le\kappa<\frac{\delta-\alpha}{d}+1$ and $c_1,c_2,c_3\ge 0$ such that $(c_2+c_3)<c_0\left(1-\frac{d(\kappa-1)+\alpha}{\delta}\right)(4C_0)^{-m}$.*

Proof. By Corollary 1.8 and Remarks 1.9 and 1.10, we only need to prove that there exists C such that

$$\frac{1}{|B|}\int_B \|t\partial_t\mathcal{W}_t1(y)-(t\partial_t\mathcal{W}_t1)_B\|_{\mathbb{F}}dy\le C\left(\frac{r}{\rho(x_0)}\right)^\beta, \tag{4.50}$$

for every ball $B=B(x_0,r)$ with $0<r\le\frac{1}{2}\rho(x_0)$ and $0<\beta<\min\{1,\delta_\mu\}$. From Minkowski's inequality, it will suffice to show that

$$\|t\partial_t\mathcal{W}_t1(y)-t\partial_t\mathcal{W}_t1(z)\|_{\mathbb{F}}\le C\left(\frac{r}{\rho(x_0)}\right)^\beta,$$

for all $y,z\in B$ and every $0\le\beta<\min\{1,\delta_\mu\}$. Note that

$$\|t\partial_t\mathcal{W}_t1(y)-t\partial_t\mathcal{W}_t1(z)\|^2_{\mathbb{F}}=\int_0^\infty\left|\int_{\mathbb{R}^d}(t\partial_t\mathcal{W}_t(y,x)-t\partial_t\mathcal{W}_t(z,x))dx\right|^2\frac{dt}{t}.$$

We proceed as in the proof of Theorem 4.9, splitting the integral in the t variable into three regions.

For $0<t\le 4r^2$, using that $\rho(y)\sim\rho(x_0)\sim\rho(z)$ since $y,z\in B$, and applying Lemma 4.7(iii) twice, we have

$$\int_0^{4r^2}\left|\int_{\mathbb{R}^d}(t\partial_t\mathcal{W}_t(y,x)-t\partial_t\mathcal{W}_t(z,x))dx\right|^2\frac{dt}{t}\lesssim\int_0^{4r^2}\left(\frac{\sqrt{t}}{\rho(x_0)}\right)^{2\delta_\mu}\frac{dt}{t}\sim\left(\frac{r}{\rho(x_0)}\right)^{2\delta_\mu}.$$

Next, if $4r^2<t\le\rho(x_0)^2$, we use (4.33) to rewrite the integral in x. Since $|y-z|\le\sqrt{t}$, we can apply Lemma 4.7(ii). Next, we use (4.34) because $t<C\rho(y)^2$ in this region and thus, for every $0<\beta<\min\{1,\delta_\mu\}$,

$$\begin{aligned}\int_{4r^2}^{\rho(x_0)^2}\left|\int_{\mathbb{R}^d}(t\partial_t\mathcal{W}_t(y,x)-t\partial_t\mathcal{W}_t(z,x))dx\right|^2\frac{dt}{t}&=\int_{4r^2}^{\rho(x_0)^2}t\left|\int_{\mathbb{R}^d}(\mathcal{W}_t(y,x)-\mathcal{W}_t(z,x))d\mu(x)\right|^2dt\\ &\lesssim|y-z|^{2\beta}\int_{4r^2}^{\rho(x_0)^2}t^{1-\beta}\left|\int_{\mathbb{R}^d}t^{-d/2}\exp\left(-c\frac{|y-x|^2}{t}\right)d\mu(x)\right|^2dt\\ &\lesssim r^{2\beta}\int_0^{\rho(x_0)^2}t^{-\beta-1}\left(\frac{\sqrt{t}}{\rho(y)}\right)^{2\delta_\mu}dt\sim\left(\frac{r}{\rho(x_0)}\right)^{2\beta}.\end{aligned}$$

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Finally, to analyze the integral when $t > \rho(x_0)^2$, we see that $|y-z| < 2r \le \rho(x_0) < \sqrt{t}$. Then, by Lemma 4.7(ii)

$$\int_{\rho(x_0)^2}^{\infty} \left| \int_{\mathbb{R}^d} (t\partial_t \mathcal{W}_t(y,x) - t\partial_t \mathcal{W}_t(z,x))dx \right|^2 \frac{dt}{t}$$
$$\lesssim \int_{\rho(x_0)^2}^{\infty} \left(\frac{|y-z|}{\sqrt{t}}\right)^{2\beta} \left(\int_{\mathbb{R}^d} t^{-d/2} \exp\left(-c\frac{|y-x|^2}{t}\right) dx\right)^2 \frac{dt}{t}$$
$$\lesssim r^{2\beta} \int_{\rho(x_0)^2}^{\infty} t^{-\beta-1} dt \sim \left(\frac{r}{\rho(x_0)}\right)^{2\beta},$$

for every $0 < \beta < \min\{1, \delta_\mu\}$, which concludes the proof of (4.50). □

4.6. **Littlewood-Paley function for the Poisson semigroup.** The Littlewood-Paley g-function associated with the Poisson semigroup is defined for $f \in L^2(\mathbb{R}^d)$ as

$$g_{\mathcal{P}}(f)(x) = \left(\int_0^\infty |t\partial_t \mathcal{P}_t f(x)|^2 \frac{dt}{t}\right)^{1/2} = \|t\partial_t \mathcal{P} t f(x)\|_{\mathbb{F}}, \quad x \in \mathbb{R}^d,$$

where $\mathbb{F} = L^2\left((0,\infty), \frac{dt}{t}\right)$, and $\mathcal{P}_t$ is the classical Poisson semigroup $\mathcal{P}_t^{1/2}$.

