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SPARSE APPROACH TO THE TWO-WEIGHT BOUNDEDNESS OF LOCAL SINGULAR OPERATORS. APPLICATIONS.

MAURICIO RAMSEYER, OSCAR SALINAS, AND JUAN MANUEL SOTTO RÍOS.

Abstract. In this paper, we study local singular integral operators on Euclidean spaces whose kernels satisfy a pointwise size condition together with Hörmander-type regularity conditions. We establish sufficient conditions of local Muckenhoupt type, reinforced by bump conditions on the weights, that guarantee two-weight norm inequalities for these operators. As applications, we obtain analogous estimates for local truncated operators and Schrödinger-type operators.

1. Introduction

Let Ω be a proper open and non empty subset of $\mathbb{R}^n$. The notation $Q = Q(x_Q, l_Q)$ means a cube Q with sides parallel to the coordinate axes, where x_Q and l_Q are the center and the length of half of its side respectively. Here, when we mention the metric d, we mean the usual d_∞. As usual we will denote by λQ the cube with same center and side length λ-times of Q.

For $0 < \beta < 1$, we consider the family of cubes well-inside of Ω defined by

$$\mathcal{F}_\beta = \{Q = Q(x_Q, l_Q) : \ x_Q \in \Omega, \ l_Q < \beta \, \mathrm{d}(x_Q, \Omega^c)\} \ ,$$

and the set

$$\mathcal{N}_\beta(E) = \bigcup_{\substack{R \cap E \neq \emptyset \\ R \in \mathcal{F}_\beta}} R \, , \tag{1.1}$$

as the cloud of E. For balls in a general metric space, this notion was introduced in [13].

Now, associated to the family $\mathcal{F}_\beta$, we introduce the definitions of the local operators we are interested in.

Definition 1.2. *Let $0 < \beta < 1$ and $f \in L^1_{loc}(\Omega)$. For $x \in \Omega$ we define the* ***β-local maximal operator*** *as*

$$M_\beta f(x) = \sup_{Q \in \mathcal{F}_\beta} \left(|Q|^{-1} \int_Q |f| \right) \chi_Q(x) \, .$$

Here, as it is usual in the classical literature, we denote by Δ the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$, that is $\Delta = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$. We proceed to define the β-local singular integral operator with which we will work throughout this article.

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Preprint

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Definition 1.3. *For $\beta \in (0,1)$ we say that T_β is a* ***β-local singular integral operator*** *if*

i) T_β is bounded on $L^2(\Omega)$.

ii) There exists a kernel $K : \Omega \times \Omega \setminus \Delta \to \mathbb{C}$ such that for $f \in L^\infty_c(\Omega)$

$$T_\beta f(x) = \int_\Omega K(x,y) f(y)\, dy\,,$$

a.e. $x \notin \operatorname{supp} f$ and $T_\beta f(x) = 0$, for x such that $\operatorname{supp} f \cap Q(x, \beta d(x,\Omega^c)) = \emptyset$. For each $x \in \Omega$, this can be written as $\operatorname{supp} K(x,\cdot) \subset Q(x, \beta d(x,\Omega^c))$.

iii) There exist constants $C > 0$ such that

a) $|K(x,y)| \leq \dfrac{C}{d(x,y)^n}$.

b) $|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C \dfrac{d(x,x')^\delta}{d(x,y)^{n+\delta}}$, for some $\delta > 0$ and whenever $2\, d(x,x') \leq d(x,y)$.

Remark 1.4. Although the definition of the local singular integral operator uses the uniform metric, it is important to note that we can replace it with the Euclidean metric since they are equivalent.

Remark 1.5. It is interesting, and useful, to note that an $\alpha-$local singular integral operator is also β-local whenever $\alpha \leq \beta$. This follows from the definition, noting that locality is only present in the kernel, more precisely in its support; therefore, by replacing α by a larger value, all the conditions continue to hold.

Remark 1.6. If a function f has a bounded support, more precisely in a β-local cube, then $T_\beta f$ also has bounded support contained in a local cube. Indeed, suppose that $\operatorname{supp} f \subset Q \in \mathcal{F}_\beta$. It follows from Definition 1.3 that $T_\beta f(x) \neq 0$ only when $Q \cap Q(x, \beta d(x,\Omega^c)) \neq \emptyset$. Recalling the definition of cloud, see Lemma 4.1, we then have

$$\operatorname{supp} T_\beta f \subset \mathcal{N}_\beta(Q)\,,$$

hence by Lemma 2.7, we have that $\operatorname{supp} T_\beta f \subset \tilde{Q} \in \mathcal{F}_{\tilde{\beta}}$.

Remark 1.7. Furthermore, if we restrict the domain of the operator to a set $E \subset \Omega$, then it is possible to consider that the function f, affected by the operator, has support dependent on the set E. To see this, take $x \in E$, and note that

$$\operatorname{supp} K(x,\cdot) \subset Q(x, \beta d(x,\Omega^c)) \subset \mathcal{N}_\beta(E)\,.$$

Then, from the definition, it is clear that

$$T_\beta f(x) = T_\beta \left(f \chi_{\mathcal{N}_\beta(E)} \right)(x)\,.$$

The one-weight boundedness problem related to the operators T_β was studied in [14] in a general metric space with a doubling Borel measure. There, the authors proved a result analogous to the well known for classical singular integrals in the euclidean setting, that is boundedness between weighted Lebesgue spaces if and only if the weight belongs to a Muckenhoupt-type class A_p.

The problem of characterizing pairs of weights (u,v) such that classical singular integrals are bounden from $L^p(\mathbb{R}^n, v)$ to $L^q(\mathbb{R}^n, u)$ is considerably more difficult and it is open. There are, however, some partial results in which conditions are imposed in the pairs of weights that are very difficult to verify. By a way of example, it

Preprint

IMAL PREPRINT # 2026-0085
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Publication date: November 13, 2026

can be seen [18]. This type of conditions are not strange. They already appear in similar problems relating to fractional maximals and integrals (see [29]). In another direction, a great deal of effort has been devoted to find sufficient and simpler conditions on (u, v) that ensure the aforementioned boundedness results.

The aim of this paper falls precisely within this line of research: to obtain two-weight boundedness of the β-local singular integral, T_β, under sufficient conditions on the weights and to show some applications of it. Before stating our main result, we introduce the classes of weights we will work with.

A function $\varphi : [0, \infty) \to [0, \infty)$ is a Young function if φ is continuous, convex and strictly increasing, $\varphi(0) = 0$ and $\varphi(t)/t \to \infty$ as $t \to \infty$. If φ and ψ are Young functions we say that φ dominates ψ if there exist $c > 0$ and $t_0 > 0$ such that $\psi(t) \leq c\varphi(t)$, for all $t \geq t_0 > 0$. Associated to φ we consider its conjugate function, namely $\overline{\varphi}$, defined by $\overline{\varphi}(t) = \sup_{s>0} \big(st - \varphi(s)\big)$. Finally, if φ is a Young function, the normalized Luxembourg norm of a locally integrable function f in Q is given by

$$\|f\|_{\varphi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \varphi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\} .$$

In the particular case $\varphi(t) = t^p$, with $1 \leq p < \infty$, we simply write $\|f\|_{p,Q}$ and it is easy to see that it is equal to $\left(\frac{1}{|Q|} \int_Q |f(x)|^p \, dx\right)^{\frac{1}{p}}$.

Moreover the following Hölder-type inequality holds (see [28])

$$\frac{1}{|Q|} \int_Q |fg| \leq C \, \|f\|_{\varphi,Q} \, \|g\|_{\overline{\varphi},Q} \, .$$

Equipped with these conventions, we are now ready to define our classes. In the three definitions that follow, we assume that $0 < \beta < 1$ and $1 < p, q < \infty$.

Definition 1.8. *We say that the pair of weights (u, v) belongs to the class $A^\beta_{p,q}$ if and only if*

$$[u, v]_{A^\beta_{p,q}} = \sup_{Q \in \mathcal{F}_\beta} |Q|^{\frac{1}{q} - \frac{1}{p}} \left\| u^{\frac{1}{q}} \right\|_{q,Q} \left\| v^{-\frac{1}{p}} \right\|_{p',Q} < \infty \, .$$

Definition 1.9. *Let ψ be a Young function. We say that the pair of weights (u, v) belongs to the class $A^\beta_{p,\psi,q}$ if and only if*

$$[u, v]_{A^\beta_{p,\psi,q}} = \sup_{Q \in \mathcal{F}_\beta} |Q|^{\frac{1}{q} - \frac{1}{p}} \left\| u^{\frac{1}{q}} \right\|_{q,Q} \left\| v^{-\frac{1}{p}} \right\|_{\psi,Q} < \infty \, .$$

Definition 1.10. *Let ϕ and ψ be Young functions. We say that the pair of weights (u, v) belongs to the class $A^\beta_{p,\psi,q,\phi}$ if and only if*

$$[u, v]_{A^\beta_{p,\psi,q,\phi}} = \sup_{Q \in \mathcal{F}_\beta} |Q|^{\frac{1}{q} - \frac{1}{p}} \left\| u^{\frac{1}{q}} \right\|_{\phi,Q} \left\| v^{-\frac{1}{p}} \right\|_{\psi,Q} < \infty \, .$$

These three definitions could have been written as one as above, but for historical reasons it was decided to include them. Only the case remains here where $\psi = t^{p'}$, in which case we will denote this class by $A^\beta_{p,q,\phi}$.

Preprint

IMAL PREPRINT # 2026-0085
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We say that a Young function φ belongs to the class B_p, with $1 < p < \infty$, if for some $c > 0$

$$\int_c^\infty \frac{\varphi(t)}{t^p}\,\frac{dt}{t} < \infty\,.$$

This condition is equivalent to the boundedness of the maximal operator defined in terms of Orlicz averages (see [23]).

In [5, Remark 5.12] the author proves that if $r > 1$ and Φ is a Young function such that $\overline{\Phi} \in B_r$ then

$$\|g\|_{r',Q} \leq C\,\|g\|_{\Phi,Q}\,.$$

As a consequence we can prove the following lemma.

Lemma 1.11. *Let $0 < \beta < 1$, $1 < p, q < \infty$ and let ϕ and ψ be Young functions. The following inclusions hold*

i) if $\overline{\psi} \in B_p$ *then* $A^\beta_{p,\psi,q} \subset A^\beta_{p,q}$ *and* $A^\beta_{p,\psi,q,\phi} \subset A^\beta_{p,q,\phi}$.

ii) if $\overline{\phi} \in B_{q'}$ *then* $A^\beta_{p,q,\phi} \subset A^\beta_{p,q}$ *and* $A^\beta_{p,\psi,q,\phi} \subset A^\beta_{p,\psi,q}$.

iii) if $\overline{\psi} \in B_p$ *and* $\overline{\phi} \in B_{q'}$ *then* $A^\beta_{p,\psi,q,\phi} \subset A^\beta_{p,q}$.

This type of function is important in this theory, given that it characterizes the boundedness of the associated Orlicz maximal operators on Lebesgue spaces. More precisely, for a Young function φ we have the associated maximal operator, defined as

$$M_\varphi f(x) = \sup_Q \|f\|_{\varphi,Q}\,\chi_Q(x)\,. \tag{1.12}$$

which is bounded on L^p if and only if $\varphi \in B_p$. (see [23], [24]).

Now, we are in position to state our main result.

Theorem 1.13. *Let $0 < \tau < 1$, $1 < p \leq q < \infty$ and u, v be two weights such that $(u, v) \in A^\tau_{p,\psi,q,\phi}$, where ϕ and ψ are Young functions such that $\overline{\phi}$ and $\overline{\psi}$ belong to $B_{q'}$ and B_p respectively. Then, for each $\beta < \tau$ we have*

$$T_\beta : L^p(\Omega, v) \to L^q(\Omega, u)\,.$$

The techniques we are going to apply to prove the above theorem can be adapted to prove a similar result concerning local singular integrals with Hörmander type conditions.

Definition 1.14. *For $\beta \in (0, 1)$ we say that $\mathcal{T}_\beta$ is a β-local singular integral operator of type $\mathcal{H}_s$, with $1 < s < \infty$, if*

i) $\mathcal{T}_\beta$ is bounded from $L^{s'}(\Omega)$ into $L^{s'\infty}(\Omega)$.

ii) There exists a kernel $K : \Omega \times \Omega \setminus \Delta \to \mathbb{C}$ such that for $f \in L^\infty_c(\Omega)$

$$\mathcal{T}_\beta f(x) = \int_\Omega K(x, y) f(y) dy\,,$$

a.e. $x \notin \operatorname{supp} f$ and $\mathcal{T}_\beta f(x) = 0$, for every x such that

$$\operatorname{supp} f \cap Q(x, \beta d(x, \Omega^c)) = \emptyset\,.$$

iii) There exists a constant $C > 0$ such that for any $R > 0$

Preprint

IMAL PREPRINT # 2026-0085
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Publication date: November 13, 2026

a) $\left(\dfrac{1}{R^n}\displaystyle\int_{R<d(x',y)<2R}|K(x,y)|^s\,dy\right)^{\frac{1}{s}}\leq\dfrac{C}{R^n}$,
where $d(x,x')<\frac{R}{2}$.

b) $\left(\dfrac{1}{R^n}\displaystyle\int_{R<d(x',y)<2R}|K(x,y)-K(x',y)|^s\,dy\right)^{\frac{1}{s}}\leq\dfrac{C}{R^n}\left(\dfrac{r}{R}\right)^{\delta}$,
for some $\delta>0$ *and whenever* $d(x,x')<r<\frac{R}{2}$.

Theorem 1.15. *Let* $0<\tau<1$, $1<s<\infty$, $s'<p\leq q<\infty$ *and* u, v *be two weights such that* $(u,v)\in A^{\tau}_{\frac{p}{s'},\psi,\frac{q}{s'},\phi}(\Omega)$, *where* ϕ *y* ψ *are Young functions such that* $\overline{\phi_{s'}}$ *and* $\overline{\psi}_{s'}$ *belong to* $B_{q'}$ *y* B_p *respectively. Then, for each* $\beta<\tau$ *we have*

$$\mathcal{T}_\beta : L^p(\Omega,v)\to L^q(\Omega,u)\,.$$

It is important to emphasize that a key tool in the proof techniques we will be using is the identification os sparse operators that enable us to control our singular integrals.

As consequence of these results we can prove boundenness results for localizations of singular integrals and singular integrals related to Schrödinger operators. With regard to this latter application, the basic framework technique consists of dividing the operators into a local and a global part. It is precisely in the local part that our main result are applied. And it is interesting to note that, even though the concept of locality used to divide the operators does not, in principle, coincides exactly with that set up in our introduction, it can nevertheless be related to it in a natural way.

The structure of the article is as follows. Section 2 contains results concerning a maximal local singular integral operator, which are interesting in themselves. Section 3 is devoted to obtain a sparse domination of T_β. Section 4 is devoted to the proof of Theorem 1.13 (poner acá lo de Hormander o bien poner que se hizo en otra sección). Finally, Section 5 contains the aforementioned applications.

For the study of T_β we remark that the case of small β and then consider the general case. However, splitting the operator, with a smaller locality parameter, it is not as simple as in the fractional case. The main difficult lies in correctly defining the resulting operators. This limitation motivates the need for the following results, which will allow us to overcome this obstacle.

Consider a smooth cut-off function $\eta:\mathbb{R}^+\to[0,1]$, such that $\eta(t)=1$ for $0<t\leq 1$ and $\eta(t)=0$ for $t\geq 2$.

Let T_β be a β-local singular integral with kernel K. For each α, such that $2\alpha<\beta$, we define the following auxiliary operator

$$T_0f(x)=\int_\Omega K(x,y)\left(1-\eta\left(\frac{d(x,y)}{\alpha\,d(x,\Omega^c)}\right)\right)f(y)\,dy\,. \tag{1.16}$$

Related to these kind of operator we can prove the following result.