By the subordination representation of the Poisson kernel for $\sigma = \frac{1}{2}$ given in (4.47), the estimates for weighted Lebesgue and BMO spaces are obtained in the same manner as those for $g_{\mathcal{W}}$. Specifically, we derive the same size and smoothness estimates corresponding to (1.8) and (1.9) considering the norm $\|\cdot\|_{\mathbb{F}}$, with parameters $c = c_0$ and $m = \frac{1}{k_0+1}$. This implies that $g_{\mathcal{P}}$ is bounded on $L^p(w)$ for $w \in H^{\rho,m^*}_{p,c^*}$, where $m^* = \frac{1}{(k_0+1)^2}$ and $c^* < c_0(2^{3k_0+7}C_0^{k_0+3})^{-m^*}$ (see [14, Theorem 4.1 and Proposition 4.2]). The proofs of the following results are omitted since they follow the same lines as the ones for $g_{\mathcal{W}}$.

Proposition 4.17. *Let $x_0, x, y \in \mathbb{R}^d$ and $t > 0$.*

(i) There exists a constant C such that

$$\|t\partial_t \mathcal{P}_t(x,y)\|_{\mathbb{F}} \le C\frac{1}{|x-y|^d} \exp\left(-c_0\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right),$$

where c_0 is the constant from Lemma 4.6(i).

(ii) For every $0 < \delta < \min\{1, \delta_\mu\}$, there exists a constant C such that

$$\|t\partial_t \mathcal{P}_t(x,y) - t\partial_t \mathcal{P} t(x_0,y)\|_{\mathbb{F}} \le C\frac{|x-x_0|^\delta}{|x-y|^{d+\delta}},$$

whenever $|x-y| > 2|x-x_0|$.

Theorem 4.18. *Let $m = \frac{1}{k_0+1}$, c_0 be the constant from Lemma 4.6(i), and $\delta = \min\{1, \delta_\mu\}$. Then, for every $0 \le \alpha < \delta$, the operator $g_{\mathcal{P}}$ is bounded on $\mathrm{BMO}^\alpha_\rho(w)$ provided that $w \in E^{\rho,m}_{\infty,c_1,c_2} \cap D^{\rho,m}_{\kappa,c_3}$ with $1 \le \kappa < \frac{\delta-\alpha}{d} + 1$ and $c_1, c_2, c_3 \ge 0$ such that $(c_2 + c_3) < c_0\left(1 - \frac{d(\kappa-1)+\alpha}{\delta}\right)(4C_0)^{-m}$.*

4.7. **Fractional integral operators.** Given $\gamma > 0$, the negative powers of $\mathcal{L}$ admit the following integral representation for $f \in L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$:

$$\mathcal{L}_\mu^{-\gamma} f(x) = \frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-t\mathcal{L}_\mu} f(x) \frac{dt}{t^{1-\gamma}} = \int_{\mathbb{R}^d} \mathcal{J}_\gamma(x,y) f(y) dy,$$

where

$$\mathcal{J}_\gamma(x,y) = \frac{1}{\Gamma(\gamma)} \int_0^\infty \mathcal{W}_t(x,y) \frac{dt}{t^{1-\gamma}}, \qquad x, y \in \mathbb{R}^d. \tag{4.51}$$

The case of negative powers for the operator $\mathcal{L}_V$ was first analyzed in [11], where weighted Lebesgue estimates for A_p^ρ were obtained, alongside the corresponding weak-type $(1, \frac{d}{d-2\gamma})$ results for $w \in A_1^\rho$.

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Subsequently, the authors in [4] established bounds from weighted Lebesgue spaces into $\mathrm{BMO}_\rho(w)$ spaces for $w \in A_p^\rho$. On the other hand, [21] obtained estimates for $\mathcal{L}_V^{-\gamma}$ between unweighted $\mathrm{BMO}_\rho^\alpha(\mathbb{R}^d)$ spaces using a $T1$-type theorem, while [9] provided $\mathrm{BMO}_\rho^\alpha(w)$ estimates using techniques adapted to the operator.

The boundedness for $\mathcal{L}_\mu^{-\gamma}$ in both $L^p(w)$ and $\mathrm{BMO}_\rho^\alpha(w)$ spaces, where w belongs to the $H_{p,c}^{\rho,m}$ weight class is given below.

Proposition 4.19. *Let $0<\gamma<d/2$. Then, $\mathcal{L}_\mu^{-\gamma}$ is an exponential SCZO operator of $(2\gamma,\infty,\delta)$ type for all $0<\delta<\min\{1,\delta_\mu\}$, with parameters $c=c_0$ (from Lemma 4.6(i)) and $m=\frac{1}{k_0+1}$.*

Proof. From Lemma 4.6(i) it follows that

$$\begin{aligned} |\mathcal{J}_\gamma(x,y)| &\le \frac{1}{\Gamma(\gamma)} \exp\left(-c_0\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right)\int_0^\infty t^{-\frac{d}{2}+\gamma-1}\exp\left(-c\frac{|x-y|^2}{t}\right)dt \\ &\lesssim \frac{1}{|x-y|^{d-2\gamma}}\exp\left(-c_0\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right), \end{aligned} \tag{4.52}$$

and (1.8) is verified.

For the smoothness condition (1.9), let $|x-y|>2|x-x_0|$. Taking into account

$$|\mathcal{J}_\gamma(x,y)-\mathcal{J}_\gamma(x_0,y)| = \frac{1}{\Gamma(\gamma)}\int_0^\infty |\mathcal{W}_t(x,y)-\mathcal{W}_t(x_0,y)|\frac{dt}{t^{1-\gamma}},$$

we consider two cases.

If $t\le|x-x_0|^2$, we apply Lemma 4.6(i) and estimate (4.39) with $N=\delta/2$, $0<\ \delta<\ \min\{1,\delta_\mu\}$,

$$\begin{aligned} |\mathcal{W}_t(x,y)| &\lesssim t^{-d/2}\exp\left(-c\frac{|x-y|^2}{t}\right)\left(\frac{|x-x_0|}{\sqrt{t}}\right)^\delta \\ &\lesssim t^{-d/2}\exp\left(-c\frac{|x-y|^2}{2t}\right)\left(\frac{|x-y|}{\sqrt{t}}\right)^{-\delta}\left(\frac{|x-x_0|}{\sqrt{t}}\right)^\delta \\ &\lesssim t^{-d/2}\left(\frac{|x-x_0|}{|x-y|}\right)^\delta\exp\left(-c\frac{|x-y|^2}{2t}\right), \end{aligned}$$

and we obtain the same bound for $|\mathcal{W}_t(x_0,y)|$ since $|x_0-y|>\frac{1}{2}|x-y|$.

For $t>|x-x_0|^2$, by Lemma 4.6(ii) and (4.39) for $N=\delta/2$, we have for $0<\delta<\min\{1,\delta_\mu\}$

$$\begin{aligned} |\mathcal{W}_t(x,y)-\mathcal{W}_t(x_0,y)| &\lesssim t^{-d/2}\left(\frac{|x-x_0|}{\sqrt{t}}\right)^\delta\exp\left(-c\frac{|x-y|^2}{t}\right) \\ &\lesssim \left(\frac{|x-x_0|}{|x-y|}\right)^\delta t^{-d/2}\exp\left(-c\frac{|x-y|^2}{t}\right). \end{aligned}$$

Then, integrating in t and making the change of variable $u=\frac{|x-y|^2}{2t}$,

$$\int_0^\infty |\mathcal{W}_t(x,y)-\mathcal{W}_t(x_0,y)|\frac{dt}{t^{1-\gamma}} \lesssim \frac{1}{|x-y|^{d-2\gamma}}\left(\frac{|x-x_0|}{|x-y|}\right)^\delta, \quad 0<\delta<\min\{1,\delta_\mu\}.$$

Finally, we prove that $\mathcal{L}_\mu^{-\gamma}$ is bounded from $L^1(\mathbb{R}^d)$ into $L^{\frac{d}{d-2\gamma},\infty}(\mathbb{R}^d)$. This follows from (4.52) which implies $|\mathcal{J}_\gamma(x,y)|\le C|x-y|^{-(d-2\gamma)}$. Thus, $|\mathcal{L}_\mu^{-\gamma}f(x)|\le CI_{2\gamma}(|f|)(x)$, where $I_{2\gamma}$ is the classical fractional integral. By the Hardy–Littlewood–Sobolev Theorem, this operator is bounded from $L^1(\mathbb{R}^d)$ to $L^{\frac{d}{d-2\gamma},\infty}(\mathbb{R}^d)$, completing the proof. ∎

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As a consequence of Proposition 4.19 we have that, for $0<\gamma<d/2$, $\mathcal{L}_\mu^{-\gamma}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$ for all $1<p<\frac{d}{2\gamma}$ and $\frac{1}{q}=\frac{1}{p}-\frac{2\gamma}{d}$, and any weight w such that $w^{-p'}\in H^{\rho,m^*}_{1+\frac{p'}{q},c}$ with $m^*=\frac{1}{(k_0+1)^2}$ and $c<c_0\frac{d}{d-2\gamma}(2^{3k_0+7}C_0^{k_0+3})^{-m^*}$. The proof of this result is omitted, as this follows the same lines as the singular case given in [14, Theorem 4.1, Proposition 4.2].