Proposition 1.17. *Let* T_β *be a* β*-local singular integral, then for each* $\theta\in(0,\beta)$ *there exists a* θ*-local singular integral operator,* T_θ, *such that the pointwise estimate*

$$|T_\beta f(x)|\leq|T_\theta f(x)|+CM_\beta f(x)\,,$$

holds, where M_β *is the* β*-local Maximal function (see 1.2).*

Preprint

IMAL PREPRINT # 2026-0085
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Proof. Given $\theta \in (0,\beta)$, we consider a smooth cut-off function η as above and $\alpha = \theta/2$. Then we define

$$T_\theta \doteq T_\beta - T_0\,,$$

where the operator T_0 is as before. This operator is a θ-local singular integral operator (see [14, Theorem 4.1]).

The pointwise estimate is a consequence of the following estimate. For $x \in \Omega$, the hypotheses on K and η implies that

$$\begin{aligned}|T_0 f(x)| &\leq C \int_{\frac{\theta}{2} d(x,\Omega^c) \leq d(x,y) \leq \beta d(x,\Omega^c)} \frac{|f(y)|}{d(x,y)^n}\, dy \\ &= \frac{C}{|Q(x,\beta d(x,\Omega^c))|} \int_{Q(x,\beta d(x,\Omega^c))} |f(y)|\, dy \leq C M_\beta f(x)\,.\end{aligned}$$

Finally

$$|T_\beta f(x)| \leq |T_\beta f(x) - T_0 f(x)| + |T_0 f(x)| \leq |T_\theta f(x)| + C M_\beta f(x)\,,$$

which completes the proof. □

Remark 1.18. Note that the constant appearing in the estimate of the last proposition depends on both β and θ.

2. Weak $(1,1)$ boundedness of T^*_β

Let T_β be a β-local maximal singular integral operator and $\varepsilon > 0$. We define the truncated β-local singular integral operator as

$$T^*_\beta f(x) = \sup_{\varepsilon > 0} \left| T^\varepsilon_\beta f(x) \right|\,,$$

where

$$T^\varepsilon_\beta f(x) = \int_{d(x,y)>\varepsilon} K(x,y) f(y)\, dy\,.$$

An important tool in the proof of the our main result is the $(1,1)$ weak type inequality for this operator. In order to do that, we follow the steps taken in [8] to prove the weak boundedness for the classical maximal singular integral operator. This leads us to the following lemma.

Lemma 2.1 (Local Cotlar inequality)**.** *Let us consider $\sigma \in (0,1]$ and let T_β be a β-local singular integral operator with $\beta < \frac{\sigma}{3}$, then there exists $\xi \in (\beta,\sigma)$ such that for any $f \in L^\infty_c(\Omega)$ and every $\nu \in (0,1]$, the following pointwise estimate holds*

$$T^*_\beta f(x) \leq C_\nu \left(M_\beta(|T_\beta|^\nu)(x)^{1/\nu} + M_\xi f(x) \right)\,, \tag{2.2}$$

for each $x \in \Omega$.

Proof. For a given $\sigma \in (0,1]$ we consider $\beta < \sigma/3$ and a β-local singular integral operator T_β. Let $x_0 \in \Omega$ and $\varepsilon > 0$. It will be sufficient to prove (2.2) for $T^\varepsilon_\beta f(x_0)$ with constant C_ν independent of ε and x_0. Moreover, since $\operatorname{supp} K(x_0,\cdot) \subset Q(x_0,\beta d(x_0,\Omega^c))$ it is sufficient to consider $\varepsilon < \beta d(x_0,\Omega^c)$.

Preprint

IMAL PREPRINT # 2026-0085
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Publication date: November 13, 2026

If we consider $Q = Q(x_0, \frac{\varepsilon}{2})$ and taking the functions $f_1 = f\chi_{2Q}$ and $f_2 \doteq f - f_1$, we get that

$$T_\beta^\varepsilon f(x_0) = \int_{d(x_0,y)>\varepsilon} K(x_0,y) f(y) dy = T_\beta f_2(x_0)\,.$$

Now consider $z \in Q \in \mathcal{F}_\beta$. We claim that there exists $\xi \in (\beta, \sigma)$ such that

$$\operatorname{supp} K(z,\cdot) \subset Q(x_0, \xi d(x_0, \Omega^c))\,.$$

To do this, let $y \in \operatorname{supp} K(z,\cdot) \subset Q_z \doteq Q(z, \beta d(z,\Omega^c))$, and since $\beta < 1$

$$\begin{aligned} d(y,x_0) &< \beta\, d(z,\Omega^c) + \frac{\varepsilon}{2} \\ &< \beta \frac{1+\beta}{1-\beta} d(x_0,\Omega^c) + \beta d(x_0,\Omega^c) \\ &\leq \frac{2\beta}{1-\beta} d(x_0,\Omega^c)\,. \end{aligned}$$

Defining $\xi \doteq \frac{2\beta}{1-\beta}$ we have that $\operatorname{supp} K(z,\cdot) \subset Q(x_0, \xi d(x_0,\Omega^c))$, furthermore since $\beta < \frac{\sigma}{3}$ it is easy to see that $\beta < \xi < \sigma$. In particular, it is also true that

$$\operatorname{supp}\{K(x_0,\cdot) - K(z,\cdot)\} \subset \operatorname{supp} K(x_0,\cdot) \cup \operatorname{supp} K(z,\cdot) \subset Q(x_0, \xi d(x_0,\Omega^c))\,.$$

On the other hand, since $y \notin 2Q$ implies $2\, d(z,x_0) < \varepsilon \leq d(x_0,y)$ it is possible to restrict the domain of integration and then use the smoothness condition of the kernel. Thus

$$\begin{aligned} |T_\beta f_2(x_0) - T_\beta f_2(z)| &= \left| \int_\Omega (K(x_0,y) - K(z,y))\, f_2(y) dy \right| \\ &\leq \int_{\varepsilon < d(x_0,y) < \xi d(x_0,\Omega^c)} |K(x_0,y) - K(z,y)|\, |f(y)| dy \\ &\leq C\, d(x_0,z)^\delta \int_{\varepsilon < d(x_0,y) < \xi d(x_0,\Omega^c)} \frac{|f(y)|}{d(x_0,y)^{n+\delta}} dy\,. \end{aligned}$$

Now, take the integer k_0 such that

$$2^{k_0}\varepsilon < \xi d(x_0,\Omega^c) \leq 2^{k_0+1}\varepsilon\,.$$

So

$$\begin{aligned} |T_\beta f_2(x_0) - T_\beta f_2(z)| &\leq C\varepsilon^\delta \left(\sum_{k=0}^{k_0-1} \int_{2^k\varepsilon < d(x_0,y) \leq 2^{k+1}\varepsilon} \frac{|f(y)|}{d(x_0,y)^{n+\delta}} dy \right. \\ &\qquad \left. + \int_{2^{k_0}\varepsilon < d(x_0,y) < \xi d(x_0,\Omega^c)} \frac{|f(y)|}{d(x_0,y)^{n+\delta}} dy \right) \\ &\leq C \left(\sum_{k=0}^{k_0-1} \frac{2^{-k\delta}}{|Q(x_0, 2^{k+1}\varepsilon)|} \int_{Q(x_0,2^{k+1}\varepsilon)} |f(y)| dy \right. \\ &\qquad \left. + \frac{2^{-k_0\delta}}{|Q(x_0, \xi d(x_0,\Omega^c))|} \int_{Q(x_0,\xi d(x_0,\Omega^c))} |f(y)| dy \right) \\ &\leq C \left(\sum_{k=0}^{k_0} 2^{-k\delta} \right) M_\xi f(x_0) \\ &\leq C\, M_\xi f(x_0)\,. \end{aligned}$$

Preprint

IMAL PREPRINT # 2026-0085
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By recalling the definition of f_2 we have that

$$\begin{aligned}\left|T_\beta^\varepsilon f(x_0)\right| &\leq |T_\beta f_2(x_0) - T_\beta f_2(z)| + |T_\beta f_2(z)| \\ &\leq CM_\xi f(x_0) + |T_\beta f(z)| + |T_\beta f_1(z)| \,. \end{aligned} \tag{2.3}$$

First, we will prove the case $\nu = 1$. If $T_\beta^\varepsilon f(x_0) = 0$ there is nothing to prove. On the other hand we consider λ such that $0 < \lambda < \left|T_\beta^\varepsilon f(x_0)\right|$ and define

$$\begin{aligned} Q_1 &= \left\{ z \in Q : |T_\beta f(z)| > \frac{\lambda}{3} \right\}, \\ Q_2 &= \left\{ z \in Q : |T_\beta f_1(z)| > \frac{\lambda}{3} \right\}, \\ Q_3 &= \begin{cases} \emptyset\,, & \text{if } CM_\xi f(x_0) \leq \frac{\lambda}{3}\,, \\ Q\,, & \text{if } CM_\xi f(x_0) > \frac{\lambda}{3}\,. \end{cases} \end{aligned}$$

Thus $Q = Q_1 \cup Q_2 \cup Q_3$. Then, since $x_0 \in Q \in \mathcal{F}_\beta$ we get that

$$|Q_1| \leq \frac{3}{\lambda} \int_{Q_1} |T_\beta f(z)|\, dz \leq \frac{3}{\lambda} |Q| \left(\frac{1}{|Q|} \int_Q |T_\beta f(z)|\, dz \right) \leq \frac{3}{\lambda} |Q|\, M_\beta(T_\beta f)(x_0).$$

On the other hand, from the weak type $(1,1)$ inequality of the operator T_β (see [14, Theorem 4.1])

$$|Q_2| = |\{z \in Q : |T_\beta f_1(z)| > \lambda/3\}| \leq \frac{3C}{\lambda} \int_{2Q} |f(z)|\, dz \leq \frac{C}{\lambda} |Q|\, M_\xi f(x_0).$$

Then, by the definition of Q_3 we can deduce that

$$|Q| \leq |Q_1| + |Q_2| + |Q_3| \leq \frac{C}{\lambda} |Q| \left(M_\beta(T_\beta f)(x_0) + M_\xi f(x_0) \right) .$$

So

$$\lambda \leq C \left(M_\beta(T_\beta f)(x_0) + M_\xi f(x_0) \right) ,$$

for all $0 < \lambda < \left|T_\beta^\varepsilon f(x_0)\right|$. The desired estimate holds by taking the supremum over such λ.

Now considering $0 < \nu < 1$, from (2.3) it follows that

$$\left|T_\beta^\varepsilon f(x_0)\right|^\nu \leq CM_\xi f(x_0)^\nu + |T_\beta f(z)|^\nu + |T_\beta f_1(z)|^\nu \,,$$

for $z \in Q$. Taking the average over the cube $Q \in \mathcal{F}_\beta$ we have

$$\left|T_\beta^\varepsilon f(x_0)\right|^\nu \leq CM_\xi f(x_0)^\nu + \frac{1}{|Q|} \int_Q |T_\beta f(z)|^\nu\, dz + \frac{1}{|Q|} \int_Q |T_\beta f_1(z)|^\nu\, dz.$$

In the last term we can use a consequence of Kolmogorov's inequality (see, e.g., [8, p. 102]).

$$\int_Q |T_\beta f_1(z)|^\nu\, dz \leq C\, |Q|^{1-\nu}\, \|f_1\|_1^\nu = C\, |Q| \left(\frac{1}{|2Q|} \int_{2Q} |f| \right)^\nu \leq C\, |Q|\, M_\xi f(x_0)^\nu.$$

Therefore

$$\left|T_\beta^\varepsilon f(x_0)\right|^\nu \leq C \left(M_\beta(|T_\beta f|^\nu)(x_0) + M_\xi f(x_0)^\nu \right) .$$

Raising the previous inequality to the power $1/\nu$ and taking the supremum over $\varepsilon > 0$, we obtain the desired result for any $x_0 \in \Omega$, with constant independent of x_0 and ε. Then the proof of lemma is complete. □

Preprint

IMAL PREPRINT # 2026-0085
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For a given cube Q_0, let $\mathcal{D}(Q_0)$ be a dyadic grid associated with Q_0. This collection consists of all dyadic cubes relative to Q_0 formed by successively partitioning Q_0 and its descendants into 2^n congruent subcubes.

We state the following fact and refer the reader to [25] for its proof.

Lemma 2.4. *Let $\tau \in (0,1)$. If $Q \in \mathcal{F}_\beta$ with $\beta < \frac{\tau}{1+\tau}$, then $\mathcal{D}(P) \subset \mathcal{F}_\tau$, for every $P \subset Q$. In particular, $P \in \mathcal{F}_\tau$.*

Now, we will consider a dyadic maximal function associated with a dyadic grid as follows.

Definition 2.5. *Let $f \in L^1_{loc}(\Omega)$ and a cube $R \subset \Omega$. Associated with the dyadic grid $\mathcal{D} = \mathcal{D}(R)$ we define the **dyadic maximal function** $M^{\mathcal{D}} f$ for every $x \in \Omega$ as*

$$M^{\mathcal{D}} f(x) = \sup_{Q \in \mathcal{D}} \left(|Q|^{-1} \int_Q |f| \right) \chi_Q(x) \,.$$

Lemma 2.6. *Let Q be a cube and $\mathcal{D} = \mathcal{D}(Q)$. Then $M^{\mathcal{D}}$ is of weak type $(1,1)$. More precisely, for all $f \in L^1$ and $\lambda > 0$ the following estimate holds*

$$\left|\left\{x \in Q : M^{\mathcal{D}} f(x) > \lambda\right\}\right| \leq \frac{1}{\lambda} \int_{\{x \in Q : M^{\mathcal{D}} f(x) > \lambda\}} |f| \,.$$

Proof. For fixed $\lambda > 0$, we define

$$\mathcal{S} = \left\{ P \in \mathcal{D} : P \text{ is maximal, and } \frac{1}{|P|} \int_P |f| > \lambda \right\} .$$

It is clear that the cubes in $\mathcal{S}$ are disjoint and

$$\left\{x \in Q : M^{\mathcal{D}} f(x) > \lambda\right\} = \bigcup_{P \in \mathcal{S}} P \,.$$

Then we can estimate

$$\left|\left\{x \in Q : M^{\mathcal{D}} f(x) > \lambda\right\}\right| = \sum_{P \in \mathcal{S}} |P| \leq \frac{1}{\lambda} \sum_{P \in \mathcal{S}} \int_P |f| = \frac{1}{\lambda} \int_{\{x \in Q : M^{\mathcal{D}} f(x) > \lambda\}} |f| \,.$$

This proves the lemma. □

In [25] the authors proved the following two lemmas

Lemma 2.7. *Let $\eta \in (0,1]$. If $\beta \in \left(0, \frac{\eta}{5}\right)$ then there exists $\tilde{\beta} \in (\beta, \eta)$ such that*

$$\mathcal{N}_\beta(R) \subset \tilde{R} = Q(x_0, \tilde{\beta}\, d(x_0, \Omega^c)) \,,$$

for every cube $R = Q(x_0, l) \in \mathcal{F}_\beta$.

In [25, Proposition 3.3], the authors proves that each $\tau \in (0,1)$ yields a constant β_0 such that, whenever $\beta < \beta_0$ and f is a locally integrable function supported on a β-local cube R, one has

$$M_\beta f(x) \leq C \sum_{k=1}^{3^n} M^{\mathcal{D}^k} f(x) \,, \tag{2.8}$$

for all $x \in \Omega$, where the dyadic grids $\mathcal{D}^k \subset \mathcal{F}\tau$ are constructed within local cubes containing the support of f.

Preprint

IMAL PREPRINT # 2026-0085
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The proof that the operator T_β^* satisfies a weak type $(1,1)$ inequality relies on the following proposition. Specifically, this result assures us that for a sufficiently small β and a function f with suitable support, the operator behaves as expected.