Theorem 4.20. *Let $0<\gamma<d/2$, $\delta=\min\{1,\delta_\mu\}$, $m=\frac{1}{k_0+1}$, and c_0 be the constant from Lemma 4.6(i). Then, for all $0\le\alpha+2\gamma<\delta$, the operator $\mathcal{L}_\mu^{-\gamma}$ is bounded from $\mathrm{BMO}_\rho^\alpha(w)$ into $\mathrm{BMO}_\rho^{\alpha+2\gamma}(w)$ provided that $w\in E^{\rho,m}_{\infty,c_1,c_2}\cap D^{\rho,m}_{\kappa,c_3}$ with $1\le\kappa<\frac{\delta-2\gamma-\alpha}{d}+1$ and $c_1,c_2,c_3\ge 0$ such that $(c_2+c_3)<c_0\left(1-\frac{d(\kappa-1)+\alpha+2\gamma}{\delta}\right)(4C_0)^{-m}$.*

Proof. By Corollary 1.8 and Remark 1.9, it suffices to show that there exists a constant C such that

$$\frac{1}{|B|^{1+2\gamma/d}}\int_B\left|\mathcal{L}_\mu^{-\gamma}1(y)-\left(\mathcal{L}_\mu^{-\gamma}1\right)_B\right|dx\le C\left(\frac{r}{\rho(x_0)}\right)^\beta, \tag{4.53}$$

for every ball $B=B(x_0,r)$ with $0<r\le\frac{1}{2}\rho(x_0)$ and $0<\beta<\min\{1,\delta_\mu\}-2\gamma$.

Given $y,z\in B$, we note that

$$|\mathcal{L}_\mu^{-\gamma}1(y)-\mathcal{L}_\mu^{-\gamma}1(z)|\le\int_0^\infty|\mathcal{W}_t1(y)-\mathcal{W}_t1(z)|\,t^{\gamma-1}dt.$$

Then, for $0<t\le\rho(x_0)^2$, we use (4.45) to obtain

$$\int_0^{\rho(x_0)^2}|\mathcal{W}_t1(y)-\mathcal{W}_t1(z)|\,t^{\gamma-1}dt\lesssim\left(\frac{r}{\rho(x_0)}\right)^{\tilde\beta}\int_0^{\rho(x_0)^2}t^{\gamma-1}dt\sim\left(\frac{r}{\rho(x_0)}\right)^{\tilde\beta}\rho(x_0)^{2\gamma},$$

provided that $0<\tilde\beta<\min\{1,\delta_\mu\}$.

When $t>\rho(x_0)^2$, we use (4.40), which yields

$$\int_{\rho(x_0)^2}^\infty|\mathcal{W}_t1(y)-\mathcal{W}_t1(z)|\,t^{\gamma-1}dt\lesssim\int_{\rho(x_0)^2}^\infty\left(\frac{r}{\sqrt{t}}\right)^{\tilde\beta}t^{\gamma-1}dt\sim\left(\frac{r}{\rho(x_0)}\right)^{\tilde\beta}\rho(x_0)^{2\gamma},$$

for all $2\gamma<\tilde\beta<\min\{1,\delta_\mu\}$.

Combining both estimates and setting $\beta=\tilde\beta-2\gamma$, it follows that (4.53) holds for every $0<\beta+2\gamma<\min\{1,\delta_\mu\}$. □

4.8. **Operators associated to a potential.** In the particular case where $d\mu(x)=V(x)\,dx$, with V being a non-negative function in the classical reverse–Hölder class RH_q for $q>\frac{d}{2}$, we can study the operators $T_\gamma=\mathcal{L}_V^{-\gamma}V^\gamma$ for $0<\gamma<d/2$. These are defined for $f\in L^2(\mathbb{R}^d)$ by

$$T_\gamma f(x)=\int_{\mathbb{R}^d}\mathcal{J}_\gamma(x,y)V^\gamma(y)f(y)dy,$$

where $\mathcal{J}_\gamma(x,y)$ is the kernel of the operator $\mathcal{L}_V^{-\gamma}$ defined in (4.51).

In the cases $\gamma=1/2$ and $\gamma=1$, $L^p(\mathbb{R}^d)$ boundedness was established in [23] for $1\le p\le 2q$ and $1\le p\le q$, respectively. Subsequently, [19] provided pointwise size and smoothness kernel estimates featuring polynomial decay associated with the critical radius function ρ. From these estimates, [4] proved the boundedness of the operator on $L^p(w)$ for weights in the A_p^ρ class. The general case $0<\gamma<d/2$ was studied in [7], where the authors obtained the boundedness of T_γ on $L^p(w)$ for $(q/\gamma)'<p<\infty$ and $w\in A^\rho_{p/(q/\gamma)'}$. They provided the estimates (1.7) for the kernel $\mathcal{J}_\gamma V^\gamma$ with $s=q/\gamma>1$ and $\delta=\min\{1,2-d/q\}$. Consequently, by proving the size condition (1.6) for the kernel $\mathcal{J}_\gamma V^\gamma$ we obtain Proposition 4.21, which is stated below. Then, following the same lines as the singular case given in [14, Theorem 4.1, Proposition 4.2] we obtain that T_γ is bounded on $L^p(w)$ for all $(q/\gamma)'<p<\infty$ and $w\in H^{\rho,m^*}_{p/(q/\gamma)',c^*}$, with $m^*=\frac{1}{(k_0+1)^2}$ and $c^*<c_02^{-(m+1)}(2^{2k_0+6}C_0^{k_0+3})^{-m^*}$. This result extends the one given in [7].

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Proposition 4.21. *Let $V \in RH_q$, $q > \frac{d}{2}$, $V \not\equiv 0$ and $0 < \gamma < \frac{d}{2}$. Then, T_γ is an exponential SCZO of $(0, \frac{q}{\gamma}, \delta)$ type, with $\delta = \min\left\{1, 2 - \frac{d}{q}\right\}$, constants $m = \frac{1}{k_0+1}$, and $c = c_0 2^{-(m+1)}$.*

Proof. As we saw before, it is enough to prove (1.6). For that, by (4.52), and using that $V \in RH_q$ it follows that

$$\begin{aligned}
&\left(\frac{1}{R^d}\int_{R<|x_0-y|\le 2R} |\mathcal{J}_\gamma(x,y)V^\gamma(y)|^{q/\gamma}dy\right)^{\gamma/q}\\
&\lesssim \frac{1}{R^{d-2\gamma}}\exp\left(-c_0 2^{-m}\left(1+\frac{R}{\rho(x)}\right)^m\right)\left(\frac{1}{R^d}\int_{B(x,3R)} V(y)dy\right)^\gamma\\
&\lesssim \frac{1}{R^{d-2\gamma}}\exp\left(-c_0 2^{-m}\left(1+\frac{R}{\rho(x)}\right)^m\right)R^{-2\gamma}\left(1+\frac{R}{\rho(x)}\right)^{\log_2(D_V)\gamma}\\
&\lesssim \frac{1}{R^d}\exp\left(-c_0 2^{-m-1}\left(1+\frac{R}{\rho(x)}\right)^m\right),
\end{aligned}$$

where we also used [14, Lemma 5.2] with $d\mu(x) = V(x)dx$ and D_V the doubling constant of $V(x)dx$. □

Additionally, in [7] a $T1$-type condition was derived, enabling the boundedness of T_γ on weighted $\mathrm{BMO}_\rho^\alpha(w)$ spaces for certain weights in A_p^ρ . Specifically, the authors proved condition (1.10) with $0 < \alpha + d(\kappa - 1) < \tilde{\delta}$, where $\tilde{\delta} = \min\{1, 2 - d/q, \gamma(2 - d/q)\}$. As a consequence of this fact, Proposition 4.21 and Theorem 1.7 we obtain the following result, which broadens the classes of weights given in [7] for $\mathrm{BMO}_\rho^\alpha(w)$.

Theorem 4.22. *Let $V \in RH_q$, $q > \frac{d}{2}$, $V \not\equiv 0$ and $0 < \gamma < \frac{d}{2}$. For $0 \le \alpha < \tilde{\delta}$, $m = \frac{1}{k_0+1}$ and $c = c_0 2^{-(m+1)}$, T_γ is bounded on $\mathrm{BMO}_\rho^\alpha(w)$ whenever $w \in E^{\rho,m}_{\frac{q}{\gamma},c_1,c_2} \cap D^{\rho,m}_{\kappa,c_3}$, with $1 \le \kappa < \frac{\tilde{\delta}-\alpha}{d} + 1$ and $(c_2 + c_3) < c\left(1 - \frac{d(\kappa-1)+\alpha}{\tilde{\delta}}\right)(4C_0)^{-m}$.*

4.9. **The operator $\mathcal{L}_\mu^{-\gamma}\nabla$.** Note that $\mathcal{L}_\mu^{-\gamma}\nabla$ can be considered to have fractional order $\nu = 2\gamma - 1$, since $\mathcal{L}_\mu^{-\gamma}$ is of order 2γ. Thus, $\nu > 0$ provided that $1/2 < \gamma \le 1$. When $\gamma = 1$,

$$\mathcal{L}_\mu^{-1}\nabla f(x) = \int_{\mathbb{R}^d} \Gamma_\mu(x,y)\nabla f(y)dy = \int_{\mathbb{R}^d} \mathcal{K}_1(x,y)f(y)dy,$$

for $f \in L^2(\mathbb{R}^d)$ and $\mathcal{K}_1(x,y) = -\nabla_2\Gamma_\mu(x,y)$, where ∇_2 denotes the gradient with respect to the second variable. But, by the symmetry of Γ_μ (see [22, Theorem 2.18]), we have $\nabla_2\Gamma_\mu(x,y) = \nabla_1\Gamma_\mu(y,x)$ and applying [1, Theorem 4.2] we have that there exist positive constants C and ϵ such that

$$|\mathcal{K}_1(x,y)| = |\nabla_1\Gamma_\mu(y,x)| \le \frac{Ce^{-\epsilon d_\mu(y,x)}}{|y-x|^{d-2}}\left(\int_{B(y,\frac{|y-x|}{2})} \frac{d\mu(z)}{|z-y|^{d-1}} + \frac{1}{|y-x|}\right), \tag{4.54}$$

for all $x, y \in \mathbb{R}^d$ with $x \ne y$. Moreover, if $\delta_\mu > 1$, then

$$|\mathcal{K}_1(x,y)| \le \frac{Ce^{-\epsilon d_\mu(y,x)}}{|y-x|^{d-1}}.$$

For $1/2 < \gamma < 1$, we use Balakrishnan's formula (see [20, Eq. (3.53), p. 286]) to express the operator as

$$\mathcal{L}_\mu^{-\gamma} = \frac{\sin(\pi\gamma)}{\pi}\int_0^\infty \lambda^{-\gamma}\mathcal{L}_{\mu+\lambda}^{-1}\,d\lambda,$$

and then

$$\mathcal{L}_\mu^{-\gamma}\nabla f(x) = \int_{\mathbb{R}^d} \mathcal{K}_\gamma(x,y)f(y)\,dy,$$