Proposition 2.9. *There exists $\beta_1 \in (0,1)$ such that for each $\beta < \beta_1$ and every function $f \in L^\infty(\Omega)$, with* $\operatorname{supp} f \subset R \in \mathcal{F}_\beta$*, we have*

$$\left|\left\{x \in \Omega : T_\beta^* f(x) > \lambda\right\}\right| \leq \frac{C}{\lambda} \|f\|_{L^1(\Omega)} ,$$

for all $\lambda > 0$.

Proof. For $\tau \in (0,1)$, let β_0 be the upper bound, mentioned earlier, for which equation (2.8) holds. and define $\beta_1 = \beta_0/5$. Setting $\beta < \beta_1$ implies that $\beta < 1/3$. Thus, for $\nu \in (0,1)$, by Lemma 2.1, there exists $\xi < 1$ such that

$$T_\beta^* f(x) \leq C \left(M_\beta \left(|T_\beta f|^\nu \right)(x)^{1/\nu} + M_\xi f(x) \right) . \tag{2.10}$$

Consequently, we only need to bound the two terms on the right-hand side of the previous inequality.

Since M_ξ is of weak type $(1,1)$, see [13, Theorem 1.1], it follows that for $\lambda > 0$

$$\left|\{x \in \Omega : M_\xi f(x) > \lambda\}\right| \leq \frac{C}{\lambda} \|f\|_{L^1(\Omega)} . \tag{2.11}$$

On the other hand, since $\operatorname{supp} f \subset R$, by Remark 1.6 we have that $T_\beta f$ is a function with support in some $\tilde{R} \in \tilde{\beta}$, for some $\tilde{\beta} \in (0, \beta_0)$. So, from (2.8) and Lemma 2.7 we can estimate

$$M_\beta \left(|T_\beta f|^\nu \right)(x) \leq M_{\tilde{\beta}} \left(|T_\beta f|^\nu \right)(x) \leq C \sum_{k=1}^{3^n} M^{\mathcal{D}^k} \left(|T_\beta f|^\nu \right)(x) , \tag{2.12}$$

where $\mathcal{D}^k$ are dyadic grids in $\mathcal{F}_{9/10}$ constructed in cubes containing the support of $|T_\beta f|^\nu$.

For simplicity, let Q^k denote the cube in which the dyadic grid $\mathcal{D}^k$ is constructed, that is $\mathcal{D}^k = \mathcal{D}(Q^k)$. Thus $\tilde{R} \subset Q^k$. Then for fixed k, we define the set

$$E = \left\{ x \in Q^k : M^{\mathcal{D}^k} \left(|T_\beta f|^\nu \right)(x) > \lambda^\nu \right\} .$$

Then since T_β is of weak type $(1,1)$ (see [14, Theorem 4.1]), it follows by Lemma 2.6 and Kolmogorov's inequality

$$|E| \leq \frac{1}{\lambda^\nu} \int_E |T_\beta f|^\nu \leq \frac{C}{\lambda^\nu} |E|^{1-\nu} \|f\|_{L^1(\Omega)}^\nu ,$$

from which we conclude

$$\left| \left\{ x \in Q^k : M^{\mathcal{D}^k} \left(|T_\beta f|^\nu \right)(x) > \lambda^\nu \right\} \right| \leq \frac{C}{\lambda} \|f\|_{L^1(\Omega)} .$$

From here and by (2.12) we have

$$\begin{aligned}
\left| \left\{ x \in \tilde{R} : M_\beta \left(|T_\beta f|^\nu \right)(x) > \lambda^\nu \right\} \right| &\leq \sum_k \left| \left\{ x \in \tilde{R} : M^{\mathcal{D}^k} \left(|T_\beta f|^\nu \right)(x) > \frac{\lambda^\nu}{C\, 3^n} \right\} \right| \\
&\leq \sum_k \left| \left\{ x \in Q^k : M^{\mathcal{D}^k} \left(|T_\beta f|^\nu \right)(x) > \frac{\lambda^\nu}{C\, 3^n} \right\} \right| \\
&\leq \frac{C}{\lambda} \|f\|_{L^1(\Omega)} . \qquad (2.13)
\end{aligned}$$

Preprint

IMAL PREPRINT # 2026-0085
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Finally, by (2.10), (2.11) and (2.13) we obtain

$$\begin{aligned}\left|\left\{x\in\Omega : T^*_\beta f(x)>\lambda\right\}\right| &= \left|\left\{x\in\tilde{R} : T^*_\beta f(x)>\lambda\right\}\right| \\ &\leq \left|\left\{x\in\tilde{R} : M_\beta\left(|T_\beta f|^\nu\right)(x)>\frac{\lambda^\nu}{(2C)^\nu}\right\}\right| + \left|\left\{x\in\tilde{R} : M_\xi f(x)>\frac{\lambda}{2C}\right\}\right| \\ &\leq \frac{C}{\lambda}\left\|f\right\|_{L^1(\Omega)}\,,\end{aligned}$$

which completes the proof. □

Now we will prove that T^*_β is strong (p,p) and weak $(1,1)$, provided β is sufficiently small.

Proposition 2.14. *Let T_β be a β-local Calderon-Zygmund Operator. Then there exists $\beta_2\in(0,1)$ such that if $\beta<\beta_2$ then the operator T^*_β is of strong type (p,p), $p>1$, and of weak type $(1,1)$.*

Proof. Let β_1 be as in the previous proposition and define $\beta_2=\beta_1/5$. Let $\beta<\beta_2$ and, without loss of generality, let $f\in L^\infty_c$. Since $\beta<1/3$, by Lemma 2.1 (with $\nu=1$) there exists $\xi<1$ such that

$$T^*_\beta f(x)\leq C\left(M_\beta\left(T_\beta f\right)(x)+M_\xi f(x)\right)\,. \tag{2.15}$$

The strong type (p,p) boundedness, with $p>1$, follows immediately from the previous inequality since M_β, M_ξ and T_β are also strong (p,p). These results can be found in [13, Theorem 1.1] and in [14, Theorem 4.1] respectively. Then we obtain

$$\left\|T^*_\beta f\right\|_{L^p}\leq C\left(\left\|M_\beta\left(T_\beta f\right)\right\|_{L^p}+\left\|M_\xi f\right\|_{L^p}\right)\leq C\left(\left\|T_\beta f\right\|_{L^p}+\left\|f\right\|_{L^p}\right)\leq C\left\|f\right\|_{L^p}\,.$$

For the weak type inequality we consider the Whitney decomposition (see Lemma 4.1)

$$\Omega=\bigcup_j R_j\,.$$

For each $x\in R_j$ and for each $\varepsilon>0$, as mentioned in Remark 1.7, we have

$$T^\varepsilon_\beta f(x)=T^\varepsilon_\beta f_j(x)\,,$$

where $f_j(x)=f(x)\chi_{\mathcal{N}_\beta(R_j)}(x)$. Thus obtaining

$$T^*_\beta f(x)=T^*_\beta f_j(x)\,.$$

Since $\beta<\beta_1/5$, by Lemma 2.7 (with $\eta=\beta_1$) we have $\operatorname{supp} f_j\subset\tilde{R}_j\in\mathcal{F}_{\tilde{\beta}}$, for some $\beta<\tilde{\beta}<\beta_1$. Applying Proposition 2.9, and taking into account Remark 1.5, we obtain for $\lambda>0$

$$\begin{aligned}\left|\left\{x\in\Omega : T^*_\beta f(x)>\lambda\right\}\right| &= \sum_j\left|\left\{x\in R_j : T^*_\beta f_j(x)>\lambda\right\}\right| \\ &= \sum_j\left|\left\{x\in R_j : T^*_{\tilde{\beta}} f_j(x)>\lambda\right\}\right| \\ &\leq \frac{C}{\lambda}\sum_j\left\|f_j\right\|_{L^1(\Omega)}\end{aligned}$$

Preprint

IMAL PREPRINT # 2026-0085
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$$\leq \frac{C}{\lambda}\int_{\Omega} f(y)\Big(\sum_j \chi_{\mathcal{N}_\beta(\tilde{R}_j)}(y)\Big)dy$$
$$\leq \frac{C}{\lambda}\,\|f\|_{L^1(\Omega)}\,,$$

where we have used the controlled overlap of the clouds constructed with the Whitney cubes (see Lemma 4.2 below). This proves the Proposition. □

Next we prove that the maximal operator T^*_β is strong (p,p) and weak $(1,1)$ regardless of the value of β.

Theorem 2.16. *Let T_β be a β-local Calderon-Zygmund Operator. Then the maximal operator T^*_β is of strong type (p,p), $p>1$, and of weak type $(1,1)$.*

Proof. Consider β_2 as in the previous proposition. If $\beta<\beta_2$ there is nothing to prove since the result follows from the previous proposition. On the other hand, if $\beta\geq\beta_2$, taking $\theta \doteq \frac{\beta_2}{2}$ and using Proposition 1.17, for $\varepsilon>0$ we have

$$\begin{aligned}\left|T^\varepsilon_\beta f(x)\right| &= \left|T_\beta\left(f\chi_{(Q(x,\varepsilon))^c}\right)(x)\right| \\ &\leq C\,M_\beta\left(f\chi_{(Q(x,\varepsilon))^c}\right)(x) + \left|T_\theta\left(f\chi_{(Q(x,\varepsilon))^c}\right)(x)\right| \\ &\leq C\,M_\beta f(x) + |T^\varepsilon_\theta f(x)| \\ &\leq C\,M_\beta f(x) + T^*_\theta f(x)\,.\end{aligned}$$

Taking the supremum over $\varepsilon>0$ we obtain

$$T^*_\beta f(x) \leq C\,M_\beta f(x) + T^*_\theta f(x)\,.$$

Then, since M_β and T^*_θ, with $\theta<\beta_2$, are strong (p,p) and weak $(1,1)$, we conclude the same for the operator T^*_β. □

3. Sparse domination of T_β.

The aim of this section is to obtain a sparse operator that allows us to bound the β-local singular integral operator pointwise, at least for small β and for suitably supported functions. To do this, we adapt to the local operators the sparse domination proved by A. Lerner in [19]. With this purpose in mind, we first need to define two auxiliary operators. Then we will prove that these new operators are of weak type $(1,1)$ and strong type (p,p), with $p>1$.

Definition 3.1 (Truncated Grand Maximal Operator)**.** *Let $\beta\in(0,1)$ and T_β be a β-local singular integral operator. For $f\in L^1(\Omega)$ and $x\in\Omega$ we define*

$$\mathcal{M}_{T_\beta} f(x) = \sup_{Q\subset\Omega}\ \operatorname*{ess\,sup}_{\xi\in Q}\left|T_\beta(f\chi_{\Omega\setminus 3Q})(\xi)\right|\chi_Q(x)\,.$$

Definition 3.2 (Localized Truncated Grand Maximal Operator)**.** *Let $\beta\in(0,1)$ and T_β be a β-local singular integral operator. For $f\in L^1(\Omega)$ and $x\in R\subset\Omega$, a cube, we define:*

$$\mathcal{M}_{T_\beta,R} f(x) = \sup_{Q\subset R}\ \operatorname*{ess\,sup}_{\xi\in Q}|T_\beta(f\chi_{3R\setminus 3Q})(\xi)|\chi_Q(x)\,.$$

We need the following technical lemma.

Preprint

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Lemma 3.3. *Let $\beta \in (0,1)$ and T_β be a β-local singular integral operator. Then for $f \in L^1(\Omega)$ the following pointwise estimates hold*

i) for a.e. x in the cube $R \subset \Omega$,

$$|T_\beta(f\chi_{3R})(x)| \leq C\,|f(x)| + \mathcal{M}_{T_\beta,R}f(x)\,.$$

ii) for all $x \in \Omega$,

$$\mathcal{M}_{T_\beta}f(x) \leq CMf(x) + T^*_\beta f(x)\,,$$

where M is the Hardy-Littlewood maximal function on $\mathbb{R}^n$.

Remark 3.4. Although f is a function defined on Ω, it can be viewed as a function on $\mathbb{R}^n$ by extending it by zero outside Ω. With this extension, we consider the expression $Mf(x)$.

Proof. To prove *i)*, consider a cube $R \subset \Omega$. Let x be an interior point of R that is also a Lebesgue point of f and a point of approximate continuity of $T_\beta(f\chi_{3R})$ (see, e.g., [12, p. 58]). This means that for each $\varepsilon > 0$ and for $s > 0$ sufficiently small such that $Q = Q(x,s) \subset \Omega$, the set defined as

$$E_s(x) = \{y \in Q : |T_\beta(f\chi_{3R})(y) - T_\beta(f\chi_{3R})(x)| < \varepsilon\}\,,$$

satisfies $\frac{|E_s(x)|}{|Q|} \to 1$ when $s \to 0$, which implies that there exists s_0 such that if $s < s_0$ we have $|Q| \approx |E_s(x)|$. Moreover, we can choose s_0 such that $3Q = Q(x,3s) \subset R$ whenever $s < s_0$. Then for almost every $y \in E_s(x)$ we estimate

$$\begin{aligned}|T_\beta(f\chi_{3R})(x)| &< |T_\beta(f\chi_{3Q})(y)| + \left|T_\beta(f\chi_{3R\setminus 3Q})(y)\right| + \varepsilon \\ &\leq |T_\beta(f\chi_{3Q})(y)| + \mathcal{M}_{T_\beta,R}f(x) + \varepsilon\,.\end{aligned}$$

Taking the essential infimum over $y \in E_s(x)$ we have

$$|T_\beta(f\chi_{3R})(x)| \leq \operatorname*{ess\,inf}_{y\in E_s(x)} |T_\beta(f\chi_{3Q})(y)| + \mathcal{M}_{T_\beta,R}f(x) + \varepsilon\,.$$

Since the operator T_β is of weak type $(1,1)$, as a consequence of the Kolmogorov's inequality, for $\nu \in (0,1)$ we obtain

$$\frac{1}{|E_s(x)|}\int_{E_s(x)} |T_\beta(f\chi_{3Q})(y)|^\nu\,dy \leq \left(\frac{C}{|E_s(x)|}\int_{3Q}|f(y)|\,dy\right)^\nu\,.$$

From this

$$\operatorname*{ess\,inf}_{y\in E_s(x)} |T_\beta(f\chi_{3Q})(y)| \leq \frac{C}{|E_s(x)|}\int_{3Q}|f(y)|\,dy\,.$$

Since $|E_s(x)| \approx |Q|$ we have

$$|T_\beta(f\chi_{3R})(x)| \leq \frac{C}{|3Q|}\int_{3Q}|f(y)|\,dy + \mathcal{M}_{T_\beta,R}f(x) + \varepsilon\,.$$

Then, by the Lebesgue Differentiation Theorem, first letting $s \to 0$ and then $\varepsilon \to 0$, we obtain that

$$|T_\beta(f\chi_{3R})(x)| \leq C\,|f(x)| + \mathcal{M}_{T_\beta,R}f(x)\,.$$

Thus we prove *i)*.

To prove *ii)*, let $Q = Q(x_0,\ell) \subset \Omega$ and $x, \xi \in Q$. Define $Q_x = Q(x,4\ell)$. It follows immediately that $3Q \subset Q_x$. Given that $\Omega \setminus 3Q = \Omega \setminus Q_x \cup Q_x \setminus 3Q$, we estimate

$$\left|T_\beta(f\chi_{\Omega\setminus 3Q})(\xi)\right| \leq \left|T_\beta(f\chi_{\Omega\setminus Q_x})(\xi) - T_\beta(f\chi_{\Omega\setminus Q_x})(x)\right| \tag{3.5}$$

Preprint

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$$+ \left|T_\beta(f\chi_{Q_x\setminus 3Q})(\xi)\right| + \left|T_\beta(f\chi_{\Omega\setminus Q_x})(x)\right| .$$

Next, we work with each of the terms in the previous inequality separately. To simplify, we will use the notation Q_z^β to denote the cube $Q(z, \beta d(z,\Omega^c))$, for each $z \in \Omega$.

For the first term, note that if $y \in \Omega \setminus Q_x$ then $2\,d(x,\xi) \le d(x,y)$. Then, using the smoothness condition on the kernel and decomposing the domain into annuli, we estimate as follows

$$\begin{aligned}
&\left|T_\beta(f\chi_{\Omega\setminus Q_x})(\xi) - T_\beta(f\chi_{\Omega\setminus Q_x})(x)\right| \\
&\qquad \le \int_{(Q_x^\beta \cup Q_\xi^\beta)\cap\{d(x,y)>4\ell\}} |K(\xi,y) - K(x,y)|\,|f(y)|\,dy \\
&\qquad \le C\ell^\delta \sum_{k=0}^{\infty} \int_{2^{k+1}Q_x\setminus 2^kQ_x} \frac{|f(y)|}{d(x,y)^{n+\delta}} dy \\
&\qquad = C\sum_{k=0}^{\infty} \frac{2^{-k\delta}}{|2^{k+1}Q_x|} \int_{2^{k+1}Q_x} |f(y)|\,dy \\
&\qquad \le CMf(x)\left(\sum_{k=0}^{\infty} 2^{-k\delta}\right) = C\,Mf(x)\,.
\end{aligned}$$

For the second term, we use the size condition on the kernel

$$\left|T_\beta(f\chi_{Q_x\setminus 3Q})(\xi)\right| \le \int_{Q_\xi^\beta\cap Q_x\setminus 3Q} |K(\xi,y)|\,|f(y)|\,dy \le \frac{C}{|Q_x|}\int_{Q_x}|f| \le CMf(x)\,.$$

The third term is simpler, as we note that it can be directly controlled by the operator T_β^*

$$\left|T_\beta(f\chi_{\Omega\setminus Q_x})(x)\right| = \left|\int_{d(x,y)>4\,\ell} K(x,y)f(y)dy\right| \le T_\beta^* f(x)\,.$$

With all these estimates in (3.5) we obtain

$$\left|T_\beta(f\chi_{\Omega\setminus 3Q})(\xi)\right| \le CMf(x) + T_\beta^* f(x)\,,$$

then taking the essential supremum over $\xi \in Q$ followed by the supremum over $Q \subset \Omega$ we conclude

$$\mathcal{M}_{T_\beta} f(x) \le CMf(x) + T_\beta^* f(x)\,,$$

thus proving *ii)* and therefore the lemma. □

As a consequence of the previous lemma, we have the following corollary.

Corollary 3.6. *The truncated grand maximal operators, $\mathcal{M}_{T_\beta}$ and $\mathcal{M}_{T_\beta,R}$, associated with the β-local singular integral T_β are of weak type* $(1,1)$

$$\mathcal{M}_{T_\beta}, \mathcal{M}_{T_\beta,R} : L^1(\Omega) \to L^{1,\infty}(\Omega)\,.$$

More precisely, we have

$$\left|\left\{x\in\Omega : \mathcal{M}_{T_\beta,R}f(x) > \lambda\right\}\right| \le \frac{C}{\lambda}\int_{3R}|f(y)|\,dy\,.$$

Proof. For each $f \in L^1(\Omega)$, by part *ii)* of the previous lemma, the proof is a consequence of the weak $(1,1)$ inequality for the Hardy-Littlewood maximal function M and the operator T_β^*. □

Preprint

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Next, we will see that there exists a sparse family $\mathcal{S}$ and an operator associated with $\mathcal{S}$ that pointwise bounds the β-local singular integral when certain conditions are met.

We proceed to define the notion of sparse family and sparse operator we will work with (see for instance [19]).

Definition 3.7. *We say that a family $\mathcal{S}$ of cubes from $\mathbb{R}^n$ is η-sparse, $0 < \eta < 1$, if for every $Q \in \mathcal{S}$, there exists a measurable set $E(Q) \subset Q$ such that $|E(Q)| \geq \eta |Q|$, and the sets $\{E(Q)\}_{Q\in\mathcal{S}}$ are pairwise disjoint.*

Definition 3.8. *Lets $f \in L^1_{loc}(\Omega)$ and $\mathcal{S}$ be a sparse family. We define the Sparse Operator by*

$$\mathcal{A}_{\mathcal{S}} f(x) = \sum_{Q\in\mathcal{S}} \left(\frac{1}{|Q|} \int_Q f(y)\, dy \right) \chi_Q(x).$$

Next, we present the theorem that provides the pointwise bound. A version in the classical non-local setting can be found in [19].

Theorem 3.9. *Let $0 < \tau < 1$, T_β be a β-local singular integral operator with $\beta < \frac{\tau}{3+\tau}$ and f be a function with support in a cube $R \in \mathcal{F}_\beta$. Then there exists a η-sparse family, $\mathcal{S} \subset \mathcal{F}_\tau$, such that for almost every point $x \in R$ we have*

$$|T_\beta f(x)| \leq C \mathcal{A}_{\mathcal{S}} |f|(x).$$

Proof. First we will prove that there exists a $\frac{1}{2}$-sparse family $\tilde{\mathcal{S}} \subset \mathcal{D}(R)$ such that

$$|T_\beta f(x)| \leq C \sum_{Q\in\tilde{\mathcal{S}}} \left(\frac{1}{|3Q|} \int_{3Q} |f| \right) \chi_Q(x)\,, \tag{3.10}$$

for almost every point $x \in R$. For this purpose the following result will be necessary.

Claim: For each $Q \subset R$, there exists a countable family of disjoint cubes $P_j \in \mathcal{D}(Q)$ such that

$$\sum_j |P_j| \leq \frac{1}{2} |Q|\,,$$

and, furthermore,

$$|T_\beta(f\chi_{3Q})(x)|\, \chi_Q \leq C \left(\frac{1}{|3Q|} \int_{3Q} |f| \right) + \sum_j \left| T_\beta(f\chi_{3P_j})(x) \right| \chi_{P_j}(x)\,,$$

for almost every $x \in Q$. We will call this family of cubes the children of Q and denote it by $\eta(Q)$.

From the previous claim, we can define the family $\tilde{\mathcal{S}}$ inductively. Let $\mathcal{S}_0 = \{R\}$ and then for each $k \in \mathbb{N}_0$ define $\mathcal{S}_{k+1} = \bigcup_{P\in\mathcal{S}_k} \eta(P)$. Furthermore, define $\mathcal{S}^k = \bigcup_{j=0}^k \mathcal{S}_j$, and finally $\tilde{\mathcal{S}} = \bigcup_{k\in\mathbb{N}_0} \mathcal{S}^k$.

By the construction, after k iterations, we have

$$|T_\beta(f\chi_{3R})(x)|\, \chi_R(x) \leq C \sum_{Q\in\mathcal{S}^k} \left(\frac{1}{|3Q|} \int_{3Q} |f| \right) \chi_Q(x) + \sum_{Q\in\mathcal{S}_{k+1}} |T_\beta(f\chi_{3Q})(x)|\, \chi_Q(x).$$

Preprint

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Since $\mathcal{S}^k \subset \tilde{\mathcal{S}}$ for each $k \in \mathbb{N}_0$, we obtain

$$|T_\beta(f\chi_{3R})(x)|\,\chi_R(x) \leq C\sum_{Q\in\tilde{\mathcal{S}}}\left(\frac{1}{|3Q|}\int_{3Q}|f|\right)\chi_Q(x) + \sum_{Q\in\mathcal{S}_{k+1}}|T_\beta(f\chi_{3Q})(x)|\,\chi_Q(x).$$

Consequently

$$\begin{aligned}|T_\beta(f\chi_{3R})(x)|\,\chi_R(x) \leq C\sum_{Q\in\tilde{\mathcal{S}}}\left(\frac{1}{|3Q|}\int_{3Q}|f|\right)\chi_Q(x)\\ + \limsup_{k\to\infty}\sum_{Q\in\mathcal{S}_k}|T_\beta(f\chi_{3Q})(x)|\,\chi_Q(x)\,.\end{aligned}$$

From the properties of $\mathcal{S}_k$, it is easy to see that for almost every point $x \in R$ we have

$$\limsup_{k\to\infty}\sum_{Q\in\mathcal{S}_k}|T_\beta(f\chi_{3Q})(x)|\,\chi_Q(x) = 0.$$

In fact, by the claim we can obtain that $|R| \geq 2^k\sum_{Q\in\mathcal{S}_k}|Q|$. Now, if we define the set

$$A = \left\{x \in R : \limsup_{k\to\infty}\sum_{Q\in\mathcal{S}_k}|T_\beta(f\chi_{\sigma Q})(x)|\,\chi_Q(x) \neq 0\right\},$$

it is obvious that, for each $k \in \mathbb{N}$, $A \subset \bigcup_{Q\in\mathcal{S}_k} Q$ and then

$$|A| \leq \sum_{Q\in\mathcal{S}_k}|Q| \leq 2^{-k}\,|R|\,,$$

from which we conclude that $|A| = 0$.

With all this we obtain that, for almost every $x \in R$,

$$|T_\beta f(x)| \leq C\sum_{Q\in\tilde{\mathcal{S}}}\left(\frac{1}{|3Q|}\int_{3Q}|f|\right)\chi_Q(x). \tag{3.11}$$

Note now that $\tilde{\mathcal{S}} \subset \mathcal{D}(R)$ and that, furthermore, it is a $\frac{1}{2}$-sparse family. Indeed, from the construction it is easy to see that $\mathcal{S}_k \subset \mathcal{D}(R)$, and hence $\tilde{\mathcal{S}} \subset \mathcal{D}(R)$.

On the other hand, if we define for each $Q \in \tilde{\mathcal{S}}$ the set

$$E_Q = Q \setminus \bigcup_{P\in\eta(Q)} P,$$

it is evident that $E_Q \subset Q$, the sets E_Q are mutually disjoint, and by the same construction $|E_Q| \geq \frac{1}{2}\,|Q|$. Thus we conclude that $\tilde{\mathcal{S}}$ is a $\frac{1}{2}$-sparse family.

Now let $\mathcal{S} = \left\{3Q : Q \in \tilde{\mathcal{S}}\right\}$. It turns out to be a $\frac{1}{2\,3^n}$-sparse family. To see this, consider $P \in \mathcal{S}$, then $P = 3Q$ for some $Q \in \tilde{\mathcal{S}}$. Defining $E_P = E_Q$, we have, inherited directly from $\tilde{\mathcal{S}}$, that the E_P are mutually disjoint and furthermore

$$|E_P| = |E_Q| \geq \frac{1}{2}\,|Q| = \frac{3^n}{2\,3^n}\,|Q| = \frac{1}{2\,3^n}\,|3Q| = \frac{1}{2\,3^n}\,|P|\,.$$

Also note that for $Q \in \tilde{\mathcal{S}} \subset \mathcal{D}(R)$, we have in particular that $Q \subset R \in \mathcal{F}_\beta$ with $\beta < \frac{\tau}{3+\tau}$ and by Lemma 2.4 applied to $\tau/3$, we obtain that $Q \in \mathcal{F}_{\tau/3}$. Then for $P \in \mathcal{S}$, since $P = 3Q$, it is immediate that $P \in \mathcal{F}_\tau$, hence $\mathcal{S} \subset \mathcal{F}_\tau$.

Preprint

IMAL PREPRINT # 2026-0085
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From (3.11) it follows that

$$|T_\beta f(x)| \leq C \sum_{P\in\mathcal{S}} \left(\frac{1}{|P|}\int_P |f(y)|\,dy\right)\chi_P(x) = C\mathcal{A}_{\mathcal{S}}\,|f|\,(x),$$

for almost every $x \in R$, with $\mathcal{S} \subset \mathcal{F}_\tau$, as we wanted to prove.

It remains only to prove the claim, but to do so we can proceed as in [19, Theorem 3.1], where we need the localized grand maximal operator $\mathcal{M}_{T_\beta,R}$ to be of weak type $(1,1)$, which we proved in Corollary 3.6, and also the pointwise estimate proved in Lemma 3.3. We omit the proof.

□

4. Proof of Theorem 1.13

In this section we prove the main result. Since the key to the procedure focuses on reducing the problem to suitable contexts, for the final proof we require the following two technical lemmas, the proofs of them can be found in [27].

Lemma 4.1. *Let Ω be an open proper subset of $\mathbb{R}^n$. Given $0 < \beta < 1$, for each $t \in \mathbb{N}$ such that $2^{-t} \leq \beta/5$, there exists a covering $\mathcal{W}_t$ of Ω by disjoint dyadic cubes belonging to $\mathcal{F}_\beta$ and satisfying the following properties*

i) If $R = R(x_R, l_R) \in \mathcal{W}_t$, then $10\,R \in \mathcal{F}_\beta$ and

$$2^{-t-3}\,d(x_R,\Omega^c) \;\leq\; l_R \;\leq\; 2^{-t-1}\,d(x_R,\Omega^c)\,.$$

ii) There is a number M, only depending on β and t, such that for any cube $Q_0 = Q(x_0, l_0) \in \mathcal{F}_\beta$ with $10Q_0 \notin \mathcal{F}_\beta$, the cardinal of the set

$$\mathcal{W}_t(Q_0) = \{R \in \mathcal{W}_t \,:\, R \cap \mathcal{N}_\beta(Q_0) \neq \emptyset\}\,,$$

is at most M. We will call the union of this cubes as

$$\mathcal{W}_{t,Q_0} = \bigcup_{R\in\mathcal{W}_t(Q_0)} R\,.$$

Lemma 4.2. *Let $\{Q_i\}$ be a pairwise disjoint collection of Whitney type cubes. Then their clouds have bounded overlapping. More precisely, for every x in Ω there exists a natural number N_0 such that*

$$\sum_i \chi_{\mathcal{N}_\beta(Q_i)}(x) \leq N_0\,.$$

Proposition 4.3. *Let $0 < \tau < 1$ and let T_β be a β-local singular integral operator with $\beta < \frac{\tau}{3+\tau}$. For $1 < p \leq q < \infty$, consider weights u and v such that $(u,v) \in A^{\mathcal{T}}_{p,\psi,q,\phi}$, where ϕ and ψ are Young functions such that $\overline{\phi} \in B_{q'}$ and $\overline{\psi} \in B_p$. Then for every function f supported on a cube $R \in \mathcal{F}_\beta$ we have*

$$\left(\int_R |T_\beta f(y)|^q u(y)\,dy\right)^{1/q} \leq C\left(\int_R |f(y)|^p v(y)\,dy\right)^{1/p}.$$

Preprint

IMAL PREPRINT # 2026-0085
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Proof. Let $R \in \mathcal{F}_\beta$ and f be a function supported on R. Since we are under the hypotheses of Theorem 3.9, there exists a sparse family $\mathcal{S} \subset \mathcal{F}_\tau$ such that

$$|T_\beta f(x)| \leq C\mathcal{A}_{\mathcal{S}}\,|f|\,(x), \tag{4.4}$$

for almost every $x \in R$.