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where

$$\mathcal{K}_\gamma(x,y) = -\frac{\sin(\pi\gamma)}{\pi}\int_0^\infty \lambda^{-\gamma}\nabla_1\Gamma_{\mu+\lambda}(y,x)d\lambda. \tag{4.55}$$

Based on these estimates, we can establish the following classification for $\mathcal{L}_\mu^{-\gamma}\nabla$.

Proposition 4.23. *Let $m_0 = \frac{1}{k_0+1}$ and $1/2 < \gamma \leq 1$.*

(i) If $\delta_\mu > 1$, $\mathcal{L}_\mu^{-\gamma}\nabla$ is an exponential SCZO of $(2\gamma-1,\infty,\delta)$ type, with parameters $c = \frac{\epsilon}{2D_1}$ and m_0.

(ii) If $0 < \delta_\mu < 1$, $\mathcal{L}_\mu^{-\gamma}\nabla$ is an exponential SCZO of $(2\gamma-1,s,\delta)$ type for $\frac{d}{d-(2\gamma-1)} < s < \frac{2-\delta_\mu}{1-\delta_\mu}$ and $\delta \in (0,1)$, with parameters $c = \frac{\epsilon}{4D_1}$ and m_0.

Here, $\delta \in (0,1)$ is as in [22, Eq. (7.29)] *and D_1 is the constant given by* [1, Lemma 2.3].

Proof. First, note that from (4.54) and (4.16) we obtain, for every $\lambda \geq 0$ and $\delta_\mu > 1$, that

$$|\nabla_1\Gamma_{\mu+\lambda}(y,x)| \lesssim \frac{e^{-\frac{\epsilon}{2}\sqrt{\omega_d}\sqrt{\lambda}|x-y|}}{|y-x|^{d-1}}\exp\left(-\frac{\epsilon}{2D_1}\left(1+\frac{|y-x|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right), \tag{4.56}$$

where $\rho(x)$ can be replaced by $\rho(y)$ due to the symmetry of the distance d_μ. It follows immediately that for $\gamma = 1$ (and thus $\nu = 2\gamma - 1 = 1$)

$$|\mathcal{K}_1(x,y)| = |\nabla_1\Gamma_\mu(y,x)| \lesssim \frac{1}{|y-x|^{d-1}}\exp\left(-\frac{\epsilon}{2D_1}\left(1+\frac{|y-x|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right).$$

In the case $1/2 < \gamma < 1$, we integrate (4.56) with respect to λ using the change of variables $u = \frac{\epsilon}{2}\sqrt{\omega_d}\sqrt{\lambda}|x-y|$, which yields

$$\begin{aligned}|\mathcal{K}_\gamma(x,y)| &\lesssim \frac{1}{|x-y|^{d-1}}\exp\left(-\frac{\epsilon}{2D_1}\left(1+\frac{|y-x|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right)\int_0^\infty \lambda^{-\gamma}e^{-\frac{\epsilon}{2}\sqrt{\omega_d}\sqrt{\lambda}|x-y|}d\lambda \\ &\lesssim \frac{1}{|x-y|^{d-(2\gamma-1)}}\exp\left(-\frac{\epsilon}{2D_1}\left(1+\frac{|y-x|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right),\end{aligned}$$

where we have used the fact that the integral converges since $\gamma < 1$.

To prove the corresponding smoothness condition, we consider $|x-x_0| < |y-x|/8$. Applying the estimate given in [22, p. 563] and using (4.56), we have

$$\begin{aligned}|\nabla_1\Gamma_{\mu+\lambda}(y,x) - \nabla_1\Gamma_{\mu+\lambda}(y,x_0)| &\lesssim \left(\frac{|x-x_0|}{|x-y|}\right)^\delta \sup_{z\in B(x,|x-y|/2)}|\nabla_1\Gamma_{\mu+\lambda}(y,z)| \\ &\lesssim \left(\frac{|x-x_0|}{|x-y|}\right)^\delta \frac{e^{-\frac{\epsilon}{4}\sqrt{\omega_d}\sqrt{\lambda}|x-y|}}{|x-y|^{d-1}}.\end{aligned}$$