We will prove the result for the operator $\mathcal{A}_{\mathcal{S}}$ using a duality argument. Let $g \in L^{q'}(\Omega, u)$ with $\|g\|_{L^{q'}(\Omega,u)} = 1$. With the generalized Hölder inequality in Orlicz spaces and the condition on the weights, we have

$$\begin{aligned}
\int_\Omega \left(\mathcal{A}_{\mathcal{S}}\,|f|\,\right) g\,u &= \sum_{Q\in\mathcal{S}} |Q|^{-1} \left(\int_Q f\right)\left(\int_Q g\,u\right) \\
&\leq C \sum_{Q\in\mathcal{S}} |Q| \left(\left\|fv^{\frac{1}{p}}\right\|_{\overline{\psi},Q} \left\|v^{-\frac{1}{p}}\right\|_{\psi,Q}\right)\left(\left\|gu^{\frac{1}{q'}}\right\|_{\overline{\phi},Q} \left\|u^{\frac{1}{q}}\right\|_{\phi,Q}\right) \\
&= C \sum_{Q\in\mathcal{S}} \left(|Q|^{\frac{1}{q}-\frac{1}{p}} \left\|u^{\frac{1}{q}}\right\|_{\phi,Q} \left\|v^{-\frac{1}{p}}\right\|_{\psi,Q}\right) \left\|fv^{\frac{1}{p}}\right\|_{\overline{\psi},Q} \left\|gu^{\frac{1}{q'}}\right\|_{\overline{\phi},Q} |Q|^{\frac{1}{p}+\frac{1}{q'}} \\
&\leq C \sum_{Q\in\mathcal{S}} \left(\left\|fv^{\frac{1}{p}}\right\|_{\overline{\psi},Q} |E_Q|^{\frac{1}{p}}\right)\left(\left\|gu^{\frac{1}{q'}}\right\|_{\overline{\phi},Q} |E_Q|^{\frac{1}{q'}}\right),
\end{aligned}$$

where in the last inequality we used a property of sparse families, namely $|Q| \approx |E_Q|$. Then, applying the discrete Hölder inequality and since $p' > q'$, we can continue the estimate as follows

$$\begin{aligned}
\int_\Omega \left(\mathcal{A}_{\mathcal{S}}\,|f|\,\right) g\,u &\leq C \left(\sum_Q \left\|fv^{\frac{1}{p}}\right\|^p_{\overline{\psi},Q} |E_Q|\right)^{\frac{1}{p}} \left(\sum_Q \left\|gu^{\frac{1}{q'}}\right\|^{p'}_{\overline{\phi},Q} |E_Q|^{\frac{p'}{q'}}\right)^{\frac{1}{p'}} \\
&\leq C \left(\sum_Q \left\|fv^{\frac{1}{p}}\right\|^p_{\overline{\psi},Q} |E_Q|\right)^{\frac{1}{p}} \left(\sum_Q \left\|gu^{\frac{1}{q'}}\right\|^{q'}_{\overline{\phi},Q} |E_Q|\right)^{\frac{1}{q'}} \\
&= C \left(\sum_Q \int_{E_Q} \left\|fv^{\frac{1}{p}}\right\|^p_{\overline{\psi},Q}\,dx\right)^{\frac{1}{p}} \left(\sum_Q \int_{E_Q} \left\|gu^{\frac{1}{q'}}\right\|^{q'}_{\overline{\phi},Q}\,dx\right)^{\frac{1}{q'}}.
\end{aligned} \tag{4.5}$$

We will estimate each sum separately. Bearing in mind that the sets E_Q are disjoints, and that $\overline{\phi} \in B_{q'}$ and $\overline{\psi} \in B_p$ imply the maximal functions $M_{\overline{\phi}}$ and $M_{\overline{\psi}}$ are bounded on $L^{q'}$ and L^p, respectively, we can estimate as follows

$$\begin{aligned}
\left(\sum_Q \int_{E_Q} \left\|fv^{\frac{1}{p}}\right\|^p_{\overline{\psi},Q}\,dx\right)^{\frac{1}{p}} &\leq \left(\int_\Omega \left(M_{\overline{\psi}}\left(fv^{\frac{1}{p}}\right)(x)\right)^p\,dx\right)^{\frac{1}{p}} \\
&\leq \left\|M_{\overline{\psi}}\left(fv^{\frac{1}{p}}\right)\right\|_{L^p(\mathbb{R}^n)} \\
&\leq C \left\|fv^{\frac{1}{p}}\right\|_{L^p(\mathbb{R}^n)} = C\,\|f\|_{L^p(\Omega,v)}\,.
\end{aligned}$$

Preprint

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Analogously, since $\|g\|_{L^{q'}(\Omega,u)} = 1$, we have

$$\left(\sum_Q \int_{E_Q} \left\|gu^{\frac{1}{q'}}\right\|_{\overline{\phi},Q}^{q'} dx\right)^{\frac{1}{q'}} \leq \left\|M_{\overline{\phi}}\left(gu^{\frac{1}{q'}}\right)\right\|_{L^{q'}(\mathbb{R}^n)} \leq C\,\|g\|_{L^{q'}(\Omega,u)} = C\,.$$

Therefore, returning to (4.5), we obtain

$$\int_\Omega \left(\mathcal{A}_{\mathcal{S}}\,|f|\,\right) g\,u \leq C\,\|f\|_{L^p(\Omega,v)}\,.$$

Thus, taking the supremum over all functions $g \in L^{q'}(\Omega,u)$, with norm equal to one, we get

$$\|\mathcal{A}_{\mathcal{S}}\,|f|\|_{L^q(\Omega,u)} \leq C\,\|f\|_{L^p(\Omega,v)}\,.$$

Then, by the pointwise domination (4.4) and taking into account the support of f, we have

$$\left(\int_R |T_\beta f|^q\,u\right)^{1/q} \leq C\,\|\mathcal{A}_{\mathcal{S}}\,|f|\|_{L^q(\Omega,u)} \leq C\left(\int_R |f|^p\,v\right)^{1/p},$$

which is what we wanted to prove. □

The following result establishes that, provided the locality parameter β is sufficiently small, the β-local singular integral operator is two-weight bounded.

Proposition 4.6. *Let $0 < \tau < 1$, $1 < p \leq q < \infty$, and let u and v be weights such that $(u,v) \in A^\tau_{p,\psi,q,\phi}$, where ϕ and ψ are Young functions with $\overline{\phi} \in B_{q'}$ and $\overline{\psi} \in B_p$. Then, if T_β is a β-local singular integral operator with $\beta < \frac{1}{5}\frac{\tau}{3+\tau}$, we have*

$$T_\beta : L^p(\Omega,v) \to L^q(\Omega,u).$$

Proof. Let $f \in L^p(\Omega,v)$ and, for $\beta < \frac{1}{5}\frac{\tau}{3+\tau}$, consider the Whitney decomposition $\{R_j\}$ of Ω given by Lemma 4.1. Note that for $x \in R_j$, by Remark 1.7, we have

$$T_\beta f(x) = T_\beta f_j(x),$$

where $f_j = f\chi_{\mathcal{N}_\beta(R_j)}$.

Now, from Lemma 2.7, with $\eta = \frac{\tau}{3+\tau}$, we have $\mathcal{N}_\beta(R_j) \subset \tilde{R}_j \in \mathcal{F}_{\tilde{\beta}}$, where $\tilde{\beta} < \frac{\tau}{3+\tau}$, So, we have that $\operatorname{supp} f_j \subset \tilde{R}_j$. Then, by Proposition 4.3 and Remark 1.5,

$$\int_{R_j} |T_\beta f|^q\,u = \int_{R_j} |T_\beta f_j|^q\,u \leq \int_{\tilde{R}_j} \left|T_{\tilde{\beta}} f_j\right|^q u \leq C\Big(\int_{\mathcal{N}_\beta(R_j)} |f|^p\,v\Big)^{q/p}.$$

Finally, by using the Whitney decomposition and the fact that $p \leq q$, we estimate

$$\begin{aligned}\int_\Omega |T_\beta f|^q\,u &= \sum_j \int_{R_j} |T_\beta f_j|^q\,u \\ &\leq C\sum_j \left(\int_{\mathcal{N}_\beta(R_j)} |f|^p\,v\right)^{q/p} \\ &\leq C\left(\int_\Omega |f|^p \left(\sum_j \chi_{\mathcal{N}_\beta(R_j)}\right) v\right)^{q/p} \leq C\,\|f\|^q_{L^p(\Omega,v)}\,,\end{aligned}$$

Preprint

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where in the last inequality we apply Lemma 4.2. Therefore we conclude

$$\|T_\beta f\|_{L^q(\Omega,u)} \leq C \|f\|_{L^p(\Omega,v)},$$

as we wanted to prove. □

Now we are in a position to prove our main result.

Proof of Theorem 1.13. Let $f \in L^p(\Omega, v)$ and $\beta < \tau$. If $\beta < \frac{1}{5}\frac{\tau}{3+\tau}$, then the result follows directly from Proposition 4.6. On the other hand, we define $\theta = \frac{1}{10}\frac{\tau}{3+\tau}$ and apply the Proposition 1.17 to split the operator as follows

$$|T_\beta f(x)| \leq |T_\theta f(x)| + C M_\beta f(x).$$

Since $A^\tau_{p,\psi,q,\phi} \subset A^\tau_{p,\psi,q}$ (see Lemma 1.11) the result follows by applying Proposition 4.6 and Theorem 1.9 in [25]. That is

$$\|T_\beta f\|_{L^q(\Omega,u)} \leq C \left(\|T_\theta f\|_{L^q(\Omega,u)} + \|M_\beta f\|_{L^q(\Omega,u)} \right) \leq C \|f\|_{L^p(\Omega,v)},$$

which completes the proof. □

5. Proof of Theorem 1.15

In order to prove the theorem we take into account that the technique involves the construction of a sparse operator. The key tool for ensuring its existence is a weak-type bound for a truncated maximal operator, which we will define below. First, however, we will present a version of Theorem 4.1 in [14] for operators of type H_s.

Proposition 5.1. *Let $\mathcal{T}_\beta$ be a β-local singular integral operator of type $\mathcal{H}_s$, then for each $\theta \in (0, \beta)$ there exists a θ-local singular integral operator $\mathcal{T}_\theta$ of type $\mathcal{H}_s$ such that for each x the inequality*

$$|\mathcal{T}_\beta f(x)| \leq |\mathcal{T}_\theta f(x)| + C M^{s'}_\beta f(x),$$

holds.

Proof. Given $\theta \in (0, \beta)$ we consider a smooth cut-off function η and the auxiliary operator $\mathcal{T}_0$ formally defined as in (1.16) with $\alpha = \theta/2$.

Let us consider $k_0 \in \mathbb{N}$ such that $2^{k_0-1}\theta < \beta \leq 2^{k_0}\theta$ and denote $Q^\beta_x = Q(x, \beta d(x, \Omega^c))$ and $Q^\theta_x = Q(x, \theta d(x, \Omega^c))$. Taking into account the support of η and the size condition on the kernel we estimate as follows

$$\begin{aligned}
|\mathcal{T}_0 f(x)| &\leq \int_{\frac{\theta}{2} d(x,\Omega^c) \leq d(x,y) \leq \beta d(x,\Omega^c)} |K(x,y)| \left| 1 - \eta\left(\frac{d(x,y)}{\frac{\theta}{2} d(x,\Omega^c)} \right) \right| |f(y)| \, dy \\
&\leq C \sum_{j=0}^{k_0} \int_{2^j Q^\theta_x \setminus 2^{j-1} Q^\theta_x} |K(x,y)| \, |f(y)| \, \chi_{Q^\beta_x} \, dy \\
&\leq C \sum_{j=0}^{k_0} \left(\int_{2^j Q^\theta_x \setminus 2^{j-1} Q^\theta_x} |K(x,y)|^s \, dy \right)^{\frac{1}{s}} \left(\int_{2^j Q^\theta_x \setminus 2^{j-1} Q^\theta_x} |f(y)|^{s'} \chi_{Q^\beta_x} \, dy \right)^{\frac{1}{s'}}
\end{aligned}$$

Preprint

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$$\leq C\sum_{j=0}^{k_0}\frac{1}{(2^{j-1}\theta d(x,\Omega^c))^{\frac{n}{s'}}}\left(\int_{2^jQ_x^\theta}|f(y)|^{s'}\chi_{Q_x^\beta}\,dy\right)^{\frac{1}{s'}}$$
$$\leq CM_\beta^{s'}f(x)\,,$$

where C only depends on θ y β. Now, by Remark 5.5, the operator $\mathcal{T}_0$ is of weak type(s',s'), and so it is $\mathcal{T}_\beta$ by hypothesis. Consequently, the operator $\mathcal{T}_\theta=\mathcal{T}_\beta-\mathcal{T}_0$ is also of weak type (s',s') and for each $x\notin\operatorname{supp}f$, $\mathcal{T}_\theta$ can be written as

$$T_\theta f(x)=\int_\Omega K(x,y)\,\eta\Big(\frac{d(x,y)}{\frac{\theta}{2}d(x,\Omega^c)}\Big)f(y)dy\,.$$

We claim that $\mathcal{T}_\theta$ is a θ-local singular integral operator of type $\mathcal{H}_s$ whose kernel is

$$K_\theta(x,y)=K(x,y)\,\eta\Big(\frac{d(x,y)}{\frac{\theta}{2}d(x,\Omega^c)}\Big)\,.$$

It is clear that $\mathcal{T}_\theta f(x)=0$, for every x such that $(\operatorname{supp}f)\cap Q_x^\beta=\emptyset$, as a consequence of the support of η. Only remains to prove the item *iii)* of Definition 1.14. The condition *a)* is trivial using $0\leq\eta\leq1$. For *b)* let $R>0$ and $x,x'\in\Omega$ such that $d(x,x')<r<\frac{R}{2}$. Thus

$$\begin{aligned}&\left(\frac{1}{R^n}\int_{R<d(x',y)<2R}|K_\theta(x,y)-K_\theta(x',y)|^s\,dy\right)^{\frac{1}{s}}\\&\leq\left(\frac{1}{R^n}\int_{R<d(x',y)<2R}|K(x,y)-K(x',y)|^s\,\eta\left(\frac{d(x,y)}{\frac{\theta}{2}d(x,\Omega^c)}\right)^s\,dy\right)^{\frac{1}{s}}\\&\quad+\left(\frac{1}{R^n}\int_{R<d(x',y)<2R}|K(x',y)|^s\left|\eta\left(\frac{d(x,y)}{\frac{\theta}{2}d(x,\Omega^c)}\right)-\eta\left(\frac{d(x',y)}{\frac{\theta}{2}d(x',\Omega^c)}\right)\right|^s\,dy\right)^{\frac{1}{s}}\\&=I+II\,.\end{aligned}\tag{5.2}$$

For I, we simply apply the Hörmander type condition for K and the fact that $0\leq\eta\leq1$ and the desired inequality is obtained. More precisely

$$I\leq\left(\frac{1}{R^n}\int_{R<d(x',y)<2R}|K(x,y)-K(x',y)|^s\,dy\right)^{\frac{1}{s}}\leq\frac{C}{R^n}\left(\frac{r}{R}\right)^\delta\,,$$

where δ is given in the definition of $\mathcal{T}_\beta$.

For the second integral, identically to what was done in the proof of Theorem 4.1 in [14] (see equation (4.1) in that paper) we get

$$\left|\eta\Big(\frac{d(x,y)}{\frac{\theta}{2}d(x,\Omega^c)}\Big)-\eta\Big(\frac{d(x',y)}{\frac{\theta}{2}d(x',\Omega^c)}\Big)\right|\leq C\frac{\beta+1}{\theta}\,\frac{d(x,x')}{d(x,\Omega^c)}\,.$$

We note that every y such that $R<d(x',y)<2R$ verifies that $2\,d(x,x')<d(x',y)$. In order to estimate the last fraction of the equation above, taking into account the support of η, it suffices to consider at least one of the following cases.

- $d(x,y)\leq\theta\,d(x,\Omega^c)$ which implies that $d(x',y)\leq2\theta\,d(x,\Omega^c)$.
- $d(x',y)\leq\theta\,d(x',\Omega^c)$ which implies that $d(x',y)\leq\frac{2\theta}{2-\theta}\,d(x,\Omega^c)$.

Preprint

IMAL PREPRINT # 2026-0085
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Thus

$$\frac{d(x',x)}{d(x,\Omega^c)} \leq C\frac{d(x',x)}{d(x',y)} \leq C\,\frac{r}{R}\,.$$

So, by the conditions *a)* we have

$$\begin{aligned} II &= \left(\frac{1}{R^n}\int_{R<d(x',y)<2R} |K(x',y)|^s \left|\eta\Big(\frac{d(x,y)}{\frac{\theta}{2}d(x,\Omega^c)}\Big) - \eta\Big(\frac{d(x',y)}{\frac{\theta}{2}d(x',\Omega^c)}\Big)\right|^s dy\right)^{\frac{1}{s}} \\ &\leq C\Big(\frac{1}{R^n}\int_{R<d(x',y)<2R} |K(x',y)|^s\,dy\Big)^{\frac{1}{s}} \left(\frac{r}{R}\right) \leq \frac{C}{R^n}\left(\frac{r}{R}\right)\,. \end{aligned}$$

From our estimate of I and II, since $r<R$ we have

$$\left(\frac{1}{R^n}\int_{R<d(x',y)<2R} |K_\theta(x,y)-K_\theta(x',y)|^s\,dy\right)^{\frac{1}{s}} \leq \frac{C}{R^n}\left(\frac{r}{R}\right)^{\tilde{\delta}}\,,$$

where $\tilde{\delta} = \min\{1,\delta\}$. We thus conclude that T_θ is a θ-local singular integral operator of type $\mathcal{H}_s$, and furthermore

$$|T_\beta f(x)| \leq |T_\beta f(x) - T_0 f(x)| + |T_0 f(x)| \leq |T_\theta f(x)| + CM_\beta^{s'} f(x)\,,$$

which is what we wanted to prove. □

Definition 5.3. *Let $1 \leq r < \infty$ and $f \in L^1_{loc}(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ we define the r-maximal operator as*

$$M^r f(x) = \sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q |f|^r\right)^{\frac{1}{r}}\chi_Q(x)\,.$$

This operator have a local version, with $\Omega \subset \mathbb{R}^n$ as usually, we mention now.

Definition 5.4. *Let $1 \leq r < \infty$, $0<\beta<1$ and $f \in L^1_{loc}(\Omega)$. For $x \in \Omega$ we define the r-local maximal operator as*

$$M^r_\beta f(x) = \sup_{Q\in\mathcal{F}_\beta}\left(\frac{1}{|Q|}\int_Q |f|^r\right)^{\frac{1}{r}}\chi_Q(x)\,.$$

Remark 5.5. We note that $M^r_\beta f = (M_\beta(|f|^r))^{1/r}$ and $M^r f = (M(|f|^r))^{1/r}$. Then M^r_β y M^r are of weak type (r,r) and strong type (p,p), for $r<p<\infty$, over Ω and $\mathbb{R}^n$ respectively, because M_β and M are of weak type $(1,1)$ and strong type (p,p), for $1<p<\infty$, respectively.

The following result follows directly from Kolmogorov's inequality.

Lemma 5.6. *Given an operator S of weak type (p,p), with $1 \leq p < \infty$, ν such that $0<\nu<p$ and a subset E of finite measure, then*

$$\int_E |Sf(y)|^\nu\,dy \leq C\,|E|^{1-\frac{\nu}{p}}\,\|f\|_p^\nu\,.$$

for some constant C, depending only on ν and p.

Now, an analogous version of Lemma 2.1 can we obtain, taking into account the hypotheses for an operator of type H^s. The proof follows the lines of that lemma.

Preprint

IMAL PREPRINT # 2026-0085
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Lemma 5.7. *Let $\sigma \in (0,1]$, if $\mathcal{T}_\beta$ is a β-local singular integral operator of type $\mathcal{H}_s$, with $1 < s < \infty$ and $\beta < \frac{\sigma}{3}$, then for any $\nu \in (0, s')$ and $f \in L^\infty_c$, there exists $\xi \in (\beta, \sigma)$ such that*

$$T^*_\beta f(x) \leq C_\nu \left(M_\beta(|T_\beta|^\nu)(x)^{1/\nu} + M^{s'}_\xi f(x) \right).$$

Proposition 5.8. *Let $\mathcal{T}_\beta$ as above. Then, the truncated maximal operator T^*_β is of strong type (p,p), $p > s'$, and weak type (s', s').*

Proof. It is sufficient to prove a weak type inequality for sufficiently small β and for a function f supported on a cube, analogous to Proposition 2.9. Then, to prove the desired result for a restricted range of values β, as it was done in Proposition 2.14. Finally, we proceed as in the proof of Theorem 2.16. □

Now, with the aim of creating a sparse operator that dominates $\mathcal{T}_\beta$, we consider a slightly different version of the truncated grand maximal operators given in Definitions 3.1 and 3.2. That is, if $\Omega \subset \mathbb{R}^n$ and $\beta \in (0,1)$, the truncated grand maximal operator of $\mathcal{T}_\beta$ applied to $f \in L^1(\Omega)$ is given by

$$\mathcal{M}_{\mathcal{T}_\beta} f(x) = \sup_{Q \subset \Omega} \operatorname*{ess\,sup}_{\xi \in Q} \left| \mathcal{T}_\beta(f \chi_{\Omega \setminus 5Q})(\xi) \right| \chi_Q(x).$$

for each $x \in \Omega$ and the localized truncated grand maximal operator of $\mathcal{T}_\beta$ over a cube $R \subset \Omega$ applied to f is given by

$$\mathcal{M}_{\mathcal{T}_\beta, R} f(x) = \sup_{Q \subset R} \operatorname*{ess\,sup}_{\xi \in Q} |\mathcal{T}_\beta(f \chi_{5R \setminus 5Q})(\xi)| \chi_Q(x).$$

for every $x \in R \subset \Omega$.

Applying a reasoning analogous to that employed in the proof of Proposition 3.3 we can get

$$|\mathcal{T}_\beta(f \chi_{5R})(x)| \leq C\,|f(x)| + \mathcal{M}_{\mathcal{T}_\beta, R} f(x),$$

for a.e. $x \in R \subset \Omega$ and

$$\mathcal{M}_{\mathcal{T}_\beta} f(x) \leq C M^{s'} f(x) + \mathcal{T}^*_\beta f(x),$$

for every $x \in \Omega$.

Based on the weak boundedness properties of $M^{s'}$ (Remark 5.5) and T^*_β (Proposition 5.8), these estimates allow us to prove the following version of Corollary 3.6, the proof of which follows the same lines and we omit.

Corollary 5.9. *The operators $\mathcal{M}_{\mathcal{T}_\beta}$ y $\mathcal{M}_{\mathcal{T}_\beta, R}$ associated to a $\mathcal{T}_\beta$ of type $\mathcal{H}_s$, are of weak type (s', s'). More precisely we have that*

$$\left| \left\{ x \in \Omega : \mathcal{M}_{\mathcal{T}_\beta, R} f(x) > \lambda \right\} \right| \leq \frac{C}{\lambda^{s'}} \int_{5R} |f(y)|^{s'} \, dy.$$

Next, we define a class of sparse operators that is slightly more general than the one seen in Section 3 and that pointwisely bounds the operators analyzed here.

Definition 5.10. *Let $f \in L^1_{loc}(\Omega)$, $\mathcal{S}$ be a sparse family and $1 \leq r < \infty$, we define the r-average sparse operator as*

$$\mathcal{A}_{r,\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q f(y)^r \, dy \right)^{\frac{1}{r}} \chi_Q(x).$$

Preprint

IMAL PREPRINT # 2026-0085
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Theorem 5.11. *Let $0 < \tau < 1$, $1 < s < \infty$, $\mathcal{T}_\beta$ be a β-local singular integral operator of type $\mathcal{H}_s$, with $\beta < \frac{\tau}{5+\tau}$, and f be a function with support in a cube $R \in \mathcal{F}_\beta$. Then there exists a sparse family, $\mathcal{S} \subset \mathcal{F}\tau$, such that for a.e. $x \in R$ we have*

$$|\mathcal{T}_\beta f(x)| \leq C\mathcal{A}_{s',\mathcal{S}} |f|(x).$$

We omit the proof, as it is similar to that of Theorem 3.9, with the obvious changes adapted to the new results. In the following result, we will denote by $\varphi_r(t)$ the composition of φ with the power t^r, with $1 < r < \infty$.

Proposition 5.12. *Let $0 < \tau < 1$, $1 < s < \infty$ and $\mathcal{T}_\beta$ be a β-local singular integral operator of type $\mathcal{H}_s$, with $\beta < \frac{\tau}{5+\tau}$. For $s' < p \leq q < \infty$, consider weights u and v such that $(u,v) \in A^\tau_{\frac{p}{s'},\psi,\frac{q}{s'},\phi}(\Omega)$, where ϕ and ψ are Young functions such that $\overline{\phi_{s'}} \in B_{q'}$ and $\overline{\psi}_{s'} \in B_p$. Then for every function f supported on a cube $R \in \mathcal{F}_\beta$ we have*

$$\left(\int_R |\mathcal{T}_\beta f|^q u\right)^{1/q} \leq C\left(\int_R |f|^p v\right)^{1/p}.$$

Proof. In view of Theorem 5.11 it is sufficient to prove the result for $\mathcal{A}_{s',\mathcal{S}}$. The proof can be obtained by following the same lines as those used in the proof of Theorem 4.3, considering that

$$\left\|u^{\frac{1}{q}}\right\|^{s'}_{\phi_{s'},Q} = \left\|u^{\frac{s'}{q}}\right\|_{\phi,Q},$$

and taking the s'-averages involved in the definitions. □

With this we obtain the analogous to Proposition 4.6.

Proposition 5.13. *Let $0 < \tau < 1$, $1 < s < \infty$, $s' < p \leq q < \infty$ and let u, v be weights such that $(u,v) \in A^\tau_{\frac{p}{s'},\psi,\frac{q}{s'},\phi}(\Omega)$, where ϕ and ψ are Young functions with $\overline{\phi_{s'}} \in B_{q'}$ and $\overline{\psi}_{s'} \in B_p$. Then, for $\mathcal{T}_\beta$ as above with $\beta < \frac{1}{5}\frac{\tau}{5+\tau}$ we have*

$$\mathcal{T}_\beta : L^p(\Omega, v) \to L^q(\Omega, u).$$

Proof of Theorem 1.15. The argument of the proof mirrors the proof of Theorem 1.13, using Proposition 5.13 instead of Proposition 4.6 for small values β, and Proposition 5.1 instead of Proposition 1.17. At this point, we have that for each $x \in \Omega$, the following holds

$$|T_\beta f(x)| \leq |T_\theta f(x)| + CM^{s'}_\beta f(x),$$

where θ is chosen to be less than $\frac{1}{5}\frac{\tau}{5+\tau}$. Therefore, using Proposition 5.13 again, it follows that

$$\|T_\theta f\|_{L^q(\Omega,u)} \leq C\|f\|_{L^p(\Omega,v)}.$$

On the other hand, it is easy to see that $(u,v) \in A^\tau_{\frac{p}{s'},\psi,\frac{q}{s'}}(\Omega)$. Then, since $M^{s'}_\beta f(x) = [M_\beta(f^{s'})(x)]^{\frac{1}{s'}}$ and using Theorem 1.9 in [25] we get that

$$\left\|M^{s'}_\beta f\right\|_{L^q(\Omega,u)} = \left\|M_\beta(f^{s'})\right\|^{\frac{1}{s'}}_{L^{q/s'}(\Omega,u)} \leq C\left\|f^{s'}\right\|^{\frac{1}{s'}}_{L^{p/s'}(\Omega,v)} = C\|f\|_{L^p(\Omega,v)},$$

and this proves the theorem. □

Preprint

IMAL PREPRINT # 2026-0085
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6. Applications

In this section, we prove results concerning to classical Calderón-Zygmund operators restricted to Ω, a proper open subset of $\mathbb{R}^n$. Since the triplet (Ω, d_∞, dx) may not necessarily be a space of homogeneous type, we cannot assert that the restricted operator is bounded between two weighted spaces. However, we can prove a two-weight bounded result for a β-localization of these operators.

6.1. **Standard conditions on the kernel.** Let T be a Calderón-Zygmund operator (C-Z) on $\mathbb{R}^n$, that is, a bounded operator on $L^2(\mathbb{R}^n)$ for which there also exists a kernel $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$ such that, for every bounded function f with compact support, the operator is given by the expression

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy \,,$$

whenever $x \notin \operatorname{supp} f$ and its kernel satisfies the following conditions:

(i) $|K(x,y)| \leq \dfrac{C}{|x-y|^n}$,

(ii) $|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C \dfrac{|x-x'|^\delta}{|x-y|^{n+\delta}}$,
for some $\delta > 0$ and whenever $2|x-x'| \leq |x-y|$.

Remark 6.1. The conditions on the kernel can be reformulated in terms of the uniform metric $d(x,y) = \|x-y\|_\infty$ and it is easy to see that both formulations are equivalent.

Given an open proper subset $\Omega \subset \mathbb{R}^n$ and $\beta \in (0,1)$ we introduce now the β-localization of a C-Z operator T as

$$T_{\beta\text{-loc}} f(x) = T\left(f \chi_{Q(x,\beta d(x,\Omega^c))}\right) \,,$$

where $d = d_\infty$. Furthermore, for a smooth cut-off function η, that is, $\eta : (0,\infty) \to [0,1]$, $\eta \in C^\infty$, with $\eta = 1$ on $(0, \frac{1}{2})$ and $\eta = 0$ on $(1,\infty)$, we introduce the smooth β-localization of T by

$$T^\eta_{\beta\text{-loc}} f(x) = T\left(f \eta\left(\frac{d(x,\cdot)}{\beta d(x,\Omega^c)}\right)\right) \,.$$

The following result was proved in [14, Proposition 5.1], and we omit its proof.

Proposition 6.2. *The operator $T^\eta_{\beta\text{-loc}}$ is a β-local singular integral operator for any $\beta \in (0,1)$ and any smooth cut-off function η as above.*

Theorem 6.3. *Let $0 < \tau < 1$, $1 < p \leq q < \infty$, and let u, v be weights such that $(u,v) \in A^\tau_{p,\psi,q,\phi}$, where ϕ and ψ are Young functions such that $\overline{\phi}$ and $\overline{\psi}$ belong to $B_{q'}$ and B_p, respectively. If T is a C-Z operator, then for each $\beta < \tau$ we have that*

$$T^\eta_{\beta\text{-loc}}, T_{\beta\text{-loc}} : L^p(\Omega, v) \to L^q(\Omega, u) \,.$$

Proof. The boundedness for $T^\eta_{\beta\text{-loc}}$ follows directly from Proposition 6.2 and Theorem 1.13.

Preprint

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On the other hand, taking into account the support of the function η and the size condition on the kernel, we estimate for $T_{\beta\text{-loc}}$ as follows

$$\begin{aligned}\left|T^{\eta}_{\beta\text{-loc}}f(x)-T_{\beta\text{-loc}}f(x)\right| &\leq \int_{\frac{\beta}{2}d(x,\Omega^c)\leq d(x,y)\leq \beta d(x,\Omega^c)} |K(x,y)|\,|f(y)|\,dy\\ &\leq \frac{C}{|Q(x,\beta d(x,\Omega^c))|}\int_{Q(x,\beta d(x,\Omega^c))}|f(y)|\,dy\\ &\leq CM_{\beta}f(x)\,,\end{aligned}$$

where M_β is a local maximal operator (see Definition 1.2 with $\gamma = 0$). Then, we obtain

$$\begin{aligned}|T_{\beta\text{-loc}}f(x)| &\leq \left|T^{\eta}_{\beta\text{-loc}}f(x)\right| + \left|T_{\beta\text{-loc}}f(x)-T^{\eta}_{\beta\text{-loc}}f(x)\right|\\ &\leq \left|T^{\eta}_{\beta\text{-loc}}f(x)\right| + CM_{\beta}f(x)\,.\end{aligned}$$

Finally, using the boundedness of $T^{\eta}_{\beta\text{-loc}}$, Lemma 1.11 and Theorem 1.9 in [25] we obtain

$$\|T_{\beta\text{-loc}}f\|_{L^q(\Omega,u)} \leq \left\|T^{\eta}_{\beta\text{-loc}}f\right\|_{L^q(\Omega,u)} + C\,\|M_{\beta}f\|_{L^q(\Omega,u)} \leq C\,\|f\|_{L^p(\Omega,v)}\,,$$

we get our result. □

6.2. **Hörmander type conditions on the kernel.** As we mentioned in Section 3, in [19] the author construct a sparse operator that pointwisely dominates a sublinear operator T provided that T is of weak type (p,p) and a certain truncated maximal operator is of weak type (q,q), where $1 \leq p \leq q < \infty$.