For $\gamma = 1$, we set $\lambda = 0$ in the inequality above, leading to

$$|\mathcal{K}_1(x,y) - \mathcal{K}_1(x_0,y)| \lesssim \frac{1}{|y-x|^{d-1}}\left(\frac{|x-x_0|}{|x-y|}\right)^\delta,$$

whenever $|x-y| > 8|x-x_0|$. For the remaining cases, we integrate as before to obtain

$$\begin{aligned}|\mathcal{K}_\gamma(x,y) - \mathcal{K}_\gamma(x_0,y)| &\lesssim \int_0^\infty \lambda^{-\gamma}|\nabla_1\Gamma_{\mu+\lambda}(y,x) - \nabla_1\Gamma_{\mu+\lambda}(y,x_0)|d\lambda \\ &\lesssim \frac{1}{|x-y|^{d-1}}\left(\frac{|x-x_0|}{|x-y|}\right)^\delta\int_0^\infty \lambda^{-\gamma}e^{-\frac{\epsilon}{4}\sqrt{\omega_d}\sqrt{\lambda}|x-y|}d\lambda \\ &\lesssim \frac{1}{|y-x|^{d-(2\gamma-1)}}\left(\frac{|x-x_0|}{|x-y|}\right)^\delta,\end{aligned}$$

provided $|x-y| > 8|x-x_0|$.

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Finally, to show that $\mathcal{L}_\mu^{-\gamma}\nabla$ is bounded from $L^1(\mathbb{R}^d)$ into $L^{\frac{d}{d-(2\gamma-1)},\infty}(\mathbb{R}^d)$, we proceed as in the case of fractional integrals. Given the pointwise estimates obtained above, for $f \in L^1(\mathbb{R}^d)$ and any $1/2 < \gamma \leq 1$, we write

$$\begin{aligned}
|\mathcal{L}_\mu^{-\gamma}\nabla f(x)| &\leq \int_{\mathbb{R}^d} |\mathcal{K}_\gamma(x,y)||f(y)|dy \\
&\lesssim \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-(2\gamma-1)}} \exp\left(-\frac{\epsilon}{2D_1}\left(1+\frac{|y-x|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right)|f(y)|dy \\
&\lesssim \int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^{d-(2\gamma-1)}}dy \\
&\lesssim I_{2\gamma-1}(|f|)(x),
\end{aligned}$$

where $I_{2\gamma-1}$ is the classical fractional integral of order $2\gamma-1$. Since the latter is of weak type $\left(1, \frac{d}{d-(2\gamma-1)}\right)$, it follows that $\mathcal{L}_\mu^{-\gamma}\nabla$ is also of this type.

To prove (ii) we need to check that the kernel $\mathcal{K}_\gamma$ satisfies the integral size condition (1.6). By (4.54) and (4.16), for any $\lambda \geq 0$ and $0 < \delta_\mu < 1$ we have

$$|\nabla_1\Gamma_{\mu+\lambda}(y,x)| \lesssim \frac{e^{-\frac{\epsilon}{4}\sqrt{\omega_d}\sqrt{\lambda}|x-y|}}{|x-y|^{d-2}} \exp\left(-\frac{\epsilon}{2D_1}\left(1+\frac{|y-x|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right)\left(\int_{B\left(y,\frac{|x-y|}{2}\right)} \frac{d\mu(z)}{|z-y|^{d-1}} + \frac{1}{|x-y|}\right).$$

This immediately yields the bound for $\gamma = 1$. For $\frac{1}{2} < \gamma < 1$, integrating against $\lambda^{-\gamma}d\lambda$ gives

$$|\mathcal{K}_\gamma(x,y)| \lesssim \frac{\exp\left(-\frac{\epsilon}{2D_1}\left(1+\frac{|y-x|}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right)}{|x-y|^{d-(2\gamma-1)-1}}\left(\int_{B\left(y,\frac{|x-y|}{2}\right)} \frac{d\mu(z)}{|z-y|^{d-1}} + \frac{1}{|x-y|}\right).$$

Then, for each $\frac{1}{2} < \gamma \leq 1$, $|x-x_0| < R/2$, and $1 < s < \frac{2-\delta_\mu}{1-\delta_\mu}$, proceeding as in the Riesz transform $\mathcal{R}_\mu^*$ (see [14, Proposition 5.5]) we have

$$\left(\frac{1}{R^d}\int_{R<|x_0-y|\leq 2R} |\mathcal{K}_\gamma(x,y)|^s dy\right)^{1/s} \lesssim \frac{1}{R^{d-(2\gamma-1)}} \exp\left(-\frac{\epsilon}{4D_1}\left(1+\frac{R}{\rho(x)}\right)^{\frac{1}{k_0+1}}\right).$$

For the smoothness condition we consider $|x-x_0| < r < \rho(x_0)$ and $0 < r < R/16$. Then proceeding as in the proof of [22, Theorem 7.18] we have an estimate for $|\nabla_1\Gamma_{\mu+\lambda}(y,x) - \nabla_1\Gamma_{\mu+\lambda}(y,x_0)|$ (see [14, p. 24] for completeness) and consequently that

$$|\mathcal{K}_\gamma(x,y) - \mathcal{K}_\gamma(x_0,y)| \lesssim \left(\frac{|x-x_0|}{|x-y|}\right)^\delta \frac{e^{-\epsilon_0\left(1+\frac{R}{\rho(x)}\right)^{\frac{1}{(k_0+1)^2}}}}{|x-y|^{d-(2\gamma-1)-1}}\left(\int_{B(y,|y-x|)} \frac{d\mu(z)}{|z-y|^{d-1}} + \frac{1}{|x-y|}\right),$$

for some positive ϵ_0, which implies condition (1.7) with $\nu = 2\gamma-1$ for $1 < s < \frac{2-\delta_\mu}{1-\delta_\mu}$.