With this in mind, and drawing inspiration from [2], we now consider β-local singular integral operators with kernels satisfying s-Hörmander type conditions, where $1 < s < \infty$ (see Definition 1.3). Using the same strategy as before with minor modifications one can prove that $\mathcal{M}_T$ is of weak type (s',s'); consequently, there exists an appropriate sparse operator that pointwisely dominates T_β, thereby yielding the two-weight boundedness of the latter.

Then, using an analogous argument to that employed in subsection 6.1 for singular integrals operators with pointwise conditions, we can study β-localizations of singular integrals with Hörmander type conditions on the kernel.

For that, let $\mathcal{T}$ be a C-Z operator on $\mathbb{R}^n$ with Hörmander type conditions. That is, for $1 < s < \infty$ we have that $\mathcal{T}$ is of weak type (s',s') and there exists a kernel $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$ such that for $f \in L^\infty_c$ the following integral representation holds

$$\mathcal{T}f(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy,\ x \notin \operatorname{supp} f\,,$$

and the kernel satisfies the size and smoothness Hörmander type conditions as in *a)* and *b)* of *iii)* in Definition 1.14 where the uniform metric can be used here.

Then, given a proper open subset Ω of $\mathbb{R}^n$ and an operator $\mathcal{T}$ as described above, for $\beta \in (0,1)$ we introduce the β-localization of $\mathcal{T}$ as

$$\mathcal{T}_{\beta\text{-loc}}f(x) = \mathcal{T}\left(f\chi_{Q(x,\beta d(x,\Omega^c))}\right),$$

Preprint

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and

$$\mathcal{T}^{\eta}_{\beta\text{-loc}} f(x) = \mathcal{T}\left(f\eta\left(\frac{d(x,\cdot)}{\beta d(x,\Omega^c)} \right) \right).$$

where η is as in the precedent subsection.

For these operators, by following a reasoning similar to that in [14, Theorem 5.1], and taking into account that the smoothed kernel satisfies Hörmander type conditions, a fact we will prove later, the following result can be obtained.

Proposition 6.4. *The operator $\mathcal{T}^{\eta}_{\beta\text{-loc}}$ is a β-local singular integral operator of type $\mathcal{H}_s$, for any $\beta \in (0,1)$ and η be a smooth cut-off function as above.*

With the foregoing in mind, we are in a position to prove that the β-localizations we defined are bounded with two weights.

Theorem 6.5. *Let $0 < \tau < 1$, $1 < s < \infty$, $s' < p \leq q < \infty$ and u, v be weights such that $(u,v) \in A^{\tau}_{\frac{p}{s'},\psi,\frac{q}{s'},\phi}(\Omega)$, where ϕ and ψ are Young functions such that $\overline{\phi_{s'}}$ and $\overline{\psi}_{s'}$ belong to $B_{q'}$ and B_p respectively. If T is a C-Z operator with Hörmander type conditions, then for each $\beta < \tau$ we have that*

$$\mathcal{T}^{\eta}_{\beta\text{-loc}}, \mathcal{T}_{\beta\text{-loc}} : L^p(\Omega, v) \to L^q(\Omega, u).$$

Proof. For $\mathcal{T}^{\eta}_{\beta\text{-loc}}$, boundedness is a direct consequence of Proposition 6.4 and 1.15. For $\mathcal{T}_{\beta\text{-loc}}$, the argument follows the same lines as the proof of Theorem 6.3, where the boundedness of a certain maximal operator is needed and it is a consequence of Theorem 1.9 in [25]. □

6.3. **Schrödinger type operators.** We analyze here a two weight boundedness result for singular integral operators associated to the Schrödinger operator

$$\mathcal{L} = -\Delta + V,$$

where V is a non negative potential function belonging to the class RH_s, for some $1 < s < \infty$, that is,

$$\left(\frac{1}{|B|} \int_B V(y)^s dy \right)^{1/s} \leq C \left(\frac{1}{|B|} \int_B V(y)\, dy \right),$$

for every ball $B \subset \mathbb{R}^n$, with $n \geq 3$.

This theory has been of interest for years, beginning with the boundedness results on Lebesgue spaces found in Shen's work [31]. Regarding the line in this paper, we must mention Bongioanni, Harboure, and the second author of the present work; in [4], they introduced new classes of weights associated with L. See also [2] for analogous results concerning extensions of these operators.

In this context, it is known that $\mathcal{L}$ have local behavior similar to that of the Laplacian. This notion of locality is determined by the critical radius function given by

$$\rho(x) = \sup\left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)\, dy \leq 1 \right\}. \tag{6.6}$$

Working with the uniform metric $d = d_\infty$, the following propositions can be rewritten as in [31].

Preprint

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Proposition 6.7. *There exist $C_0, N_0 \geq 1$ such that, for every $x, y \in \mathbb{R}^n$ we have*

$$C_0^{-1}\rho(x)\left(1+\frac{d(x,y)}{\rho(x)}\right)^{-N_0} \leq \rho(y) \leq C_0\rho(x)\left(1+\frac{d(x,y)}{\rho(x)}\right)^{\frac{N_0}{N_0+1}}, \tag{6.8}$$

where the constants involved depend only on the context.

Proposition 6.9. *Let $\sigma > 0$, $x, y \in Q(x_Q, \sigma\ell)$, where $\ell \leq \rho(x_Q)$, then*

$$\rho(x) \leq C\rho(y)\,,$$

for a constant C that depends only on σ and the constants appearing in (6.8).

For the following result, we denote the cube $Q(x, \rho(x))$ a critical cube for $x \in \mathbb{R}^n$.

Proposition 6.10 (Proposition 5 in [9] or Lemma 2.3 in [11])**.** *There exists a collection $\{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ such that the family of critical cubes $\{Q_j\}_{j\in\mathbb{N}}$ is a covering for $\mathbb{R}^n$, where $Q_j = Q(x_j, \rho(x_j))$, $j \in \mathbb{N}$. Moreover, there exist constants C and N_1 such that for any $\sigma \geq 1$ it holds that*

$$\sum_{j\in\mathbb{N}} \chi_{\sigma Q_j} \leq C\sigma^{N_1}.$$

Proposition 6.11. *(Proposition 6 in* [33]*) Let Q_x be a critical cube. Then there exists $\sigma \geq 1$, depending only on the constant in* (6.8) *such that*

$$\bigcup_{\substack{y\in\mathbb{R}^n \\ Q_y\cap Q_x\neq\emptyset}} Q_y \subset \sigma Q_x \doteq \tilde{Q}_x\,.$$

Here, we consider the following class of weights.

Definition 6.12. *Let $1 < p, q < \infty$ and $c, m \geq 0$. We say that the pair (u, v) belongs to the class $H^{\rho,m}_{p,q,c}$ if there exists $C > 0$ such that*

$$|Q|^{\frac{1}{q}-\frac{1}{p}}\left\|u^{\frac{1}{q}}\right\|_{q,Q}\left\|v^{-\frac{1}{p}}\right\|_{p',Q} \leq C\exp\left\{c\left(1+\frac{\ell}{\rho(x)}\right)^m\right\},$$

for every cube $Q = Q(x, \ell)$.

Definition 6.13. *Let $1 < p, q < \infty$, $c, m \geq 0$ and ϕ, ψ Young functions. We say that the pair (u, v) belongs to the class $H^{\rho,m}_{p,\psi,q,\phi,c}$ if there exists $C > 0$ such that*

$$|Q|^{\frac{1}{q}-\frac{1}{p}}\left\|u^{\frac{1}{q}}\right\|_{\phi,Q}\left\|v^{-\frac{1}{p}}\right\|_{\psi,Q} \leq C\exp\left\{c\left(1+\frac{\ell}{\rho(x)}\right)^m\right\},$$

for every cube $Q = Q(x, \ell)$.

The version involving norms in Lebesgue spaces can be found in [15]. Additionally, the case where the weights coincide can be found in [4].

An analogous version to Lemma 1.11 can be obtained.

Lemma 6.14. *Let $1 < p, q < \infty$, ϕ and ψ Young functions. If $\overline{\psi} \in B_p$ and $\overline{\phi} \in B_{q'}$ then*

$$H^{\rho,m}_{p,\psi,q,\phi,c} \subset H^{\rho,m}_{p,q,c}\,.$$

Moreover, these weights can be related to those given in Definition 1.10.

Preprint

IMAL PREPRINT # 2026-0085
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Lemma 6.15. *Let $\tau \in (0,1)$, $\sigma > 0$ and $x \in \mathbb{R}^n$. If $(u,v) \in H^{\rho,m}_{p,\psi,q,\phi,c}$ then*

$$(u,v) \in A^{\tau}_{p,\psi,q,\phi}(\sigma Q_x)\,,$$

where $Q_x = Q(x,\rho(x))$.

Proof. Let $\tau \in (0,1)$, $\sigma > 0$ and $x \in \mathbb{R}^n$. We consider $x_0 \in Q(x,\sigma\rho(x))$ and $\ell > 0$ such that $Q = Q(x_0,\ell) \in \mathcal{F}_\tau(\sigma Q_x)$. For these cubes, note that $\ell < \sigma\rho(x)$ and $\rho(x) \approx \rho(x_0)$ by Proposition 6.9. Therefore since $(u,v) \in H^{\rho,m}_{p,\psi,q,\phi,c}$ we get

$$\begin{aligned}|Q|^{\frac{1}{q}-\frac{1}{p}} \left\|u^{\frac{1}{q}}\right\|_{\phi,Q} \left\|v^{-\frac{1}{p}}\right\|_{\psi,Q} &\le C\exp\left(c\left(1+\frac{\ell}{\rho(x_0)}\right)^m\right)\\ &\le C\exp\left(c\left(1+\sigma\frac{\rho(x)}{\rho(x_0)}\right)^m\right) \le C\,.\end{aligned}$$

The result follows by taking supremum over all cubes in family $\mathcal{F}_\tau(\sigma Q_x)$. □

We present now the operators that we will consider, which were introduced in [10]. As usual Δ denotes the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$.

Definition 6.16. *We say that a lineal operator T is a Schrödinger-Calderón-Zygmund operator of exponential type (eSCZ) if*

(I) T is bounded in L^p, for some $1 < p < \infty$.

(II) There exists a kernel $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{R}$ such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy\,,$$

a.e $x \notin \operatorname{supp} f$ and

(a) there exist constants $c, C > 0$ and $m > 0$ such that

$$|K(x,y)| \le \frac{C}{|x-y|^n}\exp\left(-c\left(1+\frac{|x-y|}{\rho(x)}\right)^m\right).$$

(b) There exist constants $C, \delta > 0$ such that

$$|K(x,y) - K(x',y)| \le C\frac{|x-x'|^\delta}{|x-y|^{n+\delta}}\,,$$

whenever $2|x-x'| \le |x-y|$.

We consider another family of operators with Hormander type conditions.

Definition 6.17. *For $1 < s < \infty$ we say that a lineal operator T is an eSCZ operator of type $\mathcal{H}_s$ if*

(I) T is bounded from $L^{s'}$ on $L^{s',\infty}$.

(II) There exists a kernel $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{R}$ such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy\,,$$

a.e $x \notin \operatorname{supp} f$ and

(a) there exist constants $c, C > 0$ and $m > 0$ such that

$$\left(\frac{1}{R^n}\int_{R<|x'-y|<2R}|K(x,y)|^s\,dy\right)^{\frac{1}{s}} \le \frac{C}{R^n}\exp\left(-c\left(1+\frac{R}{\rho(x)}\right)^m\right),$$

whenever $|x-x'| < \frac{R}{2}$.

Preprint

IMAL PREPRINT # 2026-0085
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(b) There exist constants $C, \delta > 0$ such that

$$\left(\frac{1}{R^n}\int_{R<|x'-y|<2R} |K(x,y)-K(x',y)|^s\,dy\right)^{\frac{1}{s}} \leq \frac{C}{R^n}\left(\frac{r}{R}\right)^{\delta},$$

whenever $|x-x'|<r<\frac{R}{2}$ and $r\leq\rho(x')$.

Remark 6.18. It should be noted that the two preceding operators are also singular integrals in the classical sense.

In this context we present the following results.

Theorem 6.19. *Let $1<p\leq q<\infty$ and let ϕ and ψ be Young functions such that $\overline{\phi}$ and $\overline{\psi}$ belong to $B_{q'}$ and B_p respectively. If T is an eSCZ operator with constant $m>0$ then*

$$T: L^p(v)\to L^q(u)\,,$$

for every pair of weight $(u,v)\in H^{\rho,m_1}_{p,\psi,q,\phi,c_1}$ with $m_1<m$.

Proof. Without loss of generality, we assume that f is non-negative, essentially bounded, and its support is compact.

Let us consider $\{Q_j\}$ the critical cubes provided by Proposition 6.10. By using the Proposition 6.11 we can take $\sigma\geq 1$ such that $\cup_{x\in Q_j} Q(x,\rho(x))\subset \sigma Q_j=\tilde{Q}_j$. For each j, it is possible to take $\theta>1$ such that the critical cubes belong to a certain local family of well-inside cubes in $\Omega_j=\theta\tilde{Q}_j$. More precisely, there exists $\beta\in(0,1)$ for which $Q(x,\rho(x))\in\mathcal{F}_\beta(\Omega_j)$ for every $x\in Q_j$. For further details, this argument can be found in the proof of Theorem 5.11 in [25]. For each $x\in Q_j$ we write

$$Tf(x)=T^j_{\beta\text{-loc}}f(x)+T^j_{\beta\text{-glob}}f(x)\,,$$

where $T^j_{\beta\text{-loc}}f(x)=T\big(f\chi_{Q(x,\beta d(x,\Omega_j^c))}\big)(x)$, $T^j_{\beta\text{-glob}}f(x)=T\big(f\chi_{Q(x,\beta d(x,\Omega_j^c))^c}\big)(x)$.

Thus

$$\int_{\mathbb{R}^n}|Tf|^q\,u\leq C\sum_j\int_{Q_j}\left|T^j_{\beta\text{-loc}}f\right|^q u + C\sum_j\int_{Q_j}\left|T^j_{\beta\text{-glob}}f\right|^q u\,. \tag{6.20}$$

Based on Remark 6.18 and the definitions in Section 6.1, we see that $T^j_{\beta\text{-loc}}$ is a β-localization of T with respect to Ω_j. Thus, Lemma 6.15 and Theorem 6.3 establish the boundedness result for each Ω_j; then, using the fact that $p\leq q$ and the controlled overlap provided by Proposition 6.10, we can deduce the following estimate.

$$\begin{aligned}\sum_j\int_{Q_j}\left|T^j_{\beta\text{-loc}}f\right|^q u &\leq C\sum_j\left(\int_{\Omega_j}|f|^p\,v\right)^{\frac{q}{p}}\\ &\leq C\left(\int_{\mathbb{R}^n}|f|^p\left(\sum_j\chi_{\theta\sigma Q_j}\right)v\right)^{\frac{q}{p}}\leq C\,\|f\|^q_{L^p(v)}\,.\end{aligned} \tag{6.21}$$

For the second sum, let us remember that $Q_x=Q(x,\rho(x))\subset Q(x,\beta d(x,\Omega_j^c))$, for every $x\in Q_j$. So, using the conditions on the kernel, we get

$$\left|T^j_{\beta\text{-glob}}f(x)\right|\leq\int_{Q_x^c}|K(x,y)|\,|f(y)|\,dy$$

Preprint

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$$\leq C\sum_{k=1}^{\infty}\int_{2^kQ_x\setminus 2^{k-1}Q_x}\exp\left\{-c\left(1+\frac{|x-y|}{\rho(x)}\right)^m\right\}\frac{|f(y)|}{|x-y|^n}dy$$
$$\leq C\rho(x)^{-n}\sum_{k=1}^{\infty}\exp\left\{-c\left(1+2^{k-1}\right)^m\right\}2^{-kn}\int_{2^kQ_x}|f|.$$

Taking into account the argument given at the beginning of the proof, for each $k\in\mathbb{N}$ we have that $2^kQ_x\subset 2^k\tilde{Q}_j\doteq\tilde{Q}_j^k$. By Proposition 6.9 and by Minkowski's inequality, we have

$$\left(\sum_j\int_{Q_j}\left|T^j_{\beta\text{-glob}}f\right|^q u\right)^{1/q}$$
$$\leq C\left(\sum_j\int_{Q_j}\left(\rho(x)^{-n}\sum_k\exp\left\{-c\left(1+2^{k-1}\right)^m\right\}2^{-kn}\int_{2^kQ_x}|f|\right)^q u(x)dx\right)^{1/q}$$
$$\leq C\sum_k\left(\sum_j\int_{Q_j}\left(\rho(x)^{-n}\exp\left\{-c\left(1+2^{k-1}\right)^m\right\}2^{-kn}\int_{2^kQ_x}|f|\right)^q u(x)dx\right)^{1/q}$$
$$\leq C\sum_k\exp\left\{-c\left(1+2^{k-1}\right)^m\right\}2^{-kn}\left(\sum_j\rho(x_j)^{-nq}\left(\int_{\tilde{Q}_j^k}|f|\right)^q\left(\int_{Q_j}u\right)\right)^{1/q}.$$

Applying Hölder's inequality and rearranging terms, it follows that

$$\rho(x_j)^{-nq}\left(\int_{\tilde{Q}_j^k}|f|\right)^q\left(\int_{Q_j}u\right)$$
$$\leq\rho(x_j)^{-nq}\left(\int_{\tilde{Q}_j^k}|f|\,v^{\frac{1}{p}}v^{-\frac{1}{p}}\right)^q\left(\int_{\tilde{Q}_j^k}u\right)$$
$$\leq\rho(x_j)^{-nq}\left(\int_{\tilde{Q}_j^k}u\right)\left(\int_{\tilde{Q}_j^k}v^{-\frac{p'}{p}}\right)^{\frac{q}{p'}}\left(\int_{\tilde{Q}_j^k}|f|^p\,v\right)^{\frac{q}{p}}.$$
$$\leq\left(\rho(x_j)^{-n}\left(\int_{\tilde{Q}_j^k}u\right)^{\frac{1}{q}}\left(\int_{\tilde{Q}_j^k}v^{-\frac{p'}{p}}\right)^{\frac{1}{p'}}\right)^q\left(\int_{\tilde{Q}_j^k}|f|^p\,v\right)^{\frac{q}{p}}.$$

Since $(u,v)\in H^{\rho,m_1}_{p,q,c_1}$ and by Lemma 6.14, for each j, we can obtain that

$$\rho(x_j)^{-n}\left(\int_{\tilde{Q}_j^k}u\right)^{\frac{1}{q}}\left(\int_{\tilde{Q}_j^k}v^{-\frac{p'}{p}}\right)^{\frac{1}{p'}}$$
$$\leq C\rho(x_j)^{-n}(2^k\sigma\rho(x_j))^n\exp\left\{c_1\left(1+\frac{2^k\sigma\rho(x_j)}{\rho(x_j)}\right)^{m_1}\right\}$$
$$\leq C2^{kn}\exp\left\{c_1\left(1+2^k\sigma\right)^{m_1}\right\}.$$

Therefore, using $p\leq q$ and Proposition 6.10, the second sum can be estimates as follow

Preprint

IMAL PREPRINT # 2026-0085
ISSN 2451-7100
Publication date: November 13, 2026

$$\left(\sum_j \int_{Q_j} \left|T^j_{\beta\text{-glob}} f\right|^q u\right)^{\frac{1}{q}}$$

$$\leq C \sum_k \exp\left\{-c\left(1+2^{k-1}\right)^m\right\} \exp\left\{c_1\left(1+2^k\sigma\right)^{m_1}\right\} \left(\sum_j \left(\int_{\tilde{Q}^k_j} |f|^p v\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}$$

$$\leq C \sum_k \exp\left\{-c\left(1+2^{k-1}\right)^m\right\} \exp\left\{c_1\left(1+2^k\sigma\right)^{m_1}\right\} \left(\int_{\mathbb{R}^n} |f|^p \left(\sum_j \chi_{\tilde{Q}^k_j}\right) v\right)^{\frac{1}{p}}$$

(6.22)

$$\leq C \left(\sum_k \exp\left\{-c\left(1+2^{k-1}\right)^m\right\} \exp\left\{c_1\left(1+2^k\sigma\right)^{m_1}\right\} 2^{\frac{kN_1}{p}}\right) \|f\|_{L^p(v)}.$$

Noting that because $m_1 < m$ the series above converges. Finally, returning to (6.20), with (6.21) and (6.22) the proof of the theorem is complete. □

The following theorem provides the boundedness result for singular integrals under Hörmander-type conditions; here, $\phi_r(t) \doteq \phi(t^r)$.

Theorem 6.23. *Let $1 < s < \infty$, $s' < p \leq q < \infty$ and let ϕ and ψ be Young functions such that $\overline{\phi_{s'}}$ and $\overline{\psi}_{s'}$ belong to $B_{q'}$ and B_p respectively. If T is an eSCZ of type $\mathcal{H}_s$ operator with constant $m > 0$ then*

$$T : L^p(v) \to L^q(u),$$

for every pair of weight $(u, v) \in H^{\rho,m_1}_{\frac{p}{s'},\psi,\frac{q}{s'},\phi,c_1}$ with $m_1 < m$.

Proof. As in the previous theorem, we consider a non-negative, essentially bounded function f with compact support. We consider the decomposition into critical cubes $\{Q_j\}$ given by Proposition 6.10. It is then possible to consider dilations Ω_j of these cubes to define β-localizations of T respect to Ω_j as in Section 6.2, thereby obtaining an estimate analogous to (6.20) for these operators. For the first sum therein, we proceed as in (6.21), applying this time our new conditions on the weights and using Theorem 6.5 instead of Theorem 6.3.

For the second sum, for each j and $x \in Q_j$ we note that $Q_x = Q(x, \rho(x)) \subset Q(x, \beta d(x, \Omega_j^c))$. Thus, by using the size condition of the kernel we get

$$\left|T^j_{\beta\text{-glob}} f(x)\right| \leq \int_{Q_x^c} |K(x,y)|\,|f(y)|\,dy$$

$$\leq \sum_{k=1}^{\infty} \left(\int_{2^k Q_x \setminus 2^{k-1} Q_x} |K(x,y)|^s\,dy\right)^{\frac{1}{s}} \left(\int_{2^k Q_x \setminus 2^{k-1} Q_x} |f|^{s'}\right)^{\frac{1}{s'}}$$

$$\leq C \sum_{k=1}^{\infty} \frac{1}{\left(2^{k-1}\rho(x)\right)^{\frac{n}{s'}}} \exp\left\{-c\left(1+\frac{2^{k-1}\rho(x)}{\rho(x)}\right)^m\right\} \left(\int_{2^k Q_x} |f|^{s'}\right)^{\frac{1}{s'}}$$

Preprint

IMAL PREPRINT # 2026-0085
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$$\leq C\rho(x)^{-n/s'}\sum_{k=1}^{\infty}\exp\left\{-c\left(1+2^{k-1}\right)^m\right\}2^{-kn/s'}\left(\int_{2^kQ_x}|f|^{s'}\right)^{\frac{1}{s'}}.$$

Since $Q_x\subset\tilde{Q}_j$ (see Proposition 6.11), by setting $\tilde{Q}_j^k=2^k\tilde{Q}_j\supset 2^kQ_x$, observing that $\rho(x)\approx\rho(x_j)$ (Proposition 6.9) and by Minkowski's inequality, we can estimate

$$\begin{aligned}
&\left(\sum_j\int_{Q_j}\left|T^j_{\beta\text{-glob}}f\right|^q u\right)^{\frac{1}{q}}\\
&\leq C\left(\sum_j\int_{Q_j}\left(\rho(x)^{-n/s'}\sum_k\exp\left\{-c\left(1+2^{k-1}\right)^m\right\}2^{-kn/s'}\left(\int_{2^kQ_x}|f|^{s'}\right)^{\frac{1}{s'}}\right)^q u(x)dx\right)^{\frac{1}{q}}\\
&\leq C\sum_k\left(\sum_j\int_{Q_j}\left(\rho(x)^{-n/s'}\exp\left\{-c\left(1+2^{k-1}\right)^m\right\}2^{-kn/s'}\left(\int_{2^kQ_x}|f|^{s'}\right)^{\frac{1}{s'}}\right)^q u(x)dx\right)^{\frac{1}{q}}\\
&\leq C\sum_k\exp\left\{-c\left(1+2^{k-1}\right)^m\right\}2^{-kn/s'}\left(\sum_j\rho(x_j)^{-n\frac{q}{s'}}\left(\int_{\tilde{Q}_j^k}|f|^{s'}\right)^{\frac{q}{s'}}\left(\int_{Q_j}u\right)\right)^{\frac{1}{q}}.
\end{aligned}$$

We proceed as in the analogous estimation in the previous theorem. First by Hölder's inequality with p/s' instead of p. Then, noticing that by hypothesis, $\overline{\psi}_{s'}\in B_p$, which implies that $\overline{\psi}\in B_{\frac{p}{s'}}$. Then, it can be shown ([5]) that

$$\left\|v^{-\frac{s'}{p}}\right\|_{\left(\frac{p}{s'}\right)',\tilde{Q}_j^k}\leq C\left\|v^{-\frac{s'}{p}}\right\|_{\psi,\tilde{Q}_j^k}.$$

On the other hand, it is easy to see that $\|g\|^r_{\phi_r,Q}=\|g^r\|_{\phi,Q}$ for any cube Q, $g\in L^1_{\text{loc}}$ and $1<r<\infty$. Thus

$$\left\|u^{\frac{s'}{q}}\right\|_{\frac{q}{s'},\tilde{Q}_j^k}=\left\|u^{\frac{1}{q}}\right\|^{s'}_{q,\tilde{Q}_j^k}\leq C\left\|u^{\frac{1}{q}}\right\|^{s'}_{\phi_{s'},\tilde{Q}_j^k}=C\left\|u^{\frac{s'}{q}}\right\|_{\phi,\tilde{Q}_j^k}.$$

And using that $(u,v)\in H^{\rho,m_1}_{\frac{p}{s'},\psi,\frac{q}{s'},\phi,c_1}$ we get

$$\rho(x_j)^{-n\frac{q}{s'}}\left(\int_{\tilde{Q}_j^k}|f|^{s'}\right)^{\frac{q}{s'}}\left(\int_{Q_j}u\right)\leq C2^{-kn/s'}\exp\left\{\frac{c_1}{s'}\left(1+2^k\sigma\right)^{m_1}\right\}\left(\int_{\tilde{Q}_j^k}|f|^p v\right)^{\frac{1}{p}}.$$

Finally, reasoning as in (6.22), we can conclude that

$$\begin{aligned}
&\left(\sum_j\int_{Q_j}\left|T^j_{\beta\text{-glob}}f\right|^q u\right)^{\frac{1}{q}}\\
&\leq C\sum_k\exp\left\{-c\left(1+2^{k-1}\right)^m\right\}\exp\left\{\frac{c_1}{s'}\left(1+2^k\sigma\right)^{m_1}\right\}\left(\sum_j\left(\int_{\tilde{Q}_j^k}|f|^p v\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\\
&\leq C\left(\sum_k\exp\left\{-c\left(1+2^{k-1}\right)^m\right\}\exp\left\{\frac{c_1}{s'}\left(1+2^k\sigma\right)^{m_1}\right\}2^{\frac{kN_1}{p}}\right)\|f\|_{L^p(v)}.
\end{aligned}$$

Noting that $m_1<m$ the series above converges. Then the proof of the theorem is complete.

□

Preprint

IMAL PREPRINT # 2026-0085
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6.4. **Riesz-Schrödinger Transform.** The Riesz-Schrödinger Transforms and their adjoints are given by

$$\mathcal{R} = \nabla \mathcal{L}^{-\frac{1}{2}}, \qquad \mathcal{R}^\star = \mathcal{L}^{-\frac{1}{2}} \nabla .$$

Recall that N_0 is the constant given in Proposition 6.8, we set $m_0 = \frac{1}{N_0+1}$.

Theorem 6.24. *([10, Proposition 5.3 and 5.5]) Let $V \geq 0$ satisfying a reverse Hölder inequality of order $\alpha > \frac{n}{2}$ and $n \geq 3$. Then we have*

(i) $\mathcal{R}$ y $\mathcal{R}^\star$ are eSCZ operators with constant m_0 if $\alpha > n$.
(ii) $\mathcal{R}^\star$ is an eSCZ operator of type $\mathcal{H}_s$ with constant m_0, where $1 < s < \frac{n}{n-\alpha}$ if $\alpha < n$.

Remark 6.25. If $V \in RH_\alpha$ with $\frac{n}{2} < \alpha < n$, then $\mathcal{R}$ is not a SCZ operator (neither with pointwise conditions nor of type $\mathcal{H}_s$, for any $1 < s < \infty$). Indeed, both definitions of SCS singular integrals imply that the operator is also a classical singular integral and is therefore bounded on L^p, with $p > 1$ in the case of pointwise conditions and with $p > s'$ in the case of type $\mathcal{H}_s$. However, Shen proved in [31, Theorem 0.5] that if $V \in RH_\alpha$ with $\frac{n}{2} < \alpha < n$, then $\mathcal{R}$ is bounded on L^p for $1 < p \leq p_0$, where $\frac{1}{p_0} = \frac{1}{\alpha} - \frac{1}{n}$. Furthermore, this range is sharp ([31, Section 7]). Since there exists $p > p_0$ for which $\mathcal{R}$ is unbounded, we conclude that it cannot be a SCZ operator under any of the considered definitions.

The following results are direct consequences of the preceding theorem and of Theorems 6.19 and 6.23.

Theorem 6.26. *Let $V \in RH_\alpha$ with $\alpha > n$, $1 < p \leq q < \infty$ and ϕ, ψ be Young functions such that $\overline{\phi}$ and $\overline{\psi}$ belong to $B_{q'}$ and B_p respectively. If $(u, v) \in H^{\rho,m}_{p,\psi,q,\phi,c}$ with $m < m_0$ then*

$$\mathcal{R}, \mathcal{R}^\star : L^p(v) \to L^q(u) .$$

Theorem 6.27. *Let $V \in RH_\alpha$ with $\frac{n}{2} < \alpha < n$, s be such that $\frac{1}{s} = \frac{1}{\alpha} - \frac{1}{n}$, $s' < p \leq q < \infty$ and ϕ, ψ be Young functions such that $\overline{\phi_{s'}}$ and $\overline{\psi}_{s'}$ belong to $B_{q'}$ and B_p respectively. If $(u, v) \in H^{\rho,m}_{\frac{p}{s'},\psi,\frac{q}{s'},\phi,c}$ with $m < m_0$ then*

$$\mathcal{R}^\star : L^p(v) \to L^q(u) .$$

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