Finally, as a consequence of the estimates given above, for $f \in L^{s'}(\mathbb{R}^d)$ and $0 < \delta_\mu < 1$,

$$\begin{aligned}
|\mathcal{L}_\mu^{-\gamma}\nabla f(x)| &\lesssim \sum_{j=1}^\infty \left(\int_{2^j\rho(x)<|y-x|\leq 2^{j+1}\rho(x)} |\mathcal{K}_\gamma(x,y)|^s dy\right)^{1/s}\left(\int_{B(x,2^{j+1}\rho(x))} |f(y)|^{s'}\right)^{1/s'} \\
&\lesssim \sum_{j=1}^\infty e^{-\frac{\epsilon}{4D_1}2^{j\frac{1}{k_0+1}}} |B(x,2^{j+1}\rho(x))|^{\frac{2\gamma-1}{d}} \left(\fint_{B(x,2^{j+1}\rho(x))} |f(y)|^{s'}\right)^{1/s'}.
\end{aligned}$$

Since the classical fractional maximal operator is of weak type $(1, \frac{d}{d-(2\gamma-1)})$, we conclude that $\mathcal{L}_\mu^{-\gamma}\nabla$ is of weak type $\left(s', \frac{ds'}{d-(2\gamma-1)s'}\right)$, provided $(2\gamma-1)s' < d$. $\square$

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In the potential case $d\mu(x) = V(x)dx$ with $V \in RH_q$ and $\frac{d}{2} < q < d$, a sharper estimate for $\mathcal{L}_V^{-\gamma}\nabla$ can be obtained via [14, Proposition 5.5]. Indeed, the optimal endpoint for the Riesz transforms, $p_0 = \left(\frac{1}{q} - \frac{1}{d}\right)^{-1}$, automatically satisfies $p_0 > \frac{d}{d-(2\gamma-1)}$ for any $\frac{1}{2} < \gamma \leq 1$. This leads to the following classification.

Proposition 4.24. *Let $V \in RH_q$ with $\frac{d}{2} < q < d$, $V \not\equiv 0$, and $\frac{1}{2} < \gamma \leq 1$. Then, $\mathcal{L}_V^{-\gamma}\nabla$ is an exponential SCZO of $(2\gamma-1, p_0, \delta)$ type, with parameters $c = \frac{\epsilon}{4D_1}$ and $m = \frac{1}{k_0+1}$.*

As a consequence, and following the same lines as in the singular case given in [14, Theorem 4.1, Proposition 4.2] we establish the weighted boundedness results for $\mathcal{L}_\mu^{-\gamma}\nabla$.

Theorem 4.25. *Let $\frac{1}{2} < \gamma \leq 1$, $m = \frac{1}{k_0+1}$, $m^* = \frac{1}{(k_0+1)^2}$, $1 < p < \frac{d}{2\gamma-1}$, and $\frac{1}{q} = \frac{1}{p} - \frac{2\gamma-1}{d}$. The operator $\mathcal{L}_\mu^{-\gamma}\nabla$ is bounded from $L^p(w^p)$ to $L^q(w^q)$ for any weight w such that $w^{-p'} \in H^{\rho,m^*}_{1+\frac{p'}{q},c}$ where the constant c satisfies:*

(i) $c < \frac{\epsilon}{2D_1}\frac{d}{d-(2\gamma-1)}(2^{3k_0+7}C_0^{k_0+3})^{-m^*}$ *when* $\delta_\mu > 1$;

(ii) $c < \frac{\epsilon}{4D_1}\frac{ds'}{d-(2\gamma-1)s'}(2^{2k_0+6}C_0^{k_0+3})^{-m^*}$ *when* $0 < \delta_\mu < 1$.

Finally, since $\mathcal{L}_\mu^{-\gamma}\nabla(1) = 0$, we have the next result.

Theorem 4.26. *Let $0 < \delta_\mu < 1$, $\frac{1}{2} < \gamma \leq 1$ and $m = \frac{1}{k_0+1}$. For any $0 \leq \alpha + (2\gamma-1) < \delta$, the operator $\mathcal{L}_\mu^{-\gamma}\nabla$ is bounded from $\mathrm{BMO}_\rho^\alpha(w)$ to $\mathrm{BMO}_\rho^{\alpha+2\gamma-1}(w)$ whenever $w \in E^{\rho,m}_{s,c_1,c_2} \cap D^{\rho,m}_{\kappa,c_3}$ with $1 \leq \kappa < 1 + \frac{\delta-2\gamma-1-\alpha}{d}$ and $c_1, c_2, c_3 \geq 0$ such that $(c_2+c_3) < c_0\left(1 - \frac{d(\kappa-1)+\alpha+2\gamma-1}{\delta}\right)(4C_0)^{-m}$.*

